

# Study Guide on Property Catastrophe Risk Loads for the Society of Actuaries (SOA) Exam GIADV: Advanced Topics in General Insurance

(Based on Donald F. Mango's Paper, "[An Application of Game Theory: Property Catastrophe Risk Load](#)")

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G. Stolyarov II, ASA, ACAS, MAAA, CPCU, ARe, ARC, API, AIS, AIE, AIAF  
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**Source:** Mango, Donald F., "[An Application of Game Theory: Property Catastrophe Risk Load](#)", PCAS LXXXV, 1998.

**Problem PCRL-1.** What three elements might an output file for a catastrophe model contain, as mentioned by Mango (p. 2)?

**Solution PCRL-1.** The following elements might be contained in an output file for a catastrophe model:

1. Event identifier
2. Event probability
3. Modeled loss amount for the event for the selected portfolio of exposures.

**Problem PCRL-2.** Define the *marginal impact* of a new account on the portfolio variance or standard deviation. (Mango, p. 2)

**Solution PCRL-2.** The marginal impact is the difference between the portfolio variance or standard deviation after the addition of the new account and the portfolio variance or standard deviation before the addition of the new account.

**Problem PCRL-3.** Mango, on p. 4, posits a model where an event is denoted by identifier  $i$  and is considered an independent Poisson process with occurrence rate  $\lambda_i$ .

- (a) What is the formula for the binomial approximation of  $\lambda_i$ , where  $p_i$  is the probability of occurrence?
- (b) Is the approximation in part (a) superior for large values of  $\lambda_i$  or for small values of  $\lambda_i$ ?

**Solution PCRL-3.**

- (a)  $\lambda_i = -\ln(1-p_i)$ .
- (b) This approximation is superior for **small** values of  $\lambda_i$ . (Mango, p. 4)

**Problem PCRL-4.** The model presented by Mango (p. 4) produces a modeled loss  $L_i$  for each event  $i$ .

- (a) What is the name used for the table containing the event identifiers  $i$ , the event probabilities  $p_i$ , and the modeled losses  $L_i$ ?
- (b) What is the formula for  $E(L)$ , the expected loss?
- (c) What is the formula for  $Var(L)$ , the variance of losses?

**Solution PCRL-4.**

- (a) This table is called an **occurrence size-of-loss distribution**.
- (b)  $E(L) = \sum_i [L_i * p_i]$
- (c)  $Var(L) = \sum_i [L_i^2 * p_i * (1 - p_i)]$

**Problem PCRL-5.**

- (a) For the model presented by Mango (p. 4), what is the formula for  $Cov(L, n)$ , the covariance of an existing portfolio  $L$  (with losses  $L_i$ ) and a new account  $n$  (with losses  $n_i$ )? Use  $p_i$  to denote the probability of loss event  $i$ .
- (b) What is the formula for  $Var(L + n)$ , the variance of the combined portfolio  $(L+n)$ ?

**Solution PCRL-5.**

- (a)  $Cov(L, n) = \sum_i [L_i * n_i * p_i * (1 - p_i)]$
- (b)  $Var(L + n) = Var(L) + Var(n) + 2Cov(L, n)$

**Problem PCRL-6.** You are given the following occurrence size-of-loss distribution for Portfolio H:

Event	Loss Amount for H	Event Probability
I	100,000	0.06
II	300,000	0.03
III	500,000	0.09
IV	600,000	0.02
V	999,930	0.004

You are considering adding a new account  $m$  to this portfolio. The following is known regarding the loss amounts for each of the events previously described.

Event	Loss Amount for m
I	10,000
II	20,000
III	110,000
IV	200,000
V	1,200,000

You are using the model given by Mango (p. 4) to evaluate H and m with regard to the given loss events. This model uses a binomial approximation for each  $\lambda_i$ , which the occurrence rate of an independent Poisson process with respect to any event  $i$ .

- (a) What is  $E(H)$ , the expected loss for Portfolio H?
- (b) What is  $\text{Var}(H)$ , the variance of losses for Portfolio H?
- (c) What is  $\text{Cov}(H, m)$ , the covariance between Portfolio H and Account m?
- (d) What is  $\text{Var}(H + m)$ , the variance of the combined portfolio if Account m is added to Portfolio H?

**Solution PCRL-6.**

(a) We use the formula  $E(L) = \sum_i [L_i * p_i]$ . Here,  $E(H) = 100000 * 0.06 + 300000 * 0.03 + 500000 * 0.09 + 600000 * 0.02 + 999930 * 0.004 = \mathbf{E(H) = 75,999.72}$ .

(b) We use the formula  $\text{Var}(L) = \sum_i [L_i^2 * p_i * (1 - p_i)]$ . Here,  $\text{Var}(H) = 100000^2 * 0.06 * (1 - 0.06) + 300000^2 * 0.03 * (1 - 0.03) + 500000^2 * 0.09 * (1 - 0.09) + 600000^2 * 0.02 * (1 - 0.02) + 999930^2 * 0.004 * (1 - 0.004) = \mathbf{Var(H) = 34,697,442,260}$ .

(c) We use the formula  $\text{Cov}(L, n) = \sum_i [L_i * n_i * p_i * (1 - p_i)]$ . Here,  $\text{Cov}(H, m) = 100000 * 10000 * 0.06 * (1 - 0.06) + 300000 * 20000 * 0.03 * (1 - 0.03) + 500000 * 110000 * 0.09 * (1 - 0.09) + 600000 * 200000 * 0.02 * (1 - 0.02) + 999930 * 1200000 * 0.004 * (1 - 0.004) = \mathbf{Cov(H, m) = 11,867,965,344}$ .

(d) We use the formula  $\text{Var}(L + n) = \text{Var}(L) + \text{Var}(n) + 2\text{Cov}(L, n)$ .

We seek to find  $\text{Var}(H + m) = \text{Var}(H) + \text{Var}(m) + 2\text{Cov}(H, m)$ .

We have already found that  $\text{Var}(H) = 34,697,442,260$  and  $\text{Cov}(H, m) = 11,867,965,344$ .

We find  $\text{Var}(m) = \sum_i [m_i^2 * p_i * (1 - p_i)] = 10000^2 * 0.06 * (1 - 0.06) + 20000^2 * 0.03 * (1 - 0.03) + 110000^2 * 0.09 * (1 - 0.09) + 200000^2 * 0.02 * (1 - 0.02) + 1200000^2 * 0.004 * (1 - 0.004) = 7,529,230,000$

Thus,  $\text{Var}(H + m) = 34,697,442,260 + 7,529,230,000 + 2 * 11,867,965,344 =$

$\mathbf{Var(H + m) = 65,962,602,948}$ .

**Problem PCRL-7.** You are given the following definitions (Mango, p. 5):

$L_0$  = losses from a portfolio before a new account is added,

$L_1$  = losses from a portfolio after a new account is added,

$S_0$  = Standard deviation of  $L_0$ ,

$S_1$  = Standard deviation of  $L_1$ ,

$R_0$  = Return on the portfolio before new account is added,

$R_1$  = Return on the portfolio after new account is added,

$V$  = Needed surplus, calculated as  $z * (\text{standard deviation of loss}) - \text{expected return}$ ,

$z$  = “A distribution percentage point corresponding to the acceptable probability that the actual result will require even more surplus than allocated” (Kreps, “Reinsurer Risk Loads from Marginal Surplus Requirements”, p. 197), and

$y$  = required rate of return on marginal surplus, based on management goals, market forces, and risk appetite.

