Study Guide on Non-tail Risk Measures for the Casualty Actuarial Society (CAS) Exam 7

(Based on Gary Venter's Paper, "Non-tail Measures and Allocation of Risk Measures")

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Source: Venter, G.G., "Non-tail Measures and Allocation of Risk Measures", CAS Study Note, 1/11/2010.

This is an open-source study guide and may be revised pursuant to suggestions.

S7-NTM-1. Fill in the blanks (Venter, p. 2):
When capital is allocated to line of business by allocating a risk measure, _______ becomes a method for computing profitability across lines on a risk-adjusted basis. A typical follow-up to that is to __________ (do *what*?) in order to equalize this return.

Solution S7-NTM-1. When capital is allocated to line of business by allocating a risk measure, **return on allocated capital** becomes a method for computing profitability across lines on a risk-adjusted basis. A typical follow-up to that is to **set target profits by line** in order to equalize this return.

S7-NTM-2. Venter (p. 2) mentions two issues to address in risk pricing. What are they?

Solution S7-NTM-2. Two issues to address in risk pricing are as follows:
1. Should there be a charge for any risk taken? If so, pricing tail risk is not sufficient.
2. What is the risk aversion toward larger losses? Typically, the aversion increases more than proportionally with respect to the size of loss.

S7-NTM-3.
(a) What is a weakness of using the standard deviation for risk pricing?
(b) What is one alternative to standard deviation that is related to standard deviation?
(c) What is a weakness of the alternative in part (b) above? (Venter, p. 2)

Solution S7-NTM-3.
(a) Standard deviation treats favorable and adverse deviations equally, which is a problem for asymmetric distributions (especially where the losses have a heavy tail).
(b) The **semi-standard deviation**, which measures only adverse deviations, is an alternative.
(c) A weakness of the semi-standard deviation is that, when the distribution is skewed, it might not give enough weight to the very adverse losses.

S7-NTM-4. Describe *distortion measures* in qualitative terms. (Venter, p. 2)

Solution S7-NTM-4. Distortion measures quantify risk by adjusting the probabilities of outcomes, giving more weight to the adverse outcomes and less weight to the favorable outcomes.

S7-NTM-5. Given the excess probability p, the standard normal distribution Φ, the t-distribution function with a degrees of freedom (T_a), and the parameter λ, give the formula of the *Wang transform* G(p) that can serve as a distortion measure. (Venter, pp. 2-3)

Solution S7-NTM-5. $\mathbf{G(p) = 1 - T_a*[\Phi^{-1}(1-p) + \lambda]}$**.**

S7-NTM-6. The questions below address the Esscher transform. (Venter, p. 3)
(a) Given a random variable Y and parameter ω, define the variables c and k.
(b) What is a requirement for the Esscher transform to exist?
(c) Given probability density function f(y), what is the Esscher transform f*(y)?
(d) What is the formula for the Esscher transform E*[Y] in terms of Y, k, and c?

Solution S7-NTM-6.
(a) $\mathbf{c = S^{-1}(1/\omega)}$, and $\mathbf{k = E[e^{Y/c}]^{-1}}$.
(b) The Esscher transform can only exist if the expectation $k = E[e^{Y/c}]^{-1}$ exists.
(c) $\mathbf{f^*(y) = k*f(y)*e^{y/c}}$.
(d) $\mathbf{E^*[Y] = E[k*Y*e^{Y/c}]}$.

S7-NTM-7.
(a) Let Y be a random variable with mean μ, standard deviation σ, and probability density function f(y). Give the formula for the *quadratic transform* f*(y) with parameter c.
(b) For the quadratic transform, what generalizations can be made regarding probabilities for losses below the mean and probabilities for losses above the mean? Explain these generalizations with reference to the formula in part (a) above. (Venter, p. 3)

Solution S7-NTM-7.
(a) $f^*(y) = f(y)*[1 + \sigma^2/\mu^2 + c*y/\mu]/[1 + \sigma^2/\mu^2 + c]$
(b) The quadratic transform decreases probabilities for losses below the mean and increases probabilities for losses above the mean. If $y > \mu$, then $c*y/\mu > c$, so the numerator is greater than the denominator, and $f^*(y) > f(y)$. The opposite holds if $y < \mu$.

S7-NTM-8. Match the following descriptions by Venter (p. 3) to the appropriate transforms. The choices are (i) the Wang transform, (ii) the Esscher transform, and (iii) the quadratic transform.
(a) This transform has been found to be consistent with pricing for catastrophe reinsurance coverages.
(b) This transform was a little lighter than the market in pricing extreme-tail coverages.
(c) This transform has been shown to replicate pricing for catastrophe bonds and commercial bonds.

