Study Guide on Tails of Copulas for the Casualty Actuarial Society (CAS) Exam 7

(Based on Gary Venter's Paper, "Tails of Copulas")

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Source: Venter, G.G., "Tails of Copulas," *PCAS* LXXXIX, 2002, pp. 68-113.

This is an open-source study guide and may be revised pursuant to suggestions.

S7-TC-1. According to Venter (p. 68), "Copulas differ not so much in the degree of association they provide, but rather in" *what*?

Solution S7-TC-1. Copulas differ in **which part of the distribution the association is strongest.**

S7-TC-2. According to Venter (pp. 68-69), a copula separates the joint distribution into two contributions. What are they?

Solution S7-TC-2.
1. The marginal distributions of the individual variables
2. The interdependency of the probabilities

S7-TC-3. Of the following measures, which ones depend only on the copula and not on the marginal distributions? (Venter, pp. 68-69)
(i) Pearson linear product-moment correlation
(ii) Kendall's tau
(iii) Spearman's rank correlation

Solution S7-TC-3. The following measures depend only on the copula and not on the marginal distributions:
(ii) Kendall's tau
(iii) Spearman's rank correlation

S7-TC-4. Given marginal distributions $F_X(x)$, $F_Y(y)$, and copula function C(u, v) such that the joint distribution $F(x,y) = C(F_X(x), F_Y(y))$, then what is the conditional distribution function $F_{Y|X}(y)$ for $Y | X = x$? Express your answer with reference to C. (Venter, pp. 70-71)

Solution S7-TC-4. Let $C_u(u, v)$ be the first partial derivative of the copula with respect to u. We differentiate with respect to u, the first argument of the copula, because it is x, the first of the two random variables, that is being held constant in the conditional distribution function.
Thus, $\mathbf{F_{Y|X}(y) = C_u(F_X(x), F_Y(y))}$.

S7-TC-5.
(a) Is the linear correlation coefficient based on the covariance of two variates preserved by copulas?
(b) Is the Kendall correlation τ preserved by copulas?
(c) What does Venter consider to be the "simplest formula" for the Kendall correlation τ? (Venter, p. 71)

Solution S7-TC-5.
(a) No.
(b) Yes.
(c) $\boldsymbol{\tau = 4*E[C(u, v)] - 1}$.

S7-TC-6. Give the formula for Frank's copula C(u, v) with parameter $a \neq 0$.
(Venter, p. 72)

Solution S7-TC-6.
$C(u, v) = -(1/a)*[\ln(1 + (e^{-au}-1)*(e^{-av}-1)/(\partial(e^{-au}-1)/\partial u)] =$
$\boldsymbol{C(u, v) = -(1/a)*[\ln(1 + (e^{-au}-1)*(e^{-av}-1)/(-ae^{-au})]}$.

S7-TC-7. Give the formula for the Gumbel copula C(u, v) with parameter $a \geq 1$.
(Venter, p. 73)

Solution S7-TC-7. $\boldsymbol{C(u, v) = \exp(-[(-\ln(u))^a + (-\ln(v))^a]^{1/a})}$.

S7-TC-8. State two ways in which the Gumbel copula differs qualitatively from Frank's copula.
(Venter, pp. 72-73)

Solution S7-TC-8.
1. The Gumbel copula has more probability concentrated in the tails than Frank's copula.
2. The Gumbel copula is asymmetric and has more weight in the right tail.

S7-TC-9. (a) Give the formula for the Heavy Right Tail (HRT) copula C(u, v) with parameter $a > 0$.
(b) Give a brief qualitative description of this copula's most essential attribute. (Venter, p. 73)

Solution S7-TC-9.
(a) $\boldsymbol{C(u, v) = u + v - 1 + [(1-u)^{-1/a} + (1-v)^{-1/a} - 1]^{-a}}$.
(b) This copula has high correlation in the right tail and less correlation in the left tail than many other copulas.