- (a) What is the formula for  $V_0$ , the needed surplus *before* the new account is added?
- (b) What is the formula for  $V_1$ , the needed surplus *after* the new account is added?
- (c) What is the formula for  $r$ , the risk load charged to the new account?

- (d) What is the formula for the marginal surplus requirement,  $V_1 - V_0$ , which includes  $r$  as a term?  
 (e) What is the formula for  $r$  using the Marginal Surplus (MS) method?

**Solution PCRL-7.**

- (a)  $V_0 = z \cdot S_0 - R_0$   
 (b)  $V_1 = z \cdot S_1 - R_1$   
 (c)  $r = R_1 - R_0$   
 (d)  $V_1 - V_0 = z \cdot (S_1 - S_0) - r$   
 (e)  $r = (yz) \cdot (S_1 - S_0) / (1+y)$

**Problem PCRL-8.** You are given the following information:

- Before new Account A is added to Portfolio Q, losses from Portfolio Q are expected to be 350,000.
- After Account A is added, losses from the resulting Portfolio (Q+A) are expected to be 550,000.
- The standard deviation of Portfolio Q is 220,000.
- The standard deviation of Portfolio (Q+A) is 280,000.
- The return on Portfolio Q before Account A is added is expected to be 90,000.
- The return on Portfolio (Q+A) is expected to be 110,000.

Use the Marginal Surplus (MS) method to calculate the following:

- (a) Needed surplus for Portfolio Q before Account A is added  
 (b) Needed surplus for Portfolio (Q+A) after Account A is added  
 (c) The risk load charged to Account A  
 (d)  $z$ , the distribution percentage point corresponding to the acceptable probability that the actual result will require even more surplus than allocated  
 (e) The required rate of return on marginal surplus, which is implied in the result for the risk load.

**Solution PCRL-8.**

- (a) We use the formula  $V_0 = z \cdot S_0 - R_0 = 0.5 \cdot 220000 - 90000 = V_0 = \mathbf{20,000}$ .  
 (b) We use the formula  $V_1 = z \cdot S_1 - R_1 = 0.5 \cdot 280000 - 110000 = V_1 = \mathbf{30,000}$ .  
 (c) We use the formula  $r = R_1 - R_0 = 110000 - 90000 = r = \mathbf{20,000}$ .  
 (d) We use the formula  $V_1 - V_0 = z \cdot (S_1 - S_0) - r$ . Here,  $V_1 - V_0 = 30000 - 20000 = V_1 - V_0 = 10,000$ . Thus,  $10000 = z \cdot (S_1 - S_0) - r = z \cdot (280000 - 220000) - 20000 \rightarrow 30000 = z \cdot 60000 \rightarrow z = \mathbf{0.5}$ .  
 (e) We use the formula  $r = (yz) \cdot (S_1 - S_0) / (1+y)$ . We seek to find  $y$ .  
 Thus,  $r \cdot (1+y) = (yz) \cdot (S_1 - S_0) \rightarrow r + ry = z(S_1 - S_0)y \rightarrow r = [z(S_1 - S_0) - r]y \rightarrow y = r / [z(S_1 - S_0) - r]$ .  
 We know that  $r = 20000$ ,  $z = 0.5$ , and  $S_1 - S_0 = 60000$ .  
 Thus,  $y = 20000 / (0.5 \cdot 60000 - 20000) \rightarrow y = 20000 / (30000 - 20000) = 20000 / 10000 = y = \mathbf{2 = a 200\% rate of return on marginal surplus}$ .

**Problem PCRL-9.** You are given the following occurrence size-of-loss distribution for Portfolio H:

Event	Loss Amount for H	Event Probability
I	100,000	0.06
II	300,000	0.03
III	500,000	0.09
IV	600,000	0.02
V	999,930	0.004

You are considering adding a new account m to this portfolio. The following is known regarding the loss amounts for each of the events previously described.

Event	Loss Amount for m
I	10,000
II	20,000
III	110,000
IV	200,000
V	1,200,000

Based on Problem PCRL-6, the following information is known:

- $E(H)$ , the amount of expected losses in Portfolio H, is 75,999.72.
- $\text{Var}(H)$ , the variance of losses in Portfolio H before account m is added, is 34,697,442,260.
- $\text{Var}(H + m)$ , the variance of losses in the combined Portfolio (H+m), is 65,962,602,948.

You are also given the following:

- The needed surplus for Portfolio H, before account m is added, is 120,000.
- The distribution percentage point corresponding to the acceptable probability that the actual result will require even more surplus than allocated is 0.92.
- The return required by investors in this portfolio is 15%.

Use the Marginal Surplus (MS) method to calculate the following:

- The risk-load multiplier, which is the value by which the change in the standard deviation would be multiplied to obtain the risk load.
- The risk load charged to account m;
- The needed surplus for the combined Portfolio (H+m).

**Solution PCRL-9.** (a) We use the formula  $r = (yz) \cdot (S_1 - S_0) / (1+y)$ . Since the change in standard deviation is denoted by  $(S_1 - S_0)$ , the risk-load multiplier itself is equal to  $yz / (1+y) = 0.15 \cdot 0.92 / 1.15 = \mathbf{0.12}$ .

(b) We use the formula  $r = (yz) \cdot (S_1 - S_0) / (1+y)$ .

Here, we are given that  $y = 0.15$  and  $z = 0.92$ .

We calculate  $S_0 = \text{SD}(H) = \sqrt{[\text{Var}(H)]} = \sqrt{(34,697,442,260)} = 186272.4946$ .

We calculate  $S_1 = SD(H+m) = \sqrt{[\text{Var}(H+m)]} = \sqrt{(65,962,602,948)} = 256831.8573$ .

Thus,  $r = (0.15*0.92)*(256831.8573-186272.4946)/1.15 = \mathbf{r = 8467.123523}$ .

(c) We first consider the formula  $V_0 = z*S_0 - R_0$ . We are given  $V_0$ , the needed surplus for Portfolio H, to be 120,000. We can calculate  $R_0 = z*S_0 - V_0 = 0.92*186272.4946 - 120000 = R_0 = 51370.69503$ .

Next we use the formula  $r = R_1 - R_0 \rightarrow R_1 = r + R_0 = 8467.123523 + 51370.69503 = R_1 = 59837.81855$ .

Now we can calculate  $V_1 = z*S_1 - R_1 = 0.92*256831.8573 - 59837.81855 = 176,447.4902$ .

Thus, the needed surplus for combined Portfolio (H+m) is **176,447.4902**.

**Problem PCRL-10.** Let L be an existing portfolio and n be an account considered for addition to L. Let  $\text{Var}(L)$  and  $\text{Var}(n)$  be the variances of L and n, respectively, and let  $\text{Cov}(L, n)$  be the covariance of L and n. Let  $\lambda$  be a multiplier that is dimensioned to apply to the variance. What is the formula for the risk load r, using the Marginal Variance (MV) method? (Mango, p. 6)

**Solution PCRL-10.**

$$\mathbf{r = \lambda * [\text{Var}(n) + 2 * \text{Cov}(L, n)]}$$

**Problem PCRL-11.** For a particular Portfolio  $\Delta$ , you are considering the addition of an account  $\phi$ . You know that  $\text{Var}(\Delta) = 33,000,000$ , and  $\text{Var}(\phi) = 8,000,000$ . You also know that  $\text{Cov}(\Delta, \phi) = 12,000,000$ . Using the Marginal Variance (MV) method and a multiplier  $\lambda = 0.0003$ , what is the risk load for the addition of  $\phi$ ?

