Study Guide on Dependency Modeling for the Casualty Actuarial Society (CAS) Exam 7

(Based on Sholom Feldblum's Paper, *Dependency Modeling*)

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Source: Feldblum, S., "Dependency Modeling," CAS Study Note, September 2010.

This is an open-source study guide and may be revised pursuant to suggestions.

S7-DM-1. (a) What *kinds of distributions* did early analyses of multiple random variables emphasize? **(b)** Why? (Feldblum, p. 1)

Solution S7-DM-1. (a) Multivariate normal distributions were emphasized, **(b)** because pricing focuses on means of distributions, and multivariate normal distributions reasonably approximate the means in large samples with independent, similarly-sized random variables.

S7-DM-2. What is a fundamental qualitative difference between (i) actuarial pricing studies that use regression analysis and generalized linear models (GLMs) and (ii) analysis using copulas? (Feldblum, p. 2)

Solution S7-DM-2. Actuarial studies using regression analysis and GLMs posit causal relationships among variables. Copulas do not imply causality and instead study observed relations and their effects on the distribution of outcomes.

S7-DM-3. What kind of risk do Enterprise Risk Management (ERM) analyses consider? What can the actuary *not* choose when doing ERM analysis? (Feldblum, p. 2)

Solution S7-DM-3. ERM analyses consider overall risk to the insurer from many sources and consider the tails of distributions. The actuary cannot choose marginal distributions or select the orthogonal components.

S7-DM-4. (a) What was the approach of early Dynamic Financial Analysis (DFA) models to multivariate distributions? **(b)** What are three drawbacks to this approach? (Feldblum, p. 2)

Solution S7-DM-4.
(a) Early DFA models used separate multivariate distributions for each combination of risks.
(b) Drawbacks of early DFA models:
1. Each multivariate distribution was specific to the risks joined, and statistical tools for joining distributions without specifying the types of distributions were not yet available.
2. Multivariate distributions that join marginal distributions of the same type are easier to work with than multivariate distributions joining dissimilar marginal distributions.
3. There are complex risk dependencies; two risks can be independent near the mean but dependent near the tails. Early DFA models failed to capture this.

S7-DM-5. Why are insurance applications of ERM analysis more complex than a multivariate Normal distribution? (Feldblum, p. 3)

Solution S7-DM-5. Many insurance risks have skewed distributions with thick tails, and ERM combines risks with different distributions. Also, in adverse scenarios, correlations among risks may be stronger than in normal situations.

S7-DM-6.
Fill in the blanks: Copulas join _______ functions into _________ functions. (Feldblum, p. 3)

Solution S7-DM-6 . Copulas join **marginal distribution** functions into **multivariate distribution** functions.

S7-DM-7. (Problem based off of the Illustration in Feldblum, p. 3.) Let Y and Z be marginal distributions with y_0 and z_0 such that $F(y_0) = 1/2$ and $G(z_0) = 1/2$. Let H(y, z) be the multivariate distribution of Y and Z.

(a) If Y and Z are independent, what is the numerical value of $H(y_0, z_0)$?
(b) If Y and Z are perfectly positively correlated, then what are the constraints on Z when $Y \leq y_0$?
(c) If Y and Z are perfectly positively correlated, what is the numerical value of $H(y_0, z_0)$?
(d) If Y and Z are perfectly negatively correlated, then what are the constraints on Z when $Y \leq y_0$?
(e) If Y and Z are perfectly negatively correlated, what is the numerical value of $H(y_0, z_0)$?

Solution S7-DM-7.

(a) If Y and Z are independent, $H(y_0, z_0) = (1/2)*(1/2) =$ **¼.**
(b) If Y and Z are perfectly positively correlated, then when $Y \leq y_0$, $\mathbf{Z \leq z_0}$.
(c) If Y and Z are perfectly positively correlated, $H(y_0, z_0) =$ **½.**
(d) If Y and Z are perfectly negatively correlated, then when $Y \leq y_0$, $\mathbf{Z > z_0}$.
(e) If Y and Z are perfectly negatively correlated, $H(y_0, z_0) =$ **0**.

S7-DM-8. Given cumulative distribution functions F(y) and G(z), give the expression for joining these functions via a copula. (Feldblum, p. 3)

Solution S7-DM-8. The expression is **H(y, z) = C[F(y), G(z)]**.

S7-DM-9. Fill in the blanks with regard to copulas (Feldblum, p. 4):

(a) Each ______ distribution is mapped into its _______ function.
(b) Probabilities from ______ functions follow a _________ distribution on the interval _____.
(c) Copulas join ______ (range of numbers) ________ (type of distribution) _______ (interval for distribution) distributions into a __________ distribution.
(d) The ________ distribution (from part (c)) maps ________ (interval) into _______ (interval).