S7-TC-10.
For each of the following copulas, give the formula for Kendall's τ and solve for τ when the parameter a is equal to 2. You may use a calculator to perform integration where necessary. (Venter, pp. 72-73)
(a) Frank's copula
(b) Gumbel copula
(c) HRT copula

Solution S7-TC-10.
(a) Frank's copula: $\boldsymbol{\tau = 1 - 4/a + (4/a^2)*{}_0^a\int(t/(e^t-1))dt}$. For a = 2, $\tau = 1 - 2 + 1*{}_0^2\int(t/(e^t-1))dt$ =
-1 + 1.213894569 = **τ = 0.213894569**.
(b) Gumbel copula: **τ = 1 – 1/a**. For a = 2, **τ = 1/2**.
(c) HRT copula: **τ = 1/(2a + 1)**. For a = 2, **τ = 1/5**.

S7-TC-11.
(a) Given two Burr distributions $F(x) = 1 - (1 + (x/b)^p)^{-a}$ and
$G(y) = 1 - (1 + (y/d)^q)^{-a}$, what is the formula for the joint Burr distribution F(x, y) from the heavy right tail copula?
(b) What is the formula for the conditional distribution $F_{Y|X}(y \mid x)$ of $Y \mid X = x$? (Venter, p. 74)

Solution S7-TC-11.
(a) $\mathbf{F(x, y) = 1 - (1 + (x/b)^p)^{-a} - (1 + (y/d)^q)^{-a} + [1 + (x/b)^p + (y/d)^q]^{-a}}$.
(b) $\mathbf{F_{Y|X}(y \mid x) = 1 - (1 + (y/d_x)^q)^{-(a+1)}}$**, where** $\mathbf{d_x = d[1 + (x/b)^{p/q}]}$.

S7-TC-12.
(a) How does the right tail of the normal copula compare to the right tails of (i) the Gumbel copula, (ii) the Frank copula, and (iii) the HRT copula?
(b) The left tail of the normal copula is most similar to the left tail of which of the three copulas mentioned above? (Venter, p. 74)

Solution S7-TC-12.
(a) The right tail of the normal copula is
(i) **Lighter** than the right tail of the Gumbel copula;
(ii) **Heavier** than the right tail of the Frank copula;
(iii) **Lighter** than the right tail of the HRT copula.

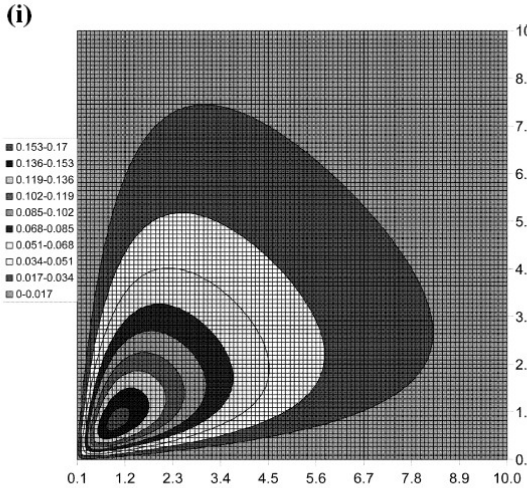
(b) The left tail of the normal copula is most similar to the left tail of the **Gumbel copula.**

S7-TC-13. For the normal copula, give the formula for Kendall's τ and solve for τ when the parameter a is equal to 0.5. (Venter, p. 75)

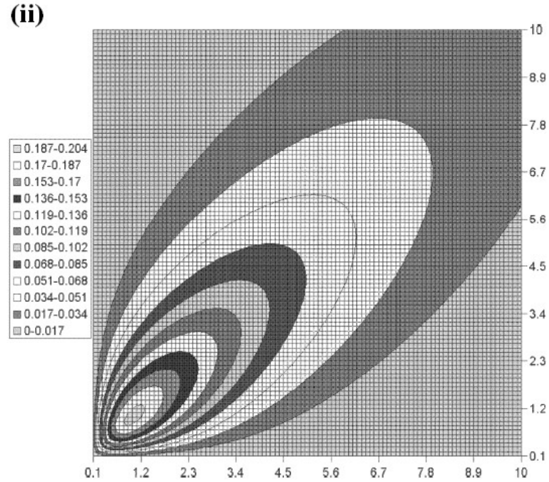
Solution S7-TC-13.
For the normal copula, **τ(a) = 2*arcsin(a)/π.** Thus, τ(0.5) = 2*arcsin(0.5)/π = **1/3**.