**Solution PCRL-11.** We use the formula  $r = \lambda * [\text{Var}(n) + 2 * \text{Cov}(L, n)]$ . Here, we calculate  $\lambda * [\text{Var}(\phi) + 2 * \text{Cov}(\Delta, \phi)] = 0.0003 * (8,000,000 + 2 * 12,000,000) = \mathbf{r = 9600}$ .

**Problem PCRL-12.** When the Marginal Variance (MV) method is used to calculate the risk load for the addition of account n to portfolio L, and the risk-load multiplier for the Marginal Surplus (MS) method is known, what is the formula for the MV risk-load multiplier in terms of the MS risk-load multiplier? (Mango, p. 8)

**Solution PCRL-12.**

**(MV risk-load multiplier) = (MS risk-load multiplier)/SD(L + n)**, where  $SD(L + n)$  is the standard deviation of the combined portfolio (L + n).

**Problem PCRL-13.** You are given the following occurrence size-of-loss distribution for Portfolio H:

Event	Loss Amount for H	Event Probability
I	100,000	0.06
II	300,000	0.03
III	500,000	0.09
IV	600,000	0.02
V	999,930	0.004

You are considering adding a new account m to this portfolio. The following is known regarding the loss amounts for each of the events previously described.

Event	Loss Amount for m
I	10,000
II	20,000
III	110,000
IV	200,000
V	1,200,000

Based on Problems PCRL-6 and PCRL-9, the following information is known:

- $E(H)$ , the amount of expected losses in Portfolio H, is 75,999.72.
- $\text{Var}(H)$ , the variance of losses in Portfolio H before account m is added, is 34,697,442,260.
- $\text{Var}(m)$ , the variance of losses in account m, is 7,529,230,000.
- $\text{Var}(H + m)$ , the variance of losses in the combined Portfolio (H+m), is 65,962,602,948.
- $\text{Cov}(H, m)$ , the covariance of losses between Portfolio H and account m, is 11,867,965,344.
- The risk=load multiplier for  $yz/(1+y)$  for the Marginal Surplus (MS) method is 0.12.

You are using the Marginal Variance (MV) method to calculate the risk load for the addition of account m.

- Find the risk-load multiplier  $\lambda$  for the MV method.
- Calculate the risk load for the addition of account m.

**Solution PCRL-13.**

(a) We use the formula (MV risk-load multiplier) = (MS risk-load multiplier)/SD(H + m). We are given that the MS risk-load multiplier is 0.12. We calculate  $\text{SD}(H + m) = \sqrt{\text{Var}(H + m)} = \sqrt{65,962,602,948} = 256831.8573$ . Thus, MV risk-load multiplier =  $0.12/256831.8573 = \lambda = \mathbf{0.000000467232}$ .

(b) We use the formula  $r = \lambda * [\text{Var}(n) + 2 * \text{Cov}(L, n)]$ . Here,  $r = \lambda * [\text{Var}(m) + 2 * \text{Cov}(H, m)] = 0.000000467232 * (7,529,230,000 + 2 * 11,867,965,344) = \mathbf{r = 14608.08356}$ .

**Problem PCRL-14.** You are given the following incomplete table for Accounts X and Y, which are being combined into a portfolio (X+Y). Account X is being added first, followed by Account Y. This table is intended for use with the Marginal Surplus (MS) method and Marginal Variance (MV) methods.

**Marginal Surplus (MS) Method**

	Account X	Account Y	Account X + Account Y	Portfolio (X+Y)
(1) Change in Standard Deviation	11000		25000	25000
(2) Risk-Load Multiplier	0.54	0.54	N/A	0.54
(3) Risk Load				

**Marginal Variance (MV) Method**

	Account X	Account Y	Account X + Account Y	Portfolio (X+Y)
(1) Change in Variance	121,000,000		625,000,000	625,000,000
(2) Risk-Load Multiplier			N/A	
(3) Risk Load				

Complete the rest of the fields in each table.

**Solution PCRL-14.** First, we address the table for the MS method. The Change in Standard Deviation from Account Y is  $SD(X + Y) - SD(X) = 25000 - 11000 = 14000$ .

The risk loads for the MS method are the products of each Change in Standard Deviation and the Risk-Load Multiplier of 0.54. The results are 5940 for Account X, 7560 for Account Y, and 13500 for Portfolio (X+Y). Also, for Account X + Account Y, the risk load will be  $5940 + 7560 = 13500$ .

For the MV method, the Change in Variance from Account Y is  $Var(X+Y) - Var(X) = 625,000,000 - 121,000,000 = 504,000,000$ .

The Risk-Load Multiplier for the MV method is equal to the Risk-Load Multiplier for the MS method, divided by  $SD(X + Y) = 0.54/25000 = 0.0000216$ .

The risk loads for the MV method are the products of each Change in Variance and the Risk-Load Multiplier of 0.0000216. The results are 2613.6 for Account X, 10886.4 for Account Y, and 13500 for Portfolio (X+Y). Also, for Account X + Account Y, the risk load will be  $2613.6 + 10886.4 = 13500$ .



The completed tables therefore look as follows:

**Marginal Surplus (MS) Method**

	Account X	Account Y	Account X + Account Y	Portfolio (X+Y)
<b>(1) Change in Standard Deviation</b>	11000	14000	25000	25000
<b>(2) Risk-Load Multiplier</b>	0.54	0.54	N/A	0.54
<b>(3) Risk Load</b>	5940	7560	13500	13500

**Marginal Variance (MV) Method**

	Account X	Account Y	Account X + Account Y	Portfolio (X+Y)
<b>(1) Change in Variance</b>	121,000,000	504,000,000	625,000,000	625,000,000
<b>(2) Risk-Load Multiplier</b>	0.0000216	0.0000216	N/A	0.0000216
<b>(3) Risk Load</b>	2613.6	10886.4	13500	13500

**Problem PCRL-15.** Using the Marginal Surplus (MS) method as applied to a portfolio with two accounts, X and Y, possible loss events  $i$  with occurrence probabilities  $p_i$  and loss amounts  $X_i$  and  $Y_i$  for each  $i$ , give formulas for the following (Mango, p. 11):

- (a) The marginal standard deviation  $\Delta SD(X)$  due to the addition of X in the build-up scenario, where X is added first.
- (b) The marginal standard deviation  $\Delta SD(X)$  due to the addition of X in the renewal scenario, where X is added after Y.

**Solution PCRL-15.**

(a)  $\Delta SD(X) = SD(X) = \sqrt{(\sum [X_i^2 * p_i * (1-p_i)])}$

(b)  $\Delta SD(X) = SD(X + Y) - SD(Y) = \sqrt{(\sum [(X_i + Y_i)^2 * p_i * (1-p_i)])} - \sqrt{(\sum [Y_i^2 * p_i * (1-p_i)])}$

**Problem PCRL-16.**

- (a) Using the Marginal Surplus (MS) method as applied to a portfolio with two accounts, X and Y, where X is added first and all loss amounts are positive, would the risk load due to X be greater during the build-up scenario or during the renewal scenario?
- (b) Using the Marginal Variance (MV) method as applied to a portfolio with two accounts, X and Y, where X is added first and all loss amounts are positive, would the risk load due to X be greater during the build-up scenario or during the renewal scenario?

