Study Guide on Extreme-Value Theory for the Casualty Actuarial Society (CAS) Exam 7

(Based on "Extreme Value Theory as a Risk Management Tool" by Embrechts, Resnick, and Samorodnitsky)

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Source: Embrechts, P.; Resnick, S.I.; and Samorodnitsky, G., "Extreme Value Theory as a Risk Management Tool," *North American Actuarial Journal*, Volume 3, Number 2, April 1999, Society of Actuaries.

This is an open-source study guide and may be revised pursuant to suggestions.

Notation from Embrechts et al.

df = distribution function
$F^{\leftarrow}$ = the inverse function of the function F
iid = independent and identically distributed
rv = random variable
$X_{n,n} = \min(X_1, \ldots, X_n)$
$X_{1,n} = \max(X_1, \ldots, X_n)$
Order statistics: $X_{n,n} \leq X_{n-1,n} \leq \ldots \leq X_{1,n}$.

Problem S7-EVT-1. Embrechts et al. give three examples on pages 30 to 31 of new products that have been developed due to increased catastrophe losses. What are they?

Solution S7-EVT-1.
1. Catastrophe (CAT) futures and PCS options
2. Covertible CAT bonds
3. Multiline, multiyear, high-layer (infrequent-event) products, credit lines, and the catastrophe risk exchange

Problem S7-EVT-2. What is a manifestation of extreme events in the realm of finance that would be apparent to the general observer? (Embrechts et al., p. 31)

Solution S7-EVT-2. Stock-market crashes / industry losses

Problem S7-EVT-3. Define **(a)** expected loss, **(b)** unexpected loss, and **(c)** stress loss (Embrechts et al., p. 31)
(d) How does Extreme Value Theory (EVT) address these three kinds of losses? List three ways. (Embrechts et al., pp. 31-32)

Solution S7-EVT-3.
(a) Expected loss: Loss that must be assumed to arise on a continuing basis as a consequence of undertaking a particular business.
(b) Unexpected loss: The unusual, though predictable, losses that the firm should be able to absorb in the normal course of its business.
(c) Stress loss: The possible, though improbable, extreme scenarios that the firm must be able to survive.
(d) 1. EVT offers a set of techniques for quantifying the boundaries between expected, unexpected, and stress losses.
2. EVT also offers a scientific language for translating management guidelines on these boundaries into actual numbers.
3. EVT helps in the modeling of default probabilities and the estimation of diversification factors in the management of bond portfolios.

Problem S7-EVT-4. (a) According to Embrechts et al. (p. 32), in classical probability theory and statistics, the results most relevant for insurance variables are based on *what*? Give a mathematical formula.
(b) Generally, when can we be confident in these kinds of results? (Embrechts et al., p. 32)

Solution S7-EVT-4. (a) The results are based on *sums* of random variables, with the sum expressible as $\mathbf{S_n = {}_{r=1}{}^{n}\sum X_r}$.
(b) We can be confident in these kinds of results when calculating on the basis of averages or when estimating results that fall "not too far" from the mean. For modeling more extreme events, classical theory is quite limited.

Problem S7-EVT-5. (Based on the exercise on pp. 32-33 of Embrechts et al.)
Suppose that, in a given portfolio, claims follow an exponential distribution with mean 350. Assume that claims are independent and identically distributed (iid).
(a) We have observed 35 claims, and the largest claim is 1000. Using classical theory, what is the probability that we have this observation and the model is still right? Show your work.
(b) We have observed 35 claims, and the largest claim is 3000. Using classical theory, what is the probability that we have this observation and the model is still right? Show your work.

Solution S7-EVT-5.
(a) By assumption, the random variables denoting each claim, X_1 through X_{35}, are iid and *each* follow an exponential distribution with df $F(x) = 1 – e^{-x/350}$.
Let $M_{35} = \max(X_1,\ldots, X_{35})$. Then the survival function of M_{35} is $\Pr(M_{35} > x) =$
$1 – \Pr(\text{all X are less than x}) = 1 – (1 – e^{-x/350})^{35}$.
We want to find $\Pr(M_{35} \geq 1000) = 1 - (1 – e^{-1000/350})^{35} =$ **0.8738363588**. So there is a substantial likelihood of the model being right and the largest claim being 1000 or greater.

(b) We want to find $Pr(M_{35} \geq 3000) = 1 - (1 - e^{-3000/350})^{35}$ = **0.0066091548**. The likelihood of experiencing a largest claim of 3000 or greater under this model is quite small if the model is right.