S7-TC-14. Based on the following plots of joint unit lognormal density (from Venter, pp. 76-79), rank the copulas displayed in descending order from heaviest right tail to lightest right tail:

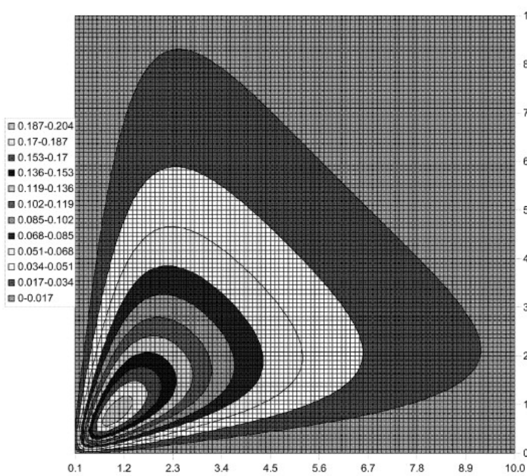
(i)



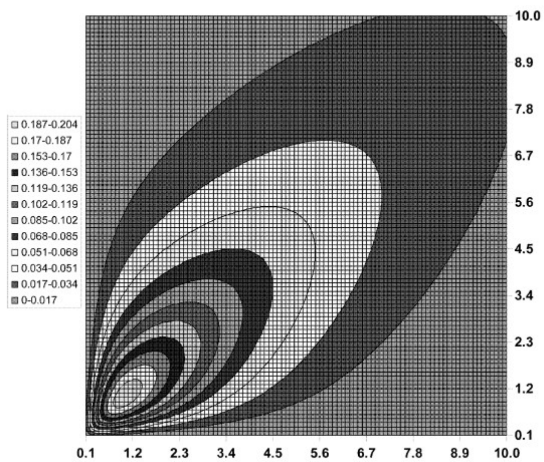
(ii)



(iii)



(iv)



Solution S7-TC-14. Here is the order from heaviest to lightest right tail:
(ii) > (iv) > (i) > (iii).
(We look at the upper right-hand corner and the extent to which the denser parts of the plots spread there.)

S7-TC-15. (a) According to Venter (p. 77), what is "the basic idea" of Rodney Kreps's Partial Perfect Correlation Copula Generator?
(b) Give a brief mathematical description of this basic idea. (Venter, pp. 77-79)

Solution S7-TC-15.
(a) "The basic idea" is to draw two perfectly correlated deviates in some cases and two uncorrelated deviates otherwise.

(b) Let h(u, v) be a symmetric function of u and v, mapping the unit square to the unit interval. Then we draw three random variables u, v, and w. If h(u, v) < w, then we separately simulate x and y as $x = F_X^{-1}(u)$ and $y = F_Y^{-1}(v)$. This corresponds to lack of correlation. If h(u, v) ≥ w, then we let $y = x = F_Y^{-1}(u)$. This corresponds to perfect correlation.

S7-TC-16. With regard to Kreps's procedure, let h(u, v) = h(u)*h(v) and let $H(x) = {}_0{}^x\int h(t)*dt$.
(a) What is the corresponding copula formula C(u, v)?
(b) Starting from C(u, v), arrive at the partial derivative of C with respect to u: $C_u(u, v)$.
(c) Using any previously applicable work, arrive at the copula density function c(u, v).
(Venter, p. 80)

Solution S7-TC-16.
(a) C(u,v) = uv – H(u)*H(v) + H(1)*H(min(u, v)).

(b) $C_u(u, v)$ = v - $H_u(u)$*H(v) + H(1)*H_u(min(u, v)).
$H_u(u)$ = h(u). Thus, $C_u(u, v)$ = v - h(u)*H(v) + H(1)*(H_u(k(u,v)), where
k(u,v) = u if u < v and k(u,v) = v if u ≥ v.
If u ≤ v, then $C_u(u, v)$ = v - h(u)*H(v) + H(1)*$H_u(u)$.
If u > v, then $C_u(u, v)$ = v - h(u)*H(v) + H(1)*$H_u(v)$.
If u ≤ v, then $C_u(u, v)$ = v - h(u)*H(v) + H(1)*h(u).
If u > v, then $C_u(u, v)$ = v - h(u)*H(v).