Solution S7-NTM-8.
(a) (ii) Esscher transform
(b) (iii) Quadratic transform
(c) (i) Wang transform

S7-NTM-9.
(a) Fill in the blanks (Venter, p. 3): A distortion measure is one that can be specified by a distribution function G(x) on the unit interval so that G(0) = _____ and G(1) = _____, and the risk measure ρ(Y) = __________, where S(y) = 1 – F(y) is the survival function of Y.
(b) From your answer in (a) above, what is another survival function associated with the distortion measure?
(c) Fill in the blank: The role G is to transform the probabilities of Y to another _______. A distortion risk measure is a transformed _______. (Venter, p. 3)

Solution S7-NTM-9.
(a) Fill in the blanks (Venter, p. 3): A distortion measure is one that can be specified by a distribution function G(x) on the unit interval so that G(0) = **0** and G(1) = **1**, and the risk measure ρ(Y) = $\int_0^\infty$**G[S(y)]dy**, where S(y) = 1 – F(y) is the survival function of Y.
(b) G[S(y)] is the other survival function.
(c) The role G is to transform the probabilities of Y to another **probability distribution**. A distortion risk measure is a transformed **mean**.

S7-NTM-10.
(a) For a distortion measure G(y), give the formula for f*(y), the density of the associated transformed probability distribution in terms of G, S(y), and f(y).
(b) Give the formula for E*[Y], the mean of the transformed distribution. This is another formula for the risk measure ρ(Y). (Hint: It may help to think of these multiple formulas as related to the multiple formulas for the ordinary expected value, E(Y).) (Venter, p. 3)

Solution S7-NTM-10.
(a) f*(y) = G'[S(y)]*f(y).
(b) E*[Y] = $\int_{-\infty}^\infty$**Y*G'[S(y)]*f(y)*dy = E[Y*G'[S(Y)]].**

S7-NTM-11.
(a) Identify two common measures that are actually distortion measures.
(b) Define a *complete risk measure* in terms of distortion risk measures. (Venter, p. 11)
(c) Why are the answers from part (a) *not* complete risk measures? (Venter, p. 3)
(d) What tail risk measures satisfy the criteria for a complete risk measure? (Venter, p. 11)

Solution S7-NTM-11.
(a) VaR and TVaR are distortion measures.
(b) A complete risk measure is one where G(p) is not constant on any interval and is an increasing function on the unit interval.
(c) VaR and TVaR are not complete risk measures because they are constant over certain intervals. For probability $p > 0.01$, it is the case for both $VaR_{0.99}$ and $TVaR_{0.99}$ that $G(p) = 1$.
(d) This is a trick question. **No** tail risk measures satisfy the criteria for a complete risk measure, because they are zero for all values below the tail.

S7-NTM-12. In what general situations are distortion measures arbitrage-free? In what situations are they not arbitrage-free? (Venter, p. 4)

Solution S7-NTM-12. Distortion measures are arbitrage-free if they transform the probabilities of underlying events. They are not arbitrage-free if they transform the probabilities of outcomes of financial deals.

S7-NTM-13.
(a) Define **marginal allocation**.
(b) Define **last-in marginal allocation** in mathematical terms for random variable $Y = \sum(X_j)$, where the X_j are business units, and $\rho(Y)$ is a risk measure on Y.
(c) Define **Aumann allocation**.
(d) Define **incremental marginal allocation** in terms of the notation from part (b) as well as the small increment ε.
(e) What does the formula from (d) become as ε approaches 0? (Venter, pp. 5-6)

Solution S7-NTM-13.
(a) Marginal allocation: Allocating in proportion to the impact of the business unit on the company risk measure.
(b) Last-in marginal allocation: The impact of a business unit is measured by **$\rho(Y) - \rho(Y - X_j)$**. This is the impact with the unit, minus the impact without the unit.
(c) Aumann allocation: The impact of a business unit is averaged over every coalition of business units that business unit can be in.
(d) Incremental marginal allocation: The impact of a business unit is measured by **$[\rho(Y) - \rho(Y - \varepsilon^*X_j)]/\varepsilon$.**
(e) As ε approaches 0, the incremental marginal allocation formula becomes the derivative of the company risk measure with respect to the volume of the business unit.

S7-NTM-14.
(a) Define **marginal decomposition.** Identify a synonym for marginal decomposition.
(b) Fill in the blanks: By Euler's theorem, marginal decomposition happens when the risk measure is homogeneous degree _, i.e., for a positive constant k it is the case that _____. (Venter, p. 6)

Solution S7-NTM-14.
(a) Marginal decomposition = Euler allocation occurs when the incremental marginal impacts add up to the whole company risk measure.
(b) By Euler's theorem, marginal decomposition happens when the risk measure is homogeneous degree **1**, i.e., for a positive constant k it is the case that **ρ(k*Y) = k*ρ(Y)**.