Solution S7-DM-9.

(a) Each **marginal** distribution is mapped into its **cumulative distribution** function (CDF).
(b) Probabilities from **cumulative distribution** functions follow a **uniform** distribution on the interval **[0, 1]**.
(c) Copulas join **$n \geq 2$ uniform [0, 1]** distributions into a **multivariate** distribution.
(d) The **multivariate** distribution maps **$[0, 1]^n$** into **[0, 1]**.

S7-DM-10. (a) Feldblum (p. 4) distinguishes the *CDF of a distribution* from the *distribution of a CDF.* Explain the distinction.

Solution S7-DM-10. The CDF of a distribution of random variable Y is F(y) and can be any of a variety of functions. But F(y) is itself uniformly distributed on the interval [0, 1].

If you graph F(y) on a standard coordinate plane, the points along the horizontal axis will reflect the CDF of the distribution, while the distribution of the CDF will be uniform from 0 to 1 and will be reflected on the vertical axis.

S7-DM-11. You are given the following data regarding average frequency and severity of losses for various groups of insureds.

Group	**Frequency**	**Severity**
A	1%	$6,000
B	3%	$3,000
C	2%	$1,000
D	4%	$8,000
E	6%	$5,000
F	5%	$4,000

(a) Calculate the Pearson product-moment correlation ρ for frequency and severity.

(b) Calculate the Kendall τ statistic.

(c) Calculate Spearman's rank correlation.

Solution S7-DM-11.
(a) ρ = [N*∑(x*y) – (∑x)*(∑y)]/[√(N∑(x^2) - ∑$(x)^2$)*√(N∑(y^2) - ∑$(y)^2$)]
(This superior formula can be found at MathBits.com.)

Let x = frequency; let y = severity. Here, N = 6 = number of observations.
N*∑(x*y) = 6*(1%*6000 + 3%*3000 + 2%*1000 + 4%*8000 + 6%*5000 + 5%*4000) = 5940.
(∑x)*(∑y) = 21%*27000 = 5670.
∑(x^2) = 0.0091.
√(N∑(x^2) - ∑$(x)^2$) = √(6*0.0091 – 0.21^2) = 0.1024695077.
∑(y^2) = 151,000,000.
√(N∑(y^2) - ∑$(y)^2$) = √(6*151,000,000 - 27000^2) = 13304.1347.
ρ = (5940 – 5670)/(0.1024695077*13304.1347) = **ρ = 0.1980534817**.

(b)
Formula: τ = 1 – (4Q/(N*(N-1)), Q is the number of swaps, N is the number of elements (here N = 6).
We find the number of swaps:
Frequency Ranking (high to low): E, F, D, B, C, A
Severity Ranking (high to low): D, A, E, F, B, C
Swap 1 (from Frequency Ranking): E, D, F, B, C, A
Swap 2: D, E, F, B, C, A
Swap 3: D, E, F, B, A, C
Swap 4: D, E, F, A, B, C
Swap 5: D, E, A, F, B, C
Swap 6: D, A, E, F, B, C → Severity Ranking.
Thus, Q = 6.
Thus, τ = 1 – (4*6/(6*5)) = **τ = 0.2**.

Solution c. We find the frequency rank and severity rank of each group:

Group	**Frequency**	**Severity**	**Frequency Rank**	**Severity Rank**	**d_j (Freq. Rank – Sev. Rank)**	**d_j^2**
E	6%	$5,000	1	3	-2	4
F	5%	$4,000	2	4	-2	4
D	4%	$8,000	3	1	2	4
B	3%	$3,000	4	5	-1	1
C	2%	$1,000	5	6	-1	1
A	1%	$6,000	6	2	4	16

We find ∑(d_j^2) = 30.
The Spearman rank correlation is ρ = 1 - ∑(d_j^2)/(N(N^2-1)/6) = 1 – (30/(6*35/6))
= 1 – 30/35 = **ρ = 1/7 = 0.1428571429**.

S7-DM-12. (a) Is the Pearson product-moment correlation a cardinal or an ordinal measure?
(b) For what types of distributions does the Pearson product-moment correlation work well?
(c) For what types of distributions does the Pearson product-moment correlation not work well? Why?
(Feldblum, p. 5)

Solution S7-DM-12.