(c) c(u, v) = $C_{uv}(u, v)$.
We start with $C_u(u, v)$,
If u ≤ v, then $C_u(u, v)$ = v - h(u)*H(v) + H(1)*h(u).
If u = v, then $C_u(u, v)$ = v - h(u)*H(v) + H(1)*h(u) and
$C_{uv}(u, v)$ = **c(u, v) = 1 – h(u)*h(v) + H(1)*h(u).**
If u ≠ v, then c(u, v) = 1 – h(u)*h(v).

S7-TC-17.
(a) Give the formula for the PP Max copula.
(b) Give the formula for Kendall's τ for the PP Max copula and solve for τ when the parameter a is equal to 0.5. (Venter, p. 81)

Solution S7-TC-17.
(a) The formula for the PP Max copula is
H(u) = (u – a) if u > a and H(u) = 0 otherwise.
(b) $\tau(a) = (1-a)^4$. For a = 0.5, $\tau(0.5) = 0.5^4$ = **0.0625**.

S7-TC-18.
(a) For a copula with random variables U and V, give the formula for L(z), the left-tail concentration function, where z is any point in the interval (0, 1). Define the formula in terms of probabilities.
(b) For a copula with random variables U and V, give the formula for R(z), the right-tail concentration function, where z is any point in the interval (0, 1). Define the formula in terms of probabilities.

(c) Now give the formula for L(z) in copula notation, with C(u, v) being the copula in question.
(d) Now give the formula for R(z) in copula notation, with C(u, v) being the copula in question.
(e) Now give the formula for L(z) in terms of a conditional probability.
(f) Now give the formula for R(z) in terms of a conditional probability.
(Venter, p. 83)

Solution S7-TC-18.
(a) $L(z) = Pr(U < z \text{ and } V < z)/z$
(b) $R(z) = Pr(U > z \text{ and } V > z)/(1-z)$
(c) $L(z) = C(z, z)/z$
(d) $R(z) = (1 - 2z + C(z, z))/(1-z)$
(e) $L(z) = Pr(U < z \mid V < z) = Pr(V < z \mid U < z)$.
(f) $R(z) = Pr(U > z \mid V > z) = Pr(V > z \mid U > z)$.

S7-TC-19. Given the product copula $\prod(u, v) = uv$, give **(a)** the left-tail concentration function L(z) for z = 0.05 and **(b)** the right-tail concentration function R(z) for z = 0.95.

Solution S7-TC-19.

(a) $L(z) = C(z, z)/z$. $L(0.05) = \prod(0.05, 0.05)/0.05 = 0.05^2/0.05 =$ **0.05.**

(b) $R(z) = (1 - 2z + C(z, z))/(1-z)$. $R(0.95) = (1 - 1.9 + 0.95^2)/0.05 =$ **0.05.**

Note the symmetry with regard to these concentration functions.

S7-TC-20. Given the maximum copula M(u, v) = min(u,v), give **(a)** the left-tail concentration function L(z) for z = 0.05 and **(b)** the right-tail concentration function R(z) for z = 0.95.

Solution S7-TC-20.
(a) $L(z) = C(z, z)/z$. $L(0.05) = M(0.05, 0.05)/0.05 = 0.05/0.05 =$ **1**.
(b) $R(z) = (1 - 2z + C(z, z))/(1-z)$. $R(0.95) = (1 - 1.9 + 0.95)/0.05 =$ **1.**
Again note the symmetry.

S7-TC-21. Given the minimum copula W(u, v) = max(0, u+v-1), give **(a)** the left-tail concentration function L(z) for z = 0.05 and **(b)** the right-tail concentration function R(z) for z = 0.95.

Solution S7-TC-21.
(a) $L(z) = C(z, z)/z$. $L(0.05) = W(0.05, 0.05)/0.05 = \max(0, 0.1-1)/0.05 =$ **0**.
(b) $R(z) = (1 - 2z + C(z, z))/(1-z)$. $R(0.95) = (1 - 1.9 + \max(0, 1.9 - 1))/0.05 =$
$(1 - 1.9 + 0.9)/0.05 = 0$.
Again note the symmetry.