**Solution PCRL-16.**

- (a) For the MS method, the risk load due to X is greater during the **build-up scenario** and less during the renewal scenario.
- (b) For the MV method, the risk load due to X is greater during the **renewal scenario** and less during the build-up scenario.

**Problem PCRL-17.** You are given the following information about two accounts, X and Q, which are being combined into a portfolio. Let  $SD(X)$  and  $SD(Q)$  denote the standard deviations of X and Q, respectively. In the build-up scenario, X is added to the portfolio first, followed by Q.

- $SD(X) = 35000$
- $SD(Q) = 26000$
- $SD(X + Q) = 50000$
- The distribution percentage point corresponding to the acceptable probability that the actual result will require even more surplus than allocated is  $z = 0.98$ .
- The required rate of return on marginal surplus is 16%.

- (a) Using the Marginal Surplus (MS) method, calculate the risk-load multiplier due to the addition of Q to the portfolio.
- (b) Using the Marginal Variance (MV) method, calculate the risk-load multiplier  $\lambda$  due to the addition of Q to the portfolio.
- (c) Complete the following table for the Marginal Surplus (MS) method:

<b>Build-Up Scenario: Q added to X: Marginal Surplus (MS) Method</b>				
	<b>Account X</b>	<b>Account Q</b>	<b>Account X + Account Q</b>	<b>Portfolio (X + Q)</b>
<b>(1) Change in SD</b>	35000			50000
<b>(2) Risk-Load Multiplier</b>			N/A	
<b>(3) Risk Load</b>				
<b>Renewal Scenario: Each Account Renewed: Marginal Surplus (MS) Method</b>				
	<b>Account X</b>	<b>Account Q</b>	<b>Account X + Account Q</b>	<b>Portfolio (X + Q)</b>
<b>(4) Change in SD</b>				50000
<b>(5) Risk-Load Multiplier</b>			N/A	
<b>(6) Risk Load</b>				
<b>(7) Build-Up Risk Load</b>				
<b>(8) Difference</b>				

(d) Complete the following table for the Marginal Variance (MV) method:

<b>Build-Up Scenario: Q added to X: Marginal Variance (MV) Method</b>				
	<b>Account X</b>	<b>Account Q</b>	<b>Account X + Account Q</b>	<b>Portfolio (X + Q)</b>
<b>(1)</b> Change in Variance				
<b>(2)</b> Risk-Load Multiplier			N/A	
<b>(3)</b> Risk Load				
<b>Renewal Scenario: Each Account Renewed: Marginal Variance (MV) Method</b>				
	<b>Account X</b>	<b>Account Q</b>	<b>Account X + Account Q</b>	<b>Portfolio (X + Q)</b>
<b>(4)</b> Change in Variance				
<b>(5)</b> Risk-Load Multiplier			N/A	
<b>(6)</b> Risk Load				
<b>(7)</b> Build-Up Risk Load				
<b>(8)</b> Difference				

**Solution PCRL-17.**

(a) We use the formula Risk-Load Multiplier =  $yz/(1+y)$ . In this case, y is given as 0.16, and z is given as 0.98. Thus, the Risk-Load Multiplier is  $0.16*0.98/1.16 = \mathbf{0.1351724138}$ .

(b) We use the formula,  $\lambda = (\text{MS Risk-Load Multiplier})/[\text{SD}(X + Q)] = 0.1351724138/50000 = \lambda = \mathbf{0.000002703448276}$ .

(c) We complete the table one row at a time.

**Row (1):** Because Q is being added to X, the change in SD due to the addition of Q is  $\text{SD}(X + Q) - \text{SD}(X) = 50000 - 35000 = 15000$ . Thus, for the sum of the marginal impacts for Account X + Account Q, we enter  $35000 + 15000 = 50000$ .

**Row (2):** We use the risk-load multiplier calculated in part (a): 0.1351724138.

**Row (3):** The Risk Load is equal to (Change in SD)\*(Risk-Load Multiplier): This product is 4731.034483 for Account X, 2027.586207 for Account Q, and 6758.62069 for Portfolio (X+Q). The value for (Account X + Account Q) is also  $4731.034483 + 2027.586207 = 6758.62069$ .

**Row (4):** For the renewal change, the marginal SD of Account Q remains 15000, but the marginal SD of Account X is calculated as  $\text{SD}(X + Q) - \text{SD}(Q) = 50000 - 26000 = 24000$ . The SD of Portfolio (X+Q) remains the same at 50000, but the sum Account X + Account Q becomes  $24000 + 15000 = 39000$ .

**Row (5):** We use the risk-load multiplier calculated in part (a): 0.1351724138.

**Row (6):** The Risk Load is equal to (Change in SD)\*(Risk-Load Multiplier): This product is

3244.137931 for Account X, 2027.586207 for Account Q, and 6758.62069 for Portfolio (X+Q). For Account X + Account Y, the sum of the renewal risk loads is  $3244.137931 + 2027.586207 = 5271.724138$ .

**Row (7):** We restate the build-up risk loads from Row (3).

**Row (8):** The difference is the value in Row (6) minus the value in Row(7) for the column in question. This difference is -1486.896552 for Account X, 0 for Account Q, -1486.896552 for (Account X + Account Q), and 0 for Portfolio (X + Q).

<b>Build-Up Scenario: Q added to X: Marginal Surplus (MS) Method</b>				
	<b>Account X</b>	<b>Account Q</b>	<b>Account X + Account Q</b>	<b>Portfolio (X + Q)</b>
<b>(1) Change in SD</b>	35000	15000	50000	50000
<b>(2) Risk-Load Multiplier</b>	0.1351724138	0.1351724138	N/A	0.1351724138
<b>(3) Risk Load</b>	4731.034483	2027.586207	6758.62069	6758.62069
<b>Renewal Scenario: Each Account Renewed: Marginal Surplus (MS) Method</b>				
	<b>Account X</b>	<b>Account Q</b>	<b>Account X + Account Q</b>	<b>Portfolio (X + Q)</b>
<b>(4) Change in SD</b>	24000	15000	39000	50000
<b>(5) Risk-Load Multiplier</b>	0.1351724138	0.1351724138	N/A	0.1351724138
<b>(6) Risk Load</b>	3244.137931	2027.586207	5271.724138	6758.62069
<b>(7) Build-Up Risk Load</b>	4731.034483	2027.586207	6758.62069	6758.62069
<b>(8) Difference</b>	-1486.896552	0	-1486.896552	0

**(d)** Complete the following table for the Marginal Variance (MV) method:

We complete the table one row at a time.

**Row (1):** Because Q is being added to X, the change in variance due to the addition of X is  $[SD(X)]^2 = 35000^2 = 1,225,000,000$ .

For Portfolio(X+Q), the variance is  $[SD(X+Q)]^2 = 50000^2 = 2,500,000,000$ .