(a) The Pearson product-moment correlation is a **cardinal** measure.
(b) The Pearson product-moment correlation works well for **thin-tailed, elliptical distributions**. Examples are the Normal and student's t distributions.
(c) The Pearson product-moment correlation does not work well for **skewed, thick-tailed distributions**, because **one outlier may distort the correlation.**

S7-DM-13. Fill in the blanks: ERM analyses with skewed distributions seek a(n) ______ (cardinal or ordinal?) measure of dependency that focuses on the _______. (Feldblum, p. 5)

Solution S7-DM-13. ERM analyses with skewed distributions seek an **ordinal** measure of dependency that focuses on the **tails.**

S7-DM-14.
(a) Is the Pearson product-moment correlation invariant under linear transformations?
(b) Is the Pearson product-moment correlation invariant under *all* monotonic transformations?
(c) Are copulas invariant under all monotonic transformations? Why or why not? (Feldblum, p. 5)

Solution S7-DM-14.
(a) Yes, the Pearson product-moment correlation is invariant under linear transformations.
(b) No, the Pearson product-moment correlation is not invariant under all monotonic transformations.
(c) Yes, copulas are invariant under all monotonic transformations, because they are dependencies based on ranks (ordinal functions) and so maintain invariance under monotone transformations.

S7-DM-15. (a) Is the Kendall τ statistic a cardinal or an ordinal measure?
(b) Is Spearman's ρ rank correlation a cardinal or an ordinal measure? (Feldblum, p. 6)

Solution S7-DM-15.
(a) The Kendall τ statistic is an **ordinal** measure.
(b) Spearman's ρ rank correlation is an **ordinal** measure.

S7-DM-16. What are three ways in which the Pearson product-moment correlation is restrictive? (Feldblum, p. 8)

Solution S7-DM-16.
1. The Pearson correlation summarizes dependence in a single number, whereas actual dependence may vary along the distribution.
2. The Pearson correlation depends on high leverage points that are far from the mean of the distribution (since it is proportional to the squared residual of each value from the mean).
3. The Pearson correlation depends on numerical values and not just the relation between the distributions. (It is cardinal rather than ordinal.)

S7-DM-17. Fill in the blanks: "Scatterplots of multivariate distributions portray the _______ distributions along with the ________ among the distributions." (Feldblum, p. 8)

Solution S7-DM-17. Scatterplots of multivariate distributions portray the **marginal** distributions along with the **dependency** among the distributions.

S7-DM-18. On page 9, Feldblum's Figure 3 gives three copulas, all with an 80% Pearson correlation or Kendall τ statistic: (a) Normal, (b) Frank, and (c) *t*.
(i) For which of these copulas could it be most accurately said that the dependency is strong in the tails and weak in the center? Why?
(ii) For which of these copulas is the dependency at the center relatively close to the dependency in the tails? Why?

Solution S7-DM-18.
(i) The **(c) *t*** copula exhibits strong dependency in the tails and weak dependency in the center. The points on the scatterplot are highly concentrated near the extremes of the x-axis and increasingly scattered near the center of the x-axis.
(ii) The **(b) Frank** copula exhibits dependencies in the center that are most similar to the dependencies at the tails. Of the three copulas, the points on the Frank copula are most closely concentrated around the 45-degree line throughout the scatterplot.

S7-DM-19. On page 9, Feldblum discusses two weaknesses of scatterplots. What are they?

Solution S7-DM-19. Two weaknesses of scatterplots are
(1) Vulnerability of human eyes to seeing patterns where none exist;
(2) Ease of confusing patterns in multivariate distributions with dependencies (analogous to seeing constellations in a night sky).

S7-DM-20. In very simple terms, what is a χ-plot? (Feldblum, p. 10)
Solution S7-DM-20. A χ-plot is a non-parametric graphical display of the dependency between two variables.

S7-DM-21. A χ-plot is constructed from a scatterplot. However, there s sometimes not a one-to-one correspondence between points on a scatterplot and points on a χ-plot. Fill in the blank: "At most __ (how many?) points in the scatterplot will not have an associated point in the χ-plot." (Feldblum, p. 10)

Solution S7-DM-21. At most **4** points in the scatterplot will not have an associated point in the χ-plot.

Formula (1): Formula for calculating χ-plot coordinates from scatterplot coordinates:

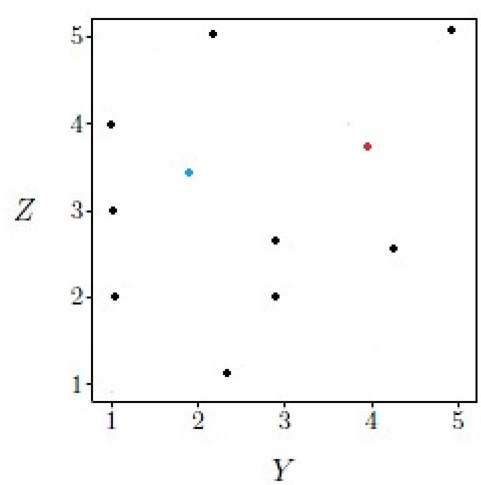
$\chi_i = (H_i - F_i*G_i)/\sqrt{(F_i*(1-F_i)*G_i*(1-G_i))}$.