S7-TC-22. How does an "LR" function combine left-tail and right-tail concentration functions L(z) and R(z)? (Venter, p. 84).

Solution S7-TC-22. An LR function is L(z) for z < ½ and R(z) for z > ½.

S7-TC-23. Prove that, for the tail concentration functions of any copula, L(0.5) = R(0.5).

Solution S7-TC-23. We know the formulas L(z) = C(z, z)/z and
R(z) = (1 – 2z + C(z, z))/(1-z).
L(0.5) = C(0.5, 0.5)/0.5
R(0.5) = (1 – 2*0.5 + C(0.5, 0.5))/(1 – 0.5) = (1 – 1 + C(0.5, 0.5))/0.5 = C(0.5, 0.5)/0.5 = L(0.5). **Q.E.D.**

This proof is important because it means that, no matter what the copula is, the LR function will not have any discontinuities at z = ½.

S7-TC-24. What feature of the right-tail concentration function can distinguish copulas from one another? (Venter, p. 84)

Solution S7-TC-24. The distinguishing feature can be **whether R(1) = 0 or R(1) > 0**. The greater the value of R(1), the heavier the tail.

S7-TC-25.
(a) Rank-order the following copulas by considering the value of the right-tail concentration function at 1 (i.e., R(1)).
Note: Some of these copulas will have R(1) = 0. These will all necessarily be in last place.
(i) Frank
(ii) Gumbel
(iii) Normal
(iv) HRT
(v) Clayton
(vi) PP Max
(vii) PP Power

(b) Which of the aforementioned copulas is the only one where L(1) > 0? (Venter, p. 84)

Solution S7-TC-25.

(a) The rank-order, in terms of descending values of R is as follows:
1. (vi) PP Max
2. (vii) PP Power
3. (iv) HRT
4. (ii) Gumbel
5. R(1) = 0 copulas: (i) Frank, (iii) Normal, (v) Clayton

(b) The **(v) Clayton** copula is the only one with L(1) > 0.

S7-TC-26. In terms of the parameter a (the same parameter as used for Kendall's τ), express the formulas for the right-tail concentration functions R(1) for each of the following copulas. Also, for each copula, find R(1) when a = 0.25. (Venter, p. 85)
(a) Gumbel copula
(b) HRT copula
(c) PP Power copula
(d) PP Max copula

Solution S7-TC-26.
(a) $\mathbf{R(1) = 2 - 2^{1/a}}$**, provided that R > 0**. For a = 0.25, the formula gives $2 - 2^4 < 0$, so **R(1) = 0**.
(b) $\mathbf{R(1) = 2^{-a}}$. For a = 0.25, the formula gives $2^{-0.25}$ = **R(1) = 0.8408964153**.
(c) R(1) = 1/(1+a). For a = 0.25, R(1) = 1/1.25 = **R(1) = 0.8.**
(d) R(1) = 1 –a. For a = 0.25, R(1) = 1 – 0.25 = **R(1) = 0.75.**

S7-TC-27.
(a) For an HRT copula, given that R(1) = 0.31, find Kendall's τ.
(b) For a PP Max copula, given that Kendall's τ is 0.55, find R(1).

Solution S7-TC-27.
(a) For an HRT copula, in terms of the parameter a, $R(1) = 2^{-a}$, and τ = 1/(2a + 1).
We are given $0.31 = 2^{-a} \rightarrow a = -\ln(0.31)/\ln(2) = a = 1.689659879$.
Thus, τ = 1/(2*1.689659879 + 1) = **τ = 0.2283459658**.
(b) For a PP Max copula, in terms of the parameter a, $\tau(a) = (1-a)^4$ and R(1) = 1 – a. Conveniently, $R(1) = [\tau(a)]^{1/4}$. In this case, $R(1) = 0.55^{1/4}$ = **R(1) = 0.86117353.**

S7-TC-28.
(a) Given the copula function C(u, v) and the copula density function c(u, v), provide the formula for J(z), also known as the cumulative tau.
(b) To what already known expression is J(1) equal? (Venter, p. 86)

Solution S7-TC-28.
(a) $\mathbf{J(z) = -1 + 4 * {}_0^z\int {}_0^z\int [C(u, v)*c(u,v)*dv*du]/[C(z, z)^2]}$
(b) J(1) = Kendall's τ.