The change in variance due to the addition of Q will be  $Var(X+Q)-Var(X) = 2,500,000,000 - 1,225,000,000 = 1,275,000,000$ . The change for (Account X + Account Q) will be  $1,225,000,000 + 1,275,000,000 = 2,500,000,000$ .

**Row (2):** We use the risk-load multiplier calculated in part (b): 0.000002703448276.

**Row (3):** The Risk Load is equal to (Change in Variance)\*(Risk-Load Multiplier): This product is 3311.724138 for Account X, 3446.896552 for Account Q, and 6758.62069 for Portfolio (X+Q). The change for (Account X + Account Q) will be  $3311.724138 + 3446.896552 = 6758.62069$ .

**Row (4):** For the renewal change, the marginal variance of Account Q remains 1,275,000,000, but the marginal variance of Account X is calculated as  $\text{Var}(X + Q) - \text{Var}(Q) = 2,500,000,000 - 26000^2 = 1,824,000,000$ . The variance of Portfolio (X+Q) remains the same at 2,500,000,000, but the sum Account X + Account Q becomes  $1,824,000,000 + 1,275,000,000 = 3,099,000,000$ .

**Row (5):** We use the risk-load multiplier calculated in part (b): 0.000002703448276.

**Row (6):** The Risk Load is equal to (Change in Variance)\*(Risk-Load Multiplier): This product is 4979.751724 for Account X, 3446.896552 for Account Q, and 6758.62069 for Portfolio (X+Q). The change for (Account X + Account Q) will be  $4979.751724 + 3446.896552 = 8426.648276$ .

**Row (7):** We restate the build-up risk loads from Row (3).

**Row (8):** The difference is the value in Row (6) minus the value in Row(7) for the column in question. This difference is 1668.027586 for Account X, 0 for Account Q, 1668.027586 for (Account X + Account Q), and 0 for Portfolio (X + Q).

<b>Build-Up Scenario: Q added to X: Marginal Variance (MV) Method</b>				
	<b>Account X</b>	<b>Account Q</b>	<b>Account X + Account Q</b>	<b>Portfolio (X + Q)</b>
<b>(1) Change in Variance</b>	1,225,000,000	1,275,000,000	2,500,000,000	2,500,000,000
<b>(2) Risk-Load Multiplier</b>	0.000002703448276	0.000002703448276	N/A	0.000002703448276
<b>(3) Risk Load</b>	3311.724138	3446.896552	6758.62069	6758.62069
<b>Renewal Scenario: Each Account Renewed: Marginal Variance (MV) Method</b>				
	<b>Account X</b>	<b>Account Q</b>	<b>Account X + Account Q</b>	<b>Portfolio (X + Q)</b>
<b>(4) Change in Variance</b>	1,824,000,000	1,275,000,000	3,099,000,000	2,500,000,000
<b>(5) Risk-Load Multiplier</b>	0.000002703448276	0.000002703448276	N/A	0.000002703448276
<b>(6) Risk Load</b>	4979.751724	3446.896552	8426.648276	6758.62069
<b>(7) Build-Up Risk Load</b>	3311.724138	3446.896552	6758.62069	6758.62069
<b>(8) Difference</b>	1668.027586	0	1668.027586	0

**Problem PCRL-18.**

- (a) Explain why, in conceptual terms, the Marginal Surplus (MS) method is not renewal-additive.
- (b) Explain why, in conceptual terms, the Marginal Variance (MV) method is not renewal-additive. (Mango, p. 11)

**Solution PCRL-18.**

(a) The MS method is not renewal-additive because the square-root operator is *sub-additive*, i.e.,  $\sqrt{X + Y} < \sqrt{X} + \sqrt{Y}$  for all X and Y. This means that the sum of the marginal standard deviations is always less than the total portfolio standard deviation, meaning that the MS method would produce a smaller renewal risk load than the build-up risk load.

(b) The MV method is not renewal-additive, because if the covariance between X and Y is greater than 0 (as it is under Mango's assumptions if all losses are positive and the probability of each event is non-negative), then  $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y) > \text{Var}(X) + \text{Var}(Y)$ . Any marginal variance for a variable would therefore be the sum of that variable's stand-alone variance and 2 times the covariance. (For instance, the marginal variance of X is  $\text{Var}(X + Y) - \text{Var}(Y) = \text{Var}(X) + 2\text{Cov}(X, Y) > \text{Var}(X)$ .) Thus, the MV method is *super-additive*, and the renewal risk load is greater than the build-up risk load.

**Problem PCRL-19.** According to Mango (p. 12), for a given portfolio of accounts, a risk-load method is *renewal-additive* if what condition holds?

**Solution PCRL-19.** A risk-load method is *renewal-additive* if the sum of the renewal risk loads calculated for each account equals the risk load calculated when the entire portfolio is treated as a single account. (Mango, p. 12)

**Problem PCRL-20.** Mango (p. 13) explains that renewal-additivity matters in a real-world context. Explain what problem would happen if (a) the Marginal Surplus (MS) method and (b) the Marginal Variance (MV) method were applied to calculating premiums for a real-world portfolio.

**Solution PCRL-20.**

(a) The MS method would result in premium quotes whose sum of risk loads would be smaller than the total required risk load, and every account would be undercharged.

(b) The MV method would result in premium quotes whose sum of risk loads would be greater than the total required risk load, and every account would be overcharged.

**Problem PCRL-21. Fill in the blanks** (Mango, p. 13): In order for the Marginal Surplus (MS) or Marginal Variance (MV) methods to be renewal-additive, one must assume a(n) \_\_\_\_\_. Since the marginal impacts depend on the \_\_\_\_\_, the \_\_\_\_\_ selected for an account could determine whether it is written or declined.

**Solution PCRL-21.** In order for the Marginal Surplus (MS) or Marginal Variance (MV) methods to be renewal-additive, one must assume an **entry order for the accounts**. Since the marginal impacts depend on the **size of the existing portfolio**, the **entry order** selected for an account could determine whether it is written or declined. (Mango, p. 13)

**Problem PCRL-22. Fill in the blanks** (Mango, p. 13): Renewal additivity reduces the renewal risk-load calculation to a(n) \_\_\_\_\_ of the \_\_\_\_\_ back to the individual accounts. The field of \_\_\_\_\_ offers a systematic, objective methodology for such \_\_\_\_\_.

**Solution PCRL-22.** Renewal additivity reduces the renewal risk-load calculation to an **allocation** of the **total portfolio amount** back to the individual accounts. The field of **game theory** offers a systematic, objective methodology for such **allocation**.

**Problem PCRL-23.** What are four features of “cooperative games with transferable utilities”? (Mango, p. 14)

**Solution PCRL-23.** The following are four features of “cooperative games with transferable utilities”:

1. Participants (players) that have some benefits or costs to share.
2. The opportunity to share benefits or costs results from cooperation of all participants or a subgroup thereof.
3. Freedom for players to engage in negotiations, bargaining, and coalition formation.
4. Conflicting player objectives, with each player wanting to secure the largest share of benefits or smallest share of costs. (Mango, p. 14)

**Problem PCRL-24.** Provide the following definitions in the context of game theory:

- (a) Define a *characteristic function*.
- (b) Define a *sub-additive function*.
- (c) Define a *super-additive function*.
- (d) Which of these functions – sub-additive or super-additive – would be used when players seek to maximize their own allocations (benefits), and which would be used when they seek to minimize their own allocations (costs)?