Definitions of variables:
i = point on scatterplot
χ_i = corresponding point on χ-plot
H_i, F_i, G_i = empirical estimates of the bivariate and marginal distribution functions relative to the ith point.

S7-DM-22.
(a) With regard to Formula (1), how else could the terms H_i and F_i*G_i in the numerator be interpreted?
(b) What does it mean if $H_i = F_i*G_i$?
(c) What does it mean if $H_i < F_i*G_i$ or $H_i > F_i*G_i$? (Feldblum, p. 10)
(d) What is the function of the denominator in Formula (1)?

Solution S7-DM-22.
(a) H_i is the empirical bivariate distribution, and F_i*G_i is the null hypothesis of independence.
(b) $H_i = F_i*G_i$ means that there is independence, since the empirical bivariate distribution is the same as the null hypothesis of independence.
(c) If $H_i < F_i*G_i$ or $H_i > F_i*G_i$, then there is dependence between the variables, because the empirical bivariate distribution does not correspond to the null hypothesis of independence.
(d) The denominator in Formula (1) serves as a **scaling factor** to ensure that the value of χ_i remains in the range [-1, 1].

S7-DM-23. Suppose you have the following scatterplot:



(a) For the blue point, find the values of H_i, F_i and G_i. (See Feldblum, p. 10)
(b) For the red point, find the values of H_i, F_i and G_i. (See Feldblum, p. 10)
(c) Find the χ-plot values of the red and blue point.

Solution S7-DM-23.

Draw a vertical line and a horizontal line passing through the blue point.

Label the quadrants as follows:
Northeast = 1
Northwest = 2
Southwest = 3
Southeast = 4

H_i is the fraction of points in the third quadrant.
F_i is the fraction of points in the second *and* third quadrants.
G_i is the fraction of points in the third *and* fourth quadrants.

There are 11 points in all, including the blue point -- 10 excluding the blue point. The distribution by quadrant relative to the blue point is as follows:
Quadrant 2: 1
Quadrant 3: 2
Quadrant 4: 4
Thus, $\mathbf{H_i}$ **=** 2/10 = **1/5**, $\mathbf{F_i}$ **=** (1+2)/10 = **3/10**, and $\mathbf{G_i}$ **=** (2+4)/10 **= 3/5**.

(b) Draw a vertical line and a horizontal line passing through the red point.

Label the quadrants as follows:
Northeast = 1
Northwest = 2
Southwest = 3
Southeast = 4

H_i is the fraction of points in the third quadrant.
F_i is the fraction of points in the second *and* third quadrants.
G_i is the fraction of points in the third *and* fourth quadrants.

There are 11 points in all, including the red point -- 10 excluding the red point. The distribution by quadrant relative to the red point is as follows:
Quadrant 2: 2
Quadrant 3: 6
Quadrant 4: 1
Thus, $\mathbf{H_i}$ **=** 6/10 = **3/5**, $\mathbf{F_i}$ **=** (6+2)/10 = **4/5**, and $\mathbf{G_i}$ **=** (6+1)/10 = **7/10**.

(c) We use the formula $\chi_i = (H_i - F_i*G_i)/\sqrt{}(F_i*(1-F_i)*G_i*(1-G_i))$.

For the red point, χ_{blue} = (1/5 - (3/10)*(3/5))/√((3/10)*(1-(3/10))*(3/5)*(1-(3/5))) = $\boldsymbol{\chi_{blue}}$ **= 0.0890870806**.

For the blue point, χ_{red} = (3/5 - (4/5)*(7/10))/√((4/5)*(1-(4/5))*(7/10)*(1-(7/10))) = $\boldsymbol{\chi_{red}}$ **= 0.2182178902**.

Formula (2): Formula for calculating λ coordinates on χ-plots :

$\lambda_i = 4*\text{sgn}(\dot{F}_i * \bar{G}_i) * \max(\dot{F}_i^2, \bar{G}_i^2)$

Definitions of variables:
$\dot{F}_i = F_i - 1/2$.
$\bar{G}_i = G_i - 1/2$.
sgn(x) = 1 if x is positive or zero, -1 if x is negative. For the point i, the sign is positive if i is in Quadrants 1 or 3, and negative if i is in Quadrants 2 or 4.

S7-DM-24. **(a)** Calculate λ_{red} for the red point in the figure for Problem S7-DM-23.

(b) Calculate λ_{blue} for the blue point in the figure for Problem S7-DM-23.

Solution S7-DM-24.

(a) We find $\dot{F}_{red} = F_{red} - 1/2 = 4/5 - 1/2 = 0.3$

We find $\bar{G}_{red} = G_{red} - 1/2 = 7/10 - 1/2 = 0.2$

Clearly, $\text{sgn}(\dot{F}_{red} * \bar{G}_{red}) = 1$ (sign is positive).