S7-TC-29. Give the formula for the joint survival function S(x, y) in terms of cumulative distribution functions $F_X(x)$, $F_Y(y)$, and F(x, y). (Venter, p. 90)

Solution S7-TC-29. $\mathbf{S(x, y) = 1 - F_X(x) - F_Y(y) + F(x, y)}$.

S7-TC-30. Give the formula for the copula survival function $C_S(u, v)$. (Venter, p. 90)

Solution S7-TC-30. $\mathbf{C_S(u, v) = 1 - u - v + C(u, v)}$.

S7-TC-31. What is the mathematical relationship between S(x, y) and $C_S(u, v)$? (Venter, p. 90)

Solution S7-TC-31. $C_S(F_X(x), F_Y(y)) = S(x, y)$.

S7-TC-32.
(a) Give the formula for a *flipped copula* $C_F(u, v)$ in terms of C_S.
(b) Give the formula for a *flipped copula* $C_F(u, v)$ in terms of C.
(c) What are the mathematical relationships between $C_F(S_X(x), S_Y(y))$ and S(x, y) and C_S?
(Venter, p. 90)

Solution S7-TC-32.
(a) $C_F(u, v) = C_S(1-u, 1-v)$.
(b) $C_F(u, v) = u + v - 1 + C(1-u, 1-v)$.
(c) $C_F(S_X(x), S_Y(y)) = C_S(F_X(x), F_Y(y)) = S(x, y)$.

S7-TC-33.
(a) Give the formula for Clayton's copula.
(b) Give the formula for Kendall's τ for Clayton's copula and find τ for a = 0.6.
(c) What is the flipped copula of Clayton's copula?
(d) Starting from the formula for Clayton's copula, develop the formula for its flipped copula.
(d) What value is necessarily the same for any given copula and its flipped copula?
(Venter, p. 90)

Solution S7-TC-33.
(a) $C(u, v) = [u^{-1/a} + v^{-1/a} - 1]^{-a}$ for $a > 0$.

(b) $\tau = 1/(2a+1)$. For a = 0.6, τ = 1/2.2 = **τ = 0.4545454545**.

(c) The flipped copula of Clayton's copula is the **heavy-right-tail (HRT) copula**.

(d) $C(u, v) = [u^{-1/a} + v^{-1/a} - 1]^{-a}$ for $a > 0$.
$C_F(u, v) = u + v - 1 + C(1-u, 1-v)$.
$C_F(u, v) = u + v - 1 + [(1-u)^{-1/a} + (1-v)^{-1/a} - 1]^{-a}$ for $a > 0$.
The above is the precise formula for the HRT copula.

(e) Kendall's τ is necessarily the same for any given copula and its flipped copula

S7-TC-34. Starting at the formula for the Gumbel copula,
$C(u, v) = \exp(-[(-\ln(u))^a + (-\ln(v))^a]^{1/a})$, for $a \geq 1$, arrive at the formula for the flipped Gumbel copula.

Solution S7-TC-34.
$C(u, v) = \exp(-[(-\ln(u))^a + (-\ln(v))^a]^{1/a})$
$C_F(u, v) = u + v - 1 + C(1-u, 1-v)$
$C_F(u, v) = u + v - 1 + \exp(-[(-\ln(1-u))^a + (-\ln(1-v))^a]^{1/a})$, for $a \geq 1$.

S7-TC-35.
(a) In the empirical study by Frees and Valdez, which copula gave the best value for the Akaike Information Criterion (AIC)?
(b) According to Venter, to what is optimizing the AIC equivalent if a copula has one parameter?
(c) Which copula from estimates by Klugman and Parsa gave similarly good results? (Venter, p. 92)

Solution S7-TC-35.
(a) The **Gumbel** copula gave the best value for the Akaike Information Criterion (AIC).
(b) In a one-parameter copula, optimizing the AIC is equivalent to **finding the copula with the highest maximum likelihood.**
(c) Klugman and Parsa estimated the **Frank** copula and produced one with $\tau = 0.31$, the same value as the Gumbel copula from the study by Frees and Valdez.