S7-NTM-15.
(a) Define **proportional allocation** and give a formula for the associated ratio $r(X_j)$.
(b) Under what condition will proportional allocation provide a marginal decomposition? (Venter, p. 6)

Solution S7-NTM-15.
(a) Proportional allocation involves allocating a risk measure by calculating the risk measure on the company and each business unit and allocating by the ratio of the unit risk to the company risk. The formula for this ratio is **$r(X_j) = \rho(Y)*\rho(X_j)/\sum(X_j)$.**
(b) Proportional allocation provides a marginal decomposition if the risk measure is the mean under a transformed probability distribution.

S7-NTM-16.
(a) Define **suitable allocation**.
(b) Why does suitable allocation make sense intuitively? (Examine the definition and try to develop a logical answer.)
(c) What is the only method that guarantees a suitable allocation? (Venter, p. 6)

Solution S7-NTM-16.
(a) Under a **suitable allocation,** allocating capital by the allocation of a risk measure, and computing the return on allocated capital, then proportionally increasing the size of a business unit that has a higher-than-average return on capital, will increase the return on capital for the firm.
(b) It would make intuitive sense that, by increasing the size of a unit that produces higher-than-average return, with all other things being equal, one would increase the return of the firm. A method of allocation that achieves contrary results may *reduce* the return of the firm.
(c) Marginal decomposition is the only method that guarantees a suitable allocation.

S7-NTM-17. Let the risk measure ρ(Y) be expressible as
$\rho(Y) = E({}_i\sum(h_i(Y)*L_i(Y)) \mid \text{ith condition on } Y)$, such that for each h, h(V + W) = h(V) + h(W), and the only restriction on the L_i values is that the aforementioned conditional expected value exists. Define the **co-measure** $r(X_j)$ for unit X_j (Venter, p. 7).

Solution S7-NTM-17. $\mathbf{r(X_j) = E({}_i\sum(h_i(X_j)*L_i(Y)) \mid \text{ith condition on } Y)}$.
(Essentially, we take the expression for $\rho(Y)$ and substitute X_j for Y in the h_i term.)

S7-NTM-18. For the co-measures of individual business units, what is the implication of the fact that each function h_i is additive? (Venter, p. 7).

Solution S7-NTM-18. The fact that each function h_i is additive implies that the co-measures of individual business units add up to the whole risk measure of the enterprise.

S7-NTM-19.
(a) Give the formula for $TVar_{0.99}$ as a risk measure $\rho(Y)$, using conditional expectation notation.
(b) Give the formula for the co-$TVaR_{0.99}$ as a co-measure $r(X_j)$ for unit X_j (Venter, p. 7).
(c) Define the risk-adjusted $TVaR_{0.99}$, or $RTVaR_{0.99}$, in terms of $TVaR_{0.99}$ and any other needed conditional notation. Let c be a multiplicative scaling constant.
(d) Treating $RTVaR_{0.99}$ as a risk measure, define the co-measure co-$RTVaR_{0.99}$. (Venter, p. 7).

Solution S7-NTM-19.
(a) $TVaR_{0.99} = \mathbf{\rho(Y) = E[Y \mid Y > F^{-1}(0.99)]}$.
(b) co-$TVaR_{0.99}$ = $\mathbf{r(X_j) = E[X_j \mid Y > F^{-1}(0.99)]}$.
(c) $\mathbf{RTVaR_{0.99} = TVaR_{0.99} + c*StDev[Y \mid F(Y) > 0.99]}$
(d) co-$RTVaR_{0.99}$ = co-($TVaR_{0.99}$ + c*StDev[Y │ F(Y) > 0.99]) =
co-$TVaR_{0.99}$ + c*(co-StDev[Y │ F(Y) > 0.99]).
Note that the co-standard deviation here is
$Cov[X_j,Y \mid F(Y) > 0.99]/StDev[Y \mid F(Y) > 0.99]$, so
$\mathbf{co\text{-}RTVaR_{0.99} = co\text{-}TVaR_{0.99} + c*Cov[X_j,Y \mid F(Y) > 0.99]/StDev[Y \mid F(Y) > 0.99]}$.

S7-NTM-20.
(a) If a co-measure does provide for a marginal allocation, what other two pricing criteria should it still meet?
(b) Suggest a risk measure whose co-measures meet these criteria. (Venter, p. 8)

Solution S7-NTM-20.
(a) The co-measure should meet the criteria of (1) not ignoring risk, even for smaller losses, and (2) increasing more than linearly as losses increase.
(b) RTVaR over a low threshold meets the criteria in part (a). Also, a weighted sum of TVaRs at different probabilities, from relatively low to relatively high, could be used.