$\max(\dot{F}_{red}^2, \bar{G}_{red}^2) = 0.3^2 = 0.09$.
Thus, $\lambda_{red} = 4*0.09 =$ $\boldsymbol{\lambda_{red} = 0.36}$.

(b) We find $\dot{F}_{blue} = F_{blue} - 1/2 = 3/10 - 1/2 = -0.2$

We find $\bar{G}_{blue} = G_{blue} - 1/2 = 3/5 - 1/2 = 0.1$

Clearly, $\text{sgn}(\dot{F}_{red} * \bar{G}_{red}) = -1$ (sign is negative).

$\max(\dot{F}_{blue}^2, \bar{G}_{blue}^2) = (-0.2)^2 = 0.04$.

Thus, $\lambda_{blue} = 4*0.04 =$ $\boldsymbol{\lambda_{blue} = 0.16}$.

S7-DM-25. On a χ-plot, what would be a sign that any given random variables in question, Y and Z, are independent? (See Feldblum, p. 11)

Solution S7-DM-25. A sign that Y and Z are independent would be that all the points are close to the horizontal line χ = 0.

S7-DM-26. On page 12, Feldblum states that "Some ERM analyses use regime switching models, with stable vs. volatile states." Elaborate on this statement by discussing when stable states would be used and when volatile states would be used.

Solution S7-DM-26. Stable states would be used in "normal" situations, with high stock returns, low volatility, and low intra-correlations among and inter-correlations between stocks and bonds. Volatile states would be used in adverse scenarios with low stock returns, high volatility, and high intra-correlations and inter-correlations of the sort described above.

S7-DM-27. On page 12, Feldblum states that a bivariate distribution H(y, z) has three elements. What are they?

Solution S7-DM-27.

(1) The marginal distribution F(y)

(2) The marginal distribution G(z)

(3) The dependency of F(y) and G(z)

S7-DM-28. Suppose that the distribution functions F(y) and G(z) have a uniform distribution on the interval [0, 1] and that copulas combine F(y) and G(z) into a joint distribution H(y, z).

What is the range of H, and what is the domain of H? (Give intervals as answers.) (Feldblum, p. 13)

Solution S7-DM-28. The range of H is **[0, 1]**, and the domain of H is $\mathbf{[0, 1]^2}$.

S7-DM-29. What are the three properties of a continuous one-dimensional distribution function? Assume that F is the distribution function, y_a is the minimum value of the distribution, and y_b is the maximum value of the distribution. Let y_c and y_d be any two other points, such that $y_c \leq y_d$. (Feldblum, p. 13)

Solution S7-DM-29.

(1) $F(y_a) = \Pr(Y \leq y_a) = 0$.
(2) $F(y_b) = \Pr(Y \leq y_b) = 1$.
(3) $y_c \leq y_d \rightarrow F(y_c) \leq F(y_d)$.

S7-DM-30. What are the four properties of a two-dimensional copula? Let H(u, v) be the joint probability distribution function, and let U and V be uniformly distributed random variables on the interval [0, 1].

Solution S7-DM-30.

(1) H(0, v) = 0 for all v.
(2) H(u, 0) = 0 for all u.

(3) H(1, v) = v for all v
(4) H(u, 1) = u for all u.

(Note that Feldblum's solution has a typo for (4) !!!)

S7-DM-31. For a bivariate distribution H(u, v), give a mathematical expression equivalent to the statement that *first differences are positive.* (Feldblum, p. 13)

Solution S7-DM-31.

If $u_0 < u_1$, then $H(u_0, v) \leq H(u_1, v)$ for all v.

If $v_0 < v_1$, then $H(u, v_0) \leq H(u, v_1)$ for all u.

S7-DM-32. For a bivariate distribution H(u, v), give a mathematical expression equivalent to the statement that *cross differences are positive.* (Feldblum, p. 13)

Solution S7-DM-32. If $u_0 < u_1$, and $v_0 < v_1$, then $\partial^2 H(u, v)/(\partial u * \partial v) \geq 0$.
(Second partial derivative with respect to u and v is 0 or positive.)

S7-DM-33. Fill in the two requirements of a copula (Feldblum, p. 14):

A 2-dimensional copula is a function C from $[0, 1]^2$ to [0, 1] such that

(1) ________, and
(2) ________

Solution S7-DM-33.

(1) For every (u, v) in $[0, 1]^2$, C(u, 0) = C(0, v) = 0, C(u, 1) = u, and C(1, v) = v.
(2) For every (u_0, v_0) and (u_1, v_1) in $[0, 1]^2$ such that $u_0 \leq u_1$ and $v_0 \leq v_1$, we have

$C(u_1, v_1) - C(u_0, v_1) - C(u_1, v_0) + C(u_0, v_0) \geq 0$.