Problem S7-EVT-6.
(a) Let $\Lambda(x) = \exp(-e^{-x})$. If $M_n = \max(X_1,\ldots, X_n)$ for n random variables, give the formula for the approximation used by Embrechts et al. (p. 33) for $Pr(M_n \leq x)$, if each of the individual random variables follows an exponential distribution with mean θ.
(b) Consider again the situation in Problem S7-EVT-5, where we observed 35 claims, and the largest claim is 1000. *Using the approximation in part (a)*, what is the probability that we have this observation and the model is still right?
(c) Consider again the situation in Problem S7-EVT-5, where we observed 35 claims, and the largest claim is 3000. *Using the approximation in part (a)*, what is the probability that we have this observation and the model is still right?

Solution S7-EVT-6.
(a) $Pr(M_n \leq x) = \Lambda[x/\theta - \ln(n)]$.
(b) $Pr(M_{35} \geq 1000) = 1 - Pr(M_{35} < 1000) = 1 - \Lambda[1000/350 - \ln(35)] =$
$1 - \Lambda[-0.6982052043] = 1 - \exp(-\exp(0.6982052043))$ = **0.8660303067**.
(c) $Pr(M_{35} \geq 3000) = 1 - Pr(M_{35} < 3000) = 1 - \Lambda[3000/350 - \ln(35)] =$
$1 - \Lambda[5.01608051] = 1 - \exp(-\exp(-5.01608051))$ = **0.0066085309**.

Problem S7-EVT-7. Complete Proposition 1 of Embrechts et al. (p. 33): Suppose $X_1,\ldots, X_n$ are iid with exponential df F with mean θ. Then for real numbers x, _________ (give equation).

(Note: The mean θ = 1/λ, where λ is often used as the parameter for the exponential distribution.)

Solution S7-EVT-7. Suppose $X_1,\ldots, X_n$ are iid with exponential df F with mean θ, then for real numbers x, $\lim_{n\to\infty}[Pr(M_n/\theta - \ln(n) \leq x)] = \Lambda(x)$, where $\Lambda(x) = \exp(-e^{-x})$.

Problem S7-EVT-8. Complete Theorem 2 of Embrechts et al. (p. 33): Suppose $X_1,\ldots, X_n$ are iid with df F and (a_n), (b_n) are constants such that for some nondegenerate limit distribution G it is the case that $\lim_{n\to\infty}[Pr((M_n - b_n)/a_n \leq x)] = G(x)$ for all real x, then G is one of the following types: __________ [Give name and equation for each type.]

Solution S7-EVT-8. Suppose $X_1,\ldots, X_n$ are iid with df F and (a_n), (b_n) are constants such that for some nondegenerate limit distribution G it is the case that $\lim_{n\to\infty}[Pr((M_n - b_n)/a_n \leq x)] = G(x)$ for all real x, then G is one of the following types:

Type I: Fréchet:
$\Phi_\alpha(x) = 0$ if $x \leq 0$;
$\Phi_\alpha(x) = \exp(-x^{-\alpha})$ if $x > 0$, such that $\alpha > 0$.

<u>Type II: Weibull</u>:
$\Psi_\alpha(x) = \exp(-(-x^{-\alpha}))$ if $x \leq 0$, such that $\alpha > 0$;
$\Psi_\alpha(x) = 1$ if $x > 0$.

<u>Type III: Gumbel</u>:
$\Lambda(x) = \exp(-e^{-x})$ for all real x.

Problem S7-EVT-9.
(a) Using the definition in Embrechts et al. (p. 33), what does it mean if distribution G is of type H?
(b) What is another name given to a type H distribution that is of one of the three types in Solution S7-EVT-8?
(c) What is the general formula for the kind of distribution in part (b)? (Embrechts et al., p. 33).

Solution S7-EVT-9.
(a) G is of type H is for some a > 0 and real value b, **G(x) = H[(x-b)/a]**, for all real x.
(b) A type H distribution that is of one of the three types in Solution S7-EVT-8 is known as an *extreme-value distribution*.
(c) $H_{\xi;\mu,\sigma}(x) = \exp[-(1 + \xi*(x-\mu)/\sigma)_+^{-1/\xi}]$, for real values of x, ξ, and μ and for σ > 0. (Embrechts et al., pp. 33-34).