S7-NTM-21. What is the **Myers-Read allocation**? (Venter, pp. 8-9)

Solution S7-NTM-21. The Myers-Read allocation is an additive marginal allocation method that requires that the value of the default put option as a fraction of expected loss be the same for each business unit. This method uses the required capital as the risk measure. The method is a marginal decomposition and meets the other criteria of risk pricing. However, it is primarily useful for allocating the frictional costs of holding capital, not for measuring the price for bearing risk.

S7-NTM-22. The following notation refers to the method of **allocation by layer** (Venter, p. 9).
$X_{i,k}$ = Loss to unit i in the kth simulation
Y_k = Total company losses, such that $Y_k = \sum(X_{i,k})$
C = Total capital to be allocated to the business unit
z = Loss layer from (z-1) cents to z cents. (This is the smallest unit possible.)
n_z = Number of simulations of Y_k that are z or greater.

(a) What is the formula for the allocation of layer z to unit i?
(b) What is the formula for the allocation of capital C to unit i?
(c) Describe two strengths and one weakness of this method.

Solution S7-NTM-22.
(a) The allocation of layer z to unit i is $(1/ n_z)* {}_{k \text{ such that } Y\_k \geq z}\sum(X_{i,k}/Y_k)$.
(b) The allocation of capital C to unit i is ${}_{z=1}{}^{C}\sum[(1/ n_z)*{}_{k \text{ such that } Y\_k \geq z}\sum(X_{i,k}/Y_k)]$.
(c) Two strengths of this method are (1) that it does not ignore small losses, because all layers contribute to the allocation, and (2) the larger losses get a greater overall weight because they contribute to the allocation for all lower layers. A weakness of this method is that it is not a marginal allocation if C is set equal to a risk measure.

S7-NTM-23.
(a) Does the Merton-Perold method require capital allocation?
(b) Describe how the Merton-Perold method views the firm and the implications for valuation.
(c) Why is the Merton-Perold method inapplicable to insurance?
(Venter, pp. 9-10)

Solution S7-NTM-23.
(a) The Merton-Perold method **does not** require capital allocation.
(b) The Merton-Perold method views the firm as providing each business unit with the option to use the firm capital if its losses exceed its premiums. The value of the option is the cost to the firm of carrying that business, and the value-added of the unit is the excess of its profits over its capital cost.
(c) The Merton-Perold method is inapplicable to insurance because of its use of a one-period timeframe and the distributions/formulas used, including the Black-Scholes formula, which do not reflect insurance risk or appropriate insurance pricing.

S7-NTM-24. What does Venter (p. 10) identify as the "most controversial and most often failing" requirement for a *coherent risk measure*? When/why is this a useful criterion?

Solution S7-NTM-24. The requirement in question is **subadditivity:** that the risk measure of a sum of independent random variables should not be greater than the sum of their risk measures. This is a useful criterion when the diversification benefit of combining business units is being measured, and one seeks to guarantee in advance that the "benefit" will not be negative. Venter believes that this is a minority of cases, and otherwise subadditivity is not a necessary requirement.

S7-NTM-25.
(a) Give two criteria for an *adapted risk measure* (Venter, p. 11).
(b) Give an example of an adapted risk measure. (Venter, p. 12).

Solution S7-NTM-25.
(a) Criteria for an adapted risk measure:
1. For a distortion measure G(p), it is the case that $G(p) \geq p$. (The risk measure is not less than the mean of the underlying random variable.)
2. In the tail, the relative risk load is unbounded. ($G'' < 0$, and at $p = 0$, $G' \to \infty$.)
(b) The **Wang transform** is an example of an adapted risk measure.

S7-NTM-26.
(a) Are all transformed distributions distortion measures? If so, explain why. If not, give a counterexample.
(b) Give the formulas for the frequency and severity transforms of a combined frequency-severity process Y.
(c) What does TVaR become when a transformed distribution is applied to it?
(Venter, p. 12)

Solution S7-NTM-26.
(a) **Not all** transformed distributions are distortion measures. The **Esscher transform** is a counterexample.
(b) Formula for frequency transform: $\boldsymbol{\lambda^* = \lambda^* E(e^{Y/c})}$
Formula for severity (Esscher) transform: $\boldsymbol{f^*(y) = f(y)^* e^{y/c}/E(e^{Y/c})}$
(c) When a transformed distribution is applied to it, TVaR becomes **WTVaR** (weighted TVaR).