S7-DM-34. For the "rectangle inequality" pertaining to copulas, express the meaning of the inequality with regard to probabilities. (Feldblum, p. 14).

Solution S7-DM-34. The "rectangle inequality" states that the probability of a point falling within the rectangle described by the vertices (u_0, v_0), (u_0, v_1), (u_1, v_0), and (u_1, v_1) is non-negative.

S7-DM-35. (a) Let H(u, v) = 2uv. Is H(u, v) a copula? Why or why not?

(b) Is ∏(u, v) = uv a copula? Why or why not?

Solution S7-DM-35. (a) H(u, v) = 2uv is **not** a copula, because H(u, 1) = 2*u*1 = 2u. In order for H to be a copula, it would need to be the case that H(u, 1) = u.

(b) ∏(u, v) = uv is a copula.

It satisfies property (1):

$\prod(u, 0) = \prod(0, v) = 0$ and $\prod(u, 1) = u$ and $\prod(1, v) = v$.

It satisfies property (2):

$\prod(u_1, v_1) - \prod(u_0, v_1) - \prod(u_1, v_0) + \prod(u_0, v_0) = u_1 * v_1 - u_0 * v_1 - u_1 * v_0 + u_0 * v_0 =$

$(u_1 - u_0)*(v_1 - v_0)$.

Since $u_0 \leq u_1$ and $v_0 \leq v_1$, it is the case that $(u_1 - u_0) \geq 0$ and $(v_1 - v_0) \geq 0$, so

$(u_1 - u_0)*(v_1 - v_0) \geq 0$.

S7-DM-36. Give the formula for the **additive/maximum copula** and show that it is a copula.

Solution S7-DM-36. The additive/maximum copula is somewhat unusually named, as the formula involves a minimum: **M(u, v) = min(u, v).**

It satisfies property (1):

M(u, 0) = min(u, 0) = M(0, v) = min(0, v) = 0.

M(u, 1) = u (since u ≤ 1), and M(1, v) = v (since v ≤ 1).

It satisfies property (2):

$M(u_1, v_1) - M(u_0, v_1) - M(u_1, v_0) + M(u_0, v_0) =$

$\min(u_1, v_1) - \min(u_0, v_1) - \min(u_1, v_0) + \min(u_0, v_0)$

Assume, without loss of generality, that $u_0 \leq v_0$.

Then $\min(u_0, v_0) = u_0$ and $\min(u_0, v_1) = u_0$. Thus, $\min(u_1, v_1) - \min(u_0, v_1) - \min(u_1, v_0) + \min(u_0, v_0)$ $= \min(u_1, v_1) - \min(u_1, v_0) - 0 + 0 = \min(u_1, v_1) - \min(u_1, v_0)$.

Either $u_1 \leq v_0$ or $u_1 > v_0$.

If $u_1 \leq v_0$, then $\min(u_1, v_1) - \min(u_1, v_0) = u_1 - u_1 = 0$ and the inequality holds.

If $u_1 > v_0$, then $\min(u_1, v_1) - \min(u_1, v_0) = \min(u_1, v_1) - v_0 \geq 0$, since both u_1 and v_1 are greater than or equal to v_0 by this assumption. In all cases, we have shown that the "rectangle inequality" holds.
Q.E.D.

S7-DM-37. What does the maximum copula assume with regard to correlation between the marginal distribution functions? (Feldblum, p. 16.)

Solution S7-DM-37. The maximum copula assumes **perfect correlation** between the marginal distribution functions.

S7-DM-38. Give the formula for the **minimum copula** and show that it is a copula.

Solution S7-DM-38. Again we have a copula that is unusually named. The formula for the minimum copula is **W(u, v) = max(0, u + v – 1)**.

It satisfies property (1):

Without loss of generality:

W(u, 0) = max(0, u + 0 – 1) = 0, since u ≤ 1.

W(u, 1) = max(0, u + 1 – 1) = max(0, u) = u.

It satisfies property (2):

$W(u_1, v_1) - W(u_0, v_1) - W(u_1, v_0) + W(u_0, v_0) =$

$\max(0, u_1 + v_1 - 1) - \max(0, u_0 + v_1 - 1) - \max(0, u_1 + v_0 - 1) + \max(0, u_0 + v_0 - 1)$

There are five possibilities:

(i) $u_1 + v_1 \leq 1$

(ii) $u_1 + v_1 > 1$ and $u_0 + v_1 \leq 1$ and $u_1 + v_0 \leq 1$

(iii) $u_1 + v_1 > 1$ and $u_0 + v_1 > 1$ and $u_1 + v_0 \leq 1$

(iv) $u_1 + v_1 > 1$ and $u_0 + v_1 > 1$ and $u_1 + v_0 > 1$ and $u_0 + v_0 \leq 1$

(v) $u_0 + v_0 \leq 1$

Scenario (i):

$\max(0, u_1 + v_1 - 1) - \max(0, u_0 + v_1 - 1) - \max(0, u_1 + v_0 - 1) + \max(0, u_0 + v_0 - 1) =$

$0 - 0 - 0 + 0 = 0$. The rectangle inequality holds.