(Mango, p. 15)

**Solution PCRL-24.**

- (a) A *characteristic function* associates a real number  $v(S)$  to each coalition or group  $S$  of players.
- (b) A *sub-additive function* is such that  $v(S) + v(T) > v(S \cup T)$  for every disjoint  $S$  and  $T$ .
- (c) A *super-additive function* is such that  $v(S) + v(T) < v(S \cup T)$  for every disjoint  $S$  and  $T$ .
- (d) **To maximize allocations (benefits):** A super-additive function is used.  
**To minimize allocations (costs):** A sub-additive function is used.

(Mango, p. 15)

**Problem PCRL-25. Fill in the blanks** (Mango, pp. 15-16): In a coalition, a player’s marginal impact depends on its \_\_\_\_\_. To achieve fairness and stability, a new member should receive an allocation amount between its \_\_\_\_\_ and its \_\_\_\_\_ on the \_\_\_\_\_, but it is not clear where in between these two amounts the allocation should be.

**Solution PCRL-25.** In a coalition, a player’s marginal impact depends on its **entry order**. To achieve fairness and stability, a new member should receive an allocation amount between its **stand-alone value** and its **full marginal impact** on the **coalition characteristic function**, but it is not clear where in between these two amounts the allocation should be.

**Problem PCRL-26.** Mango (p. 16) mentions four considerations that should be balanced in determining how much to allocate to each member of a coalition. What are these four considerations?

**Solution PCRL-26.** Mango discusses the need to balance the following considerations:

1. Stability
2. Incentives to split from the group
3. Bargaining power
4. Marginal impact to the coalition characteristic-function value

**Problem PCRL-27.**

(a) What two conditions has game theory developed for legitimate allocation methods? Give the names used in game theory to describe each condition.

(b) What is the name used for the set of acceptable ranges of allocations for each player, which adhere to these two conditions?

(Mango, p. 16)

**Solution PCRL-27.**

(a) **Condition 1. Individual Rationality:** Legitimate allocation methods must be additive: the sum of players’ allocations must equal the total amount to be allocated. No player is worse off for having joined the coalition.

**Condition 2. Collective Rationality:** A coalition should be stable: there should not be incentives for a single player or sub-group to split from the group and form a faction. No sub-group would be better off on its own.

(b) The *core* of the game is the set of acceptable ranges of allocations for each player, which adhere to these two conditions. (Mango, p. 16)

**Problem PCRL-28.** Fill out the following table to demonstrate how various elements of game theory translate into elements for calculating a property catastrophe risk load. (Mango, p. 17)

Game Theory	Property Catastrophe Risk Load
Player	
Coalition	
Characteristic Function	



**Solution PCRL-28.** This table is provided by Mango, p. 17:

Game Theory	Property Catastrophe Risk Load
Player	Account
Coalition	Portfolio
Characteristic Function	Portfolio Variance or Standard Deviation

**Problem PCRL-29.**

- (a) Explain how the risk-load problem fits within the framework of game theory.
- (b) Using the current definition of marginal impact of a renewal account, what would need to be assumed in order to make the risk-load allocation additive? (Mango, pp. 17-18)

**Solution PCRL-29.**

- (a) The “players” – i.e., holders of individual accounts, want to minimize their allocations of the risk load. The goal of the allocation is to fairly and objectively assign risk loads in proportion to their contribution to the total. (Mango, p. 17)
- (b) An **entry order** would need to be assumed in order to make the risk-load allocation additive. The results of the allocation would be heavily dependent on the selected order. (Mango, p. 18)

**Problem PCRL-30.** What three desirable characteristics does the Shapley allocation method have? (Mango, p. 18)

**Solution PCRL-30.** The Shapley allocation method is:

1. Additive;
2. At the centroid of the core;
3. Order-independent. (Mango, p. 18)

**Problem PCRL-31. Fill in the blanks** (Mango, p. 18): The allocation under the Shapley method (the Shapley value) equals the average of the \_\_\_\_\_ taken over all possible \_\_\_\_\_, which are the different orders in which a new member could have been added to the coalition or a new account could have been added to a portfolio.

**Solution PCRL-31.** The allocation under the Shapley method (the Shapley value) equals the average of the **marginal impacts** taken over all possible **entrance permutations**, which are the different orders in which a new member could have been added to the coalition or a new account could have been added to a portfolio.

**Problem PCRL-32.** Suppose that an account n is considered for addition into portfolio L.

- (a) What is the general formula for the Shapley value due to the addition of account n? (Mango, p. 19)
- (b) If the variance of n is 393500 and the covariance of n with portfolio L is 99999, what is the Shapley value due to the addition of account n?

**Solution PCRL-32.**

- (a) **Shapley Value = Var(n) + Cov(L, n).**
- (b) Here, Var(n) = 393500, and Cov(L, n) = 99999, so **Shapley Value = 493499.**

**Problem PCRL-33.** You are given a portfolio with three accounts, A, B, and C, and any possible entry order from a prior vantage point. You know the following:  
 $\text{Var}(A) = 355000$ ;  $\text{Var}(B) = 326500$ ;  $\text{Var}(C) = 119990$ ;  $\text{Cov}(C, A) = 55000$ ;  $\text{Cov}(C, B) = 66000$ .

- (a) What is the marginal variance due to the entry of Account C, if the following entry order is assumed: C, B, A?
- (b) What is the marginal variance due to the entry of Account C, if the following entry order is assumed: A, C, B?
- (c) What is the marginal variance due to the entry of Account C, if the following entry order is assumed: B, C, A?
- (d) What is the marginal variance due to the entry of Account C, if the following entry order is assumed: A, B, C?
- (e) What is the Shapley value for Account C?

**Solution PCRL-33.**

- (a) Because C is being added first, the marginal variance is  $\text{Var}(C) = 119990$ .
- (b) C is being added after A, so 2 times the covariance of C with A gets added to  $\text{Var}(C)$  to calculate the marginal variance:  $\text{Var}(C) + 2\text{Cov}(C, A) = 119990 + 2*55000 = 229990$ .
- (c) C is being added after B, so 2 times the covariance of C with B gets added to  $\text{Var}(C)$  to calculate the marginal variance:  $\text{Var}(C) + 2\text{Cov}(C, B) = 119990 + 2*66000 = 251990$ .
- (d) C is being added after both A and B, so 2 times the covariance of C with A *and* 2 times the covariance of C with B get added to  $\text{Var}(C)$  to calculate the marginal variance:  $\text{Var}(C) + 2\text{Cov}(C, A) + 2\text{Cov}(C, B) = 119990 + 2*55000 + 2*66000 = 361990$ .
- (e) The Shapley value is  $\text{Var}(C) + \text{Cov}(C, \text{Portfolio except for C}) = \text{Var}(C) + \text{Cov}(C, A) + \text{Cov}(C, B) = 119990 + 55000 + 66000 = \text{Shapley Value} = 240990$ .