Problem S7-EVT-10.
(a) With regard to the general formula for an extreme-value distribution in Solution S7-EVT-9, suppose that ξ > 0. To which of the three types of extreme-value distributions does this correspond?
(b) Now suppose that ξ < 0. To which of the three types of extreme-value distributions does this correspond?
(c) Now suppose that ξ = 0. To which of the three types of extreme-value distributions does this correspond?

Solution S7-EVT-10.
(a) If ξ > 0, the distribution is **Type I: Fréchet**.
(b) If ξ < 0, the distribution is **Type II: Weibull**.
(c) If ξ = 0, the distribution is **Type III: Gumbel**.

Problem S7-EVT-11. For each of the three types of extreme-value distributions, what can be said regarding their bounds, if any? (Embrechts et al., p. 34)

Solution S7-EVT-11.
Type I: Fréchet distributions have a **finite lower bound**.
Type II: Weibull distributions have a **finite upper bound**.
Type III: Gumbel distributions are **unbounded on both sides**.

Problem S7-EVT-12. According to Embrechts et al. (p. 34), which of the three extreme-value distributions is most applicable to insurance, reinsurance, and finance?

Solution S7-EVT-12. Type I: Fréchet distributions are most applicable to insurance, reinsurance, and finance.

Problem S7-EVT-13. (a) For a general df F, what is definition given by Embrechts et al. (p. 34) for the *inverse* of F, $F^{\leftarrow}(t)$, for $0 < t < 1$? **Note:** inf({x}) is the greatest lower bound of a set of numbers x.
(b) What is the definition of x_p, the p-quantile of F, for $0 < p < 1$?

Solution S7-EVT-13.
(a) $\mathbf{F^{\leftarrow}(t) = \inf(\{x \in R \mid F(x) \geq t\})}$, for $0 < t < 1$.
(b) $\mathbf{x_p = F^{\leftarrow}(p)}$, for $0 < p < 1$.

Problem S7-EVT-14. Complete Theorem 3 of Embrechts et al. (p. 34) by filling in the blanks: Suppose $X_1, \ldots, X_n$ are iid with df F satisfying $\lim_{t\to\infty}(1-F(tx))/(1-F(t)) = x^{-\alpha}$, for $x > 0$ and $\alpha > 0$. Then for $x > 0$, ________ = _____, where b_n = ______ and a_n = ______.

Solution S7-EVT-14.
Suppose $X_1, \ldots, X_n$ are iid with df F satisfying $\lim_{t\to\infty}(1-F(tx))/(1-F(t)) = x^{-\alpha}$, for $x > 0$ and $\alpha > 0$. Then for $x > 0$, $\mathbf{\lim_{n\to\infty}[Pr((M_n-b_n)/a_n) \leq x] = \Phi_\alpha(x)}$, where $b_n = \mathbf{0}$ and $a_n = \mathbf{F^{\leftarrow}(1 - 1/n)}$.

Problem S7-EVT-15.
(a) In their example on p. 34 of the application of extreme-value theory to the pricing of an excess reinsurance treaty, what formula is used to determine the attachment point u_t of the treaty as a t-year event corresponding to a specific claim event with claim-size distribution function F?
(b) What is u_t expressed as a quantile?
(c) Fill in the blanks regarding Theorem 4 of Embrechts et al. Suppose that $X_1, \ldots, X_n$ are iid with df F and $F_u(x) = Pr(X - u \leq x \mid X > u)$. For a large u, $F_u(x)$ follows the generalized Pareto distribution $G_{\xi,\beta}(x)$ = ________ for β _____.

Solution S7-EVT-15.
(a) $\mathbf{u_t = F^{\leftarrow}(1 - 1/t)}$.
(b) $\mathbf{u_t = x_{1-1/t}}$. This is the (1-1/t)th quantile of F.
(c) For a large u, $F_u(x)$ follows the generalized Pareto distribution
$G_{\xi,\beta}(x) = \mathbf{1 - (1 + \xi x/\beta)^{-1/\xi}}$, for $\beta \mathbf{> 0}$.

Problem S7-EVT-16.
(a) According to Embrechts et al. (p. 39), estimating attachment points and retentions in reinsurance is similar to using *what* measure in finance?
(b) What is a shortcoming of this measure?
(c) What measure is an improvement and how?

Solution S7-EVT-16.
(a) Estimating attachment points and retentions in reinsurance is similar to using **VaR** (value at risk) in finance.
(b) VaR is not coherent and lacks subadditivity, creating inconsistencies in the risk capital determination.
(c) **TVaR** (tail value at risk = conditional VaR = mean excess) is an improvement and is coherent.