Scenario (ii):

$\max(0, u_1 + v_1 - 1) - \max(0, u_0 + v_1 - 1) - \max(0, u_1 + v_0 - 1) + \max(0, u_0 + v_0 - 1) =$

$u_1 + v_1 - 1 - 0 - 0 + 0 = u_1 + v_1 - 1 > 0$. The rectangle inequality holds.

Scenario (iii):

$\max(0, u_1 + v_1 - 1) - \max(0, u_0 + v_1 - 1) - \max(0, u_1 + v_0 - 1) + \max(0, u_0 + v_0 - 1) =$

$u_1 + v_1 - 1 - (u_0 + v_1 - 1) - 0 + 0 = u_1 - u_0 \geq 0$. The rectangle inequality holds.

Scenario (iv):

$\max(0, u_1 + v_1 - 1) - \max(0, u_0 + v_1 - 1) - \max(0, u_1 + v_0 - 1) + \max(0, u_0 + v_0 - 1) =$

$u_1 + v_1 - 1 - (u_0 + v_1 - 1) - (u_1 + v_0 - 1) + 0 = 1 - u_0 - v_0 = 1 - (u_0 + v_0) \geq 0$. The rectangle inequality holds.

Scenario (v):

$\max(0, u_1 + v_1 - 1) - \max(0, u_0 + v_1 - 1) - \max(0, u_1 + v_0 - 1) + \max(0, u_0 + v_0 - 1) =$

$u_1 + v_1 - 1 - (u_0 + v_1 - 1) - (u_1 + v_0 - 1) + (u_0 + v_0 - 1) = 0$. (All the terms cancel out.)

The rectangle inequality holds.

We have shown that the rectangle inequality holds in all possible scenarios. **Q. E. D.**

S7-DM-39. What does the minimum copula assume with regard to correlation between the marginal distribution functions? (Feldblum, p. 16.)

Solution S7-DM-39. The minimum copula assumes **perfect negative correlation** between the marginal distribution functions.

S7-DM-40. Express the inequality corresponding to the Fréchet-Hoeffding bounds with regard to copulas. Define all notation. (Feldblum, p. 18.)

Solution S7-DM-40. The inequality is **W(u, v) ≤ C(u, v) ≤ M(u, v)**, where

u and v are values of random variables U and V,

C(u, v) is any copula within the Fréchet-Hoeffding bounds,

W(u, v) is the minimum copula,

M(u, v) is the additive/maximum copula.

Note: This inequality sheds light on why the maximum and minimum copulas are named the way they are. The naming is in line with the copulas' places in this inequality.

S7-DM-41. Based on the descriptions below, identify the following copulas: (i) product or independence copula, (ii) minimum copula, (iii) additive/maximum copula. (Feldblum, pp. 18-19)

(a) This copula has a sloped-pyramid shape.

(b) This copula has a hyperbolic shape.

(c) This copula has a concave surface along one diagonal and a convex surface along the other.

(d) This copula is like a square that is bent along one of the diagonals.
(e) This copula rises linearly from (u, v) = (0, 0) to (u, v) = (1, 1)
(f) For half of the unit square, this copula has a value of zero.
(g) For this copula and variables u and v and marginal distribution F, F(u) = F(1-v).

Solution S7-DM-41.

(a) Maximum copula

(b) Product/independence copula

(c) Product/independence copula

(d) Additive/minimum copula

(e) Maximum copula

(f) Additive/minimum copula

(g) Additive/minimum copula

S7-DM-42. What do the curves on a contour diagram represent? (Feldblum, p. 20)

Solution S7-DM-42. Each curve on a contour diagram represents points with the same copula value in the unit square.

S7-DM-43. Match the description of curves on a contour diagram to the following copulas: (i) product or independence copula, (ii) minimum copula, (iii) additive/maximum copula.

(a) Curves are straight diagonal lines.

(b) Curves are hyperbolas of varying concavity.

(c) Curves are L-shaped.

Solution S7-DM-43.

(a) Additive/minimum copula

(b) Product/independence copula

(c) Maximum copula

S7-DM-44. Let u and v be cumulative probabilities of a standard normal distribution with mean $\mu = 0$ and variance = 1. Let ρ be the correlation of the two distributions, and let Φ_ρ be the standard bivariate normal cumulative distribution function with correlation ρ. Give the formula for $C_\rho(u, v)$, which is the bivariate normal or Gaussian copula.