**Problem PCRL-34.** What are the key similarity and the key difference between the formula for marginal variance and the formula for the Shapley value? (Mango, p. 19)

**Solution PCRL-34.**

**Key Similarity:** Both formulas include a term for the account variance.

**Key Difference:** The formula for marginal variance adds *two times* the covariance of the account with the portfolio, while the formula for the Shapley value adds the covariance of the account with the portfolio *only once*. (Mango, p. 19)

**Problem PCRL-35.** According to Mango (p. 20), how can the risk-load question be framed in a game-theoretical light?

**Solution PCRL-35.** The question becomes: *How do accounts share their mutual covariance for purposes of calculating risk load?* (Mango, p. 20)

**Problem PCRL-36.** In what situations would the Shapley method's approach of sharing mutual account covariance *not* be a reasonable allocation approach? (Mango, pp. 20-21)

**Solution PCRL-36.** The Shapley method allocates mutual covariance equally across all accounts. If one account has significantly more losses for each event than another account, then the account with fewer losses will get too much of the shared covariance relative to the risk it poses, while the account with more losses will get too little of the shared covariance.

**Problem PCRL-37.** Let  $p_i$  be the probability of event  $i$ , and let  $X$  and  $Y$  be two accounts that are being combined into a portfolio. Let  $W_i^X(X, Y)$  denote the weight used to determine account  $X$ 's share of the mutual covariance between  $X$  and  $Y$  for event  $i$ .

- (a) What is the formula for  $\text{CovShare}_i^X(X, Y)$ ,  $X$ 's share of the mutual covariance?
- (b) What is the formula for  $\text{CovShare}_i^Y(X, Y)$ ,  $Y$ 's share of the mutual covariance?
- (c) What is the formula for  $\text{CovShare}_{\text{Tot}}^X$ , the total covariance-share allocation for account  $X$  over all events?
- (d) Explain how the Shapley method is a specific example of the generalized covariance-sharing method. (Mango, p. 21)

**Solution PCRL-37.**

- (a)  $\text{CovShare}_i^X(X, Y) = W_i^X(X, Y) * 2 * X_i * Y_i * p_i * (1 - p_i)$
- (b)  $\text{CovShare}_i^Y(X, Y) = [1 - W_i^X(X, Y)] * 2 * X_i * Y_i * p_i * (1 - p_i)$
- (c)  $\text{CovShare}_{\text{Tot}}^X = \sum_Y \sum_i [\text{CovShare}_i^X(X, Y)]$ , where  $\sum_Y$  is the sum over every other account  $Y$  in the portfolio.
- (d) The Shapley method can be expressed within the framework of the generalized covariance-sharing method by setting  $W_i^X(X, Y) = 50\%$  for all  $X, Y$ , and  $i$ . (Mango, p. 21)

**Problem PCRL-38.** For the specific Covariance Share (CS) method, let  $p_i$  be the probability of event  $i$ , and let  $X$  and  $Y$  be two accounts that are being combined into a portfolio. Let  $W_i^X(X, Y)$  denote the weight used to determine account  $X$ 's share of the mutual covariance between  $X$  and  $Y$  for event  $i$ . Let  $X_i$  and  $Y_i$  be the loss amounts for accounts  $X$  and  $Y$ , respectively, due to event  $i$ . What is the formula for  $W_i^X(X, Y)$  under the specific CS method? (Mango, p. 21)

**Solution PCRL-38.**  $W_i^X(X, Y) = X_i / (X_i + Y_i)$ .

**Problem PCRL-39.** Suppose that, for event  $i$ , the loss for account  $X$  would be 900, and the loss for account  $Y$  would be 1500. For the specific Covariance Share (CS) method, what would be the weight  $W_i^X(X, Y)$  assigned to determine account  $X$ 's share of the mutual covariance between  $X$  and  $Y$  for event  $i$ ?

**Solution PCRL-39.** We use the formula  $W_i^X(X, Y) = X_i / (X_i + Y_i)$ , where  $X_i = 900$  and  $Y_i = 1500$ . Thus,  $W_i^X(X, Y) = 900 / (900 + 1500) = W_i^X(X, Y) = \mathbf{0.375}$ .

**Problem PCRL-40.** You are given the following information about two accounts,  $X$  and  $Q$ , which are being combined into a portfolio. Let  $\text{Var}(X)$  and  $\text{Var}(Q)$  denote the variances of  $X$  and  $Q$ , respectively. In the build-up scenario,  $X$  is added to the portfolio first, followed by  $Q$ .

- $\text{Var}(X) = 1,225,000,000$
- $\text{Var}(Q) = 676,000,000$
- $\text{Var}(X + Q) = 2,500,000,000$

- The risk-load multiplier is 0.000002703448276.
- For each event  $i$ , Account X is expected to have 1.5 times the losses of Account Q.

(a) Complete the following table for the Shapley method:

<b>Build-Up Scenario: Q added to X: Shapley Method</b>				
	<b>Account X</b>	<b>Account Q</b>	<b>Account X + Account Q</b>	<b>Portfolio (X + Q)</b>
<b>(1) Change in Variance</b>	1,225,000,000			2,500,000,000
<b>(2) Risk-Load Multiplier</b>	0.000002703448276	0.000002703448276	N/A	0.000002703448276
<b>(3) Risk Load</b>				
<b>Renewal Scenario: Each Account Renewed: Shapley Method</b>				
	<b>Account X</b>	<b>Account Q</b>	<b>Account X + Account Q</b>	<b>Portfolio (X + Q)</b>
<b>(4) Change in Variance</b>				2,500,000,000
<b>(5) Risk-Load Multiplier</b>	0.000002703448276	0.000002703448276	N/A	0.000002703448276
<b>(6) Risk Load</b>				
<b>(7) Build-Up Risk Load</b>				
<b>(8) Difference</b>				

(b) Complete the following table for the specific Covariance Share (CS) method:

<b>Build-Up Scenario: Q added to X: Covariance Share (CS) Method</b>				
	<b>Account X</b>	<b>Account Q</b>	<b>Account X + Account Q</b>	<b>Portfolio (X + Q)</b>
<b>(1) Change in Variance</b>	1,225,000,000			2,500,000,000
<b>(2) Risk-Load Multiplier</b>	0.000002703448276	0.000002703448276	N/A	0.000002703448276
<b>(3) Risk Load</b>				
<b>Renewal Scenario: Each Account Renewed: Covariance Share (CS) Method</b>				
	<b>Account X</b>	<b>Account Q</b>	<b>Account X + Account Q</b>	<b>Portfolio (X + Q)</b>
<b>(4) Change in Variance</b>				2,500,000,000
<b>(5) Risk-Load Multiplier</b>	0.000002703448276	0.000002703448276	N/A	0.000002703448276
<b>(6) Risk Load</b>				
<b>(7) Build-Up Risk Load</b>				
<b>(8) Difference</b>				

(c) What important property do the renewal risk loads calculated for both the Shapley method and the CS method demonstrate?

**Solution PCRL-40.**

(a) First, we find  $\text{Cov}(X, Q)$ .  $\text{Var}(X + Q) = \text{Var}(X) + \text{Var}(Q) + 2\text{Cov}(X, Q)$ .  
Thus,  $\text{Cov}(X, Q) = [\text{Var}(X + Q) - \text{Var}(X) - \text{Var}(Q)]/2 = (2,500,000,000 - 1,225,000,000 - 676,000,000)/2 = \text{Cov}(X, Q) = 299,500,000$ .

**Row (1):** Using the Shapley method, the marginal variance of Q will be  $\text{Var}(Q) + \text{Cov}(X, Q) = 676,000,000 + 299,500,000 = 975,500,000$ . The sum of marginal variances for (Account X + Account Q) will be  $1,225,000,000 + 975,500,000 = 2,200,500,000$ .