S7-TC-36. Fill in the blanks (Venter, p. 92).
(a) A convenient way to compare heavy-tailed severity fits is to look at the ______ and the ______ of the tail.
(b) The [latter of these two qualities] can be quantified as the _______ ________ ________ that does not converge.

Solution S7-TC-36.

(a) A convenient way to compare heavy-tailed severity fits is to look at the **median** and the **heaviness** of the tail.

(b) The heaviness of the tail can be quantified as the **smallest positive moment** that does not converge.

S7-TC-37. Fill in the blank (Venter, p. 93): The HRT copula is similar to the _____ copula in the left tail and the _____ copula in the right tail.

Solution S7-TC-37. The HRT copula is similar to the **Frank** copula in the left tail and the **Gumbel** copula in the right tail.

S7-TC-38. Which distribution built from the HRT copula could be useful for analyzing excess-of-loss reinsurance where the data are scarce? (Venter, p. 93)

Solution S7-TC-38. The **joint Burr distribution**.

S7-TC-39. For simulated hurricane losses, briefly describe ways to empirically estimate the following values (Venter, p. 94):
(a) Kendall's tau
(b) Values of an empirical copula.
(c) Cumulative tau (J(z)) (Venter, p. 96)

Solution S7-TC-39.
(a) For Kendall's tau, compute the average value of sign[$(u_i-u_j)(v_i-v_j)$] over all pairs of observations (u_i, v_i) and (u_j, v_j) for $i < j$.
(b) To estimate values of an empirical copula where there are n points, for each given point, count the points that are less than the given point in both variables and then divide that count by (n-1). This provides the estimate of the copula value at the given point.
(c) For each value z, after estimating the copula values for individual points where u < z and v < z, find the average of the estimates (call it A), multiply by 4, divide by C(z, z), and then subtract one, and find the estimate of J(z): $J(z) \approx 4A/C(z,z) - 1$.

S7-TC-40. For data with symmetrical tails, **(a)** which copulas among the ones discussed by Venter would fit the data best? **(b)** What problem still exists with those copulas? (Venter, p. 95)

Solution S7-TC-40.
(a) The **normal** and **Frank** copulas fit symmetrical tails best.
(b) The normal and Frank copulas have the problem of being too light in the extreme tails.

S7-TC-41.
(a) Give the formula for $M(z) = E(V \mid U < z)$, the conditional expected value of V given that U = z.
(b) What is M(1) for every copula? (Venter, p. 105)

Solution S7-TC-41.
(a) $\mathbf{M(z) = {}_0^z\!\int {}_0^1\!\int (1/z)*v*c(u,v)*dv*du}$.
(b) For every copula, $M(1) = E(V \mid U < 1) = E(V) =$ **M(1) = 1/2**. This is because V is uniformly distributed on (0, 1) and so has an unconditional expectation of 1/2.

S7-TC-42.
(a) Give the formula for Joe's copula.
(b) Give the formulas for (i) Kendall's τ, (ii) R(1), and (iii) L(0) for Joe's copula.
(c) Joe's copula is a generalized case of two specific copulas that were discussed earlier. What are these two copulas, and how do they arise from Joe's copula? (Venter, p. 112)

Solution S7-TC-42.
(a) $\mathbf{C(u, v) = (1 + [(u^{-a} - 1)^c + (v^{-a} - 1)^c]^{1/c})^{-1/a}}$**, for a > 0 and c ≥ 1.**
(b) (i) $\boldsymbol{\tau = 1 - 2/(ca + 2c)}$; (ii) $\mathbf{R(1) = 2 - 2^{1/c}}$; (iii) $\mathbf{L(0) = 2^{-1/(ac)}}$.
(c) Joe's copula is a generalized case of the **Gumbel** and **Clayton** copulas. The Gumbel copula is the limit of Joe's copula as a approaches 0. The Clayton copula is the case of Joe's copula where c = 1. In that case, a in Joe's copula is (1/a) in the standard formula for Clayton's copula.