**Row (3):** The Risk Load is equal to (Change in Variance)\*(Risk-Load Multiplier): This product is 3311.724138 for Account X, 2637.213793 for Account Q, and 6758.62069 for Portfolio (X + Q). For (Account X + Account Q), we take the sum  $3311.724138 + 2637.213793 = 5948.937931$ .

**Row (4):** In the renewal scenario, the Shapley approach is used with both Account X and Account Q. For Account Q, the marginal variance remains 975,500,000. For Account X, the marginal variance is  $\text{Var}(X) + \text{Cov}(X, Q) = 1,225,000,000 + 299,500,000 = 1,524,500,000$ . Thus, the sum of marginal variances for (Account X + Account Q) will be  $975,500,000 + 1,524,500,000 = 2,500,000,000$ .

**Row (6):** The Risk Load is equal to (Change in Variance)\*(Risk-Load Multiplier): This product is 4121.406897 for Account X, 2637.213793 for Account Q, and 6758.62069 for Portfolio (X + Q). For (Account X + Account Q), we take the sum  $4121.406897 + 2637.213793 = 6758.62069$ .

**Row (7):** We restate the build-up risk loads from Row (3).

**Row (8):** We take the difference between the value in Row (6) and the corresponding value in Row (7). The answer is 809.4689656 for Account X, 0 for Account Q, 809.4689656 for (Account X + Account Q), and 0 for Portfolio (X + Q).

<b>Build-Up Scenario: Q added to X: Shapley Method</b>				
	<b>Account X</b>	<b>Account Q</b>	<b>Account X + Account Q</b>	<b>Portfolio (X + Q)</b>
<b>(1)</b> Change in Variance	1,225,000,000	975,500,000	2,200,500,000	2,500,000,000
<b>(2)</b> Risk-Load Multiplier	0.000002703448276	0.000002703448276	N/A	0.000002703448276
<b>(3)</b> Risk Load	3311.724138	2637.213793	5948.937931	6758.62069
<b>Renewal Scenario: Each Account Renewed: Shapley Method</b>				
	<b>Account X</b>	<b>Account Q</b>	<b>Account X + Account Q</b>	<b>Portfolio (X + Q)</b>
<b>(4)</b> Change in Variance	1,524,500,000	975,500,000	2,500,000,000	2,500,000,000
<b>(5)</b> Risk-Load Multiplier	0.000002703448276	0.000002703448276	N/A	0.000002703448276
<b>(6)</b> Risk Load	4121.406897	2637.213793	6758.62069	6758.62069
<b>(7)</b> Build-Up Risk Load	3311.724138	2637.213793	5948.937931	6758.62069
<b>(8)</b> Difference	809.4689656	0	809.4689656	0

**(b)** First, we compute the weight to determine X's share of the mutual covariance. We use the formula  $W_i^X(X, Q) = X_i / (X_i + Q_i)$ . We are given that  $X_i = 1.5Q_i$  for each  $i$ . Thus, the weight becomes  $1.5Q_i / (1.5Q_i + Q_i) = 1.5 / 2.5 = 3/5 = W_i^X(X, Q) = 0.6$ . It follows that Q's share of the mutual covariance is  $W_i^Q(X, Q) = 1 - W_i^X(X, Q) = 1 - 0.6 = 0.4$ .

During the renewal scenario (or whenever one account is added to the portfolio where the other account is already present), the marginal variance of X will be  $\text{Var}(X) + 0.6 * 2 * \text{Cov}(X, Q) = \text{Var}(X) + 1.2 * \text{Cov}(X, Q) = 1,225,000,000 + 1.2 * 299,500,000 = 1,584,400,000$ . This is the entry for Account X in Row (4).

The marginal variance of Q will be  $\text{Var}(Q) + 0.4 * 2 * \text{Cov}(X, Q) = \text{Var}(Q) + 0.8 * \text{Cov}(X, Q) = 676,000,000 + 0.8 * 299,500,000 = 915,600,000$ . This is the entry for Account Q in both Row (1) and Row (4).

In Row (1), the sum of marginal variances for (Account X + Account Q) will be  $1,225,000,000 + 915,600,000 = 2,140,600,000$ .

**Row (3):** The Risk Load is equal to (Change in Variance)\*(Risk-Load Multiplier): This product is 3311.724138 for Account X, 2475.277241 for Account Q, and 6758.62069 for Portfolio (X + Q). For (Account X + Account Q), we take the sum  $3311.724138 + 2475.277241 = 5787.001379$ .

**Row (6):** The Risk Load is equal to (Change in Variance)\*(Risk-Load Multiplier): This product is 4283.343448 for Account X, 2475.277241 for Account Q, and 6758.62069 for Portfolio (X + Q). For (Account X + Account Q), we take the sum  $4283.343448 + 2475.277241 = 6758.62069$ .

**Row (7):** We restate the build-up risk loads from Row (3).

**Row (8):** We take the difference between the value in Row (6) and the corresponding value in Row (7). The answer is 971.6193103 for Account X, 0 for Account Q, 971.6193103 for (Account X + Account Q), and 0 for Portfolio (X + Q).

<b>Build-Up Scenario: Q added to X: Covariance Share (CS) Method</b>				
	<b>Account X</b>	<b>Account Q</b>	<b>Account X + Account Q</b>	<b>Portfolio (X + Q)</b>
<b>(1) Change in Variance</b>	1,225,000,000	915,600,000	2,140,600,000	2,500,000,000
<b>(2) Risk-Load Multiplier</b>	0.000002703448276	0.000002703448276	N/A	0.000002703448276
<b>(3) Risk Load</b>	3311.724138	2475.277241	5787.001379	6758.62069
<b>Renewal Scenario: Each Account Renewed: Covariance Share (CS) Method</b>				
	<b>Account X</b>	<b>Account Q</b>	<b>Account X + Account Q</b>	<b>Portfolio (X + Q)</b>
<b>(4) Change in Variance</b>	1,584,400,000	915,600,000	2,500,000,000	2,500,000,000
<b>(5) Risk-Load Multiplier</b>	0.000002703448276	0.000002703448276	N/A	0.000002703448276
<b>(6) Risk Load</b>	4283.343448	2475.277241	6758.62069	6758.62069
<b>(7) Build-Up Risk Load</b>	3311.724138	2475.277241	5787.001379	6758.62069
<b>(8) Difference</b>	971.6193103	0	971.6193103	0

(c) The renewal risk loads calculated for both the Shapley method and the CS method demonstrate the property of *renewal additivity*. The sum of the renewal risk loads of X and Q is equal to the risk load of the portfolio (X + Q).

**Problem PCRL-41. Fill in the blanks** (Mango, p. 25): In practice, risk-load allocation methods among accounts could vary based on whether new accounts are being priced or renewals are being considered. The ideal solution might involve using a \_\_\_\_\_ method for the pricing of new accounts, and a \_\_\_\_\_ method for renewals.

**Solution PCRL-41.** In practice, risk-load allocation methods among accounts could vary based on whether new accounts are being priced or renewals are being considered. The ideal solution might involve using a **marginal** method for the pricing of new accounts, and a **renewal-additive** method for renewals. (Mango, p. 25)