Solution S7-DM-44. The formula for the bivariate normal copula is $\mathbf{C_\rho(u, v) = \Phi_\rho(\Phi^{-1}(u), \Phi^{-1}(v))}$.
Note: This page (http://en.wikipedia.org/wiki/Copula_%28statistics%29) gives a better-defined formula than Feldblum.

S7-DM-45.

(a) Describe a situation where a t-copula would be used instead of a bivariate normal copula.

(b) Describe qualitatively how a t-copula differs most from a bivariate normal copula.

(c) Describe how the difference in part (b) is affected as the number of degrees of freedom for the t-copula increases. (Feldblum, p. 21)

Solution S7-DM-45.

(a) When we do not know the volatility of the normal distributions, we can estimate the volatility from the observed data by using a t-distribution for each random variable.
(b) The biggest difference is in the tails of the copulas. The t-copula has greater tail dependency at the extreme tails.

(c) As the number of degrees of freedom increases, the t-copula more closely approaches a bivariate normal copula, so the difference is diminished.

S7-DM-46. Is the geometric mean of the product and minimum copulas

H(u, v) = √(∏(u, v)*W(u, v)) itself a copula? Why or why not? (Feldblum, p. 21)

Solution S7-DM-46.

H(u, v) = √(u*v*max(0, u+v-1)) is not a copula. It does not satisfy the rectangle inequality:

We consider the points (0.5, 0.5) and (0.8, 0.8).

H(0.8, 0.8) - H(0.5, 0.8) - H(0.8, 0.5) + H(0.5, 0.5) =

√(0.8*0.8*0.6) - √(0.5*0.8*0.3) - √(0.8*0.5*0.3) + √(0.5*0.5*0) = -0.0731429876 < 0, which is a counterexample to the hypothesis that this copula always satisfies the rectangle inequality.

Thus, H(u, v) is not a copula.

S7-DM-47.

(a) Give the formula for the Farlie-Gumbel-Morgenstern (FGM) copula.

(b) Show that the FGM copula satisfies property (1) of copulas.

Solution S7-DM-47.

(a) The formula for the FGM copula is $C_\varphi(u, v) = uv(1 + \varphi(1-u)(1-v))$, where φ is a parameter.

(b) Without loss of generality, $C_\varphi(u, 0) = u*0*(1 + \varphi(1-u)(1-0)) = 0$.

$C_\varphi(u, 1) = u*1*(1 + \varphi(1-u)(1-1)) = u*(1 + 0) = u$.

S7-DM-48. What are three ways in which non-modeled catastrophes differ from the most commonly thought-of catastrophes? (Feldblum, p. 23)

Solution S7-DM-48.

1. While non-modeled catastrophes can cause severe losses in several lines, they are not modeled separately.

2. Non-modeled catastrophes are included in the attritional losses for each line, not the catastrophe events.

3. Non-modeled catastrophes do not affect correlations among lines of business but greatly increase value at risk, tail value at risk, and expected policyholder deficit.

S7-DM-49. Give three examples of non-modeled catastrophes. (Feldblum, p. 23).

Solution S7-DM-49.

1. Severe snowstorms

2. Hot and dry summers

3. Systematic risks with regard to rising medical, wage, or social inflation

S7-DM-50. For pricing reinsurance excess-of-loss treaties, why is not feasible to rely on historical experience alone? (Feldblum, p. 24)

Solution S7-DM-50. The data are sparse, and the real value of attachment points and treaty limits changes with inflation.

S7-DM-51. (a) In pricing reinsurance excess-of-loss treaties, name a distribution that a reinsurer could fit to losses and another distribution that the reinsurer could fit to ALAE.

(b) Explain why these distributions are not perfectly correlated. (Feldblum, p. 24)

Solution S7-DM-51.

(a) The reinsurer could fit a **lognormal** distribution to losses and a **Pareto** distribution to ALAE.

(b) These distributions are not perfectly correlated because a strong defense (large ALAE) could lead to the acquittal of the insured (zero loss payment).

Feldblum Exercise 2. Which of the following is true?

1. If $C(u, v) = uv$, X is independent of Y.

2. If $C(u, v) = \min(u, v)$, $G(y) = F(x)$.

3. If $C(u, v) = \max(0, u + v - 1)$, $G(y) = -F(x)$

A. 1 and 2 only

B. 1 and 3 only

C. 2 and 3 only

D. 1, 2, and 3

E. None are true.

Solution to Feldblum Exercise 2.

A. 1 and 2 only

$C(u, v) = uv$ is the independence copula and expresses the independence of the random variables.

$C(u, v) = \min(u, v)$ is the maximum copula and expresses perfect positive correlation.

$C(u, v) = \max(0, u + v - 1)$ is the minimum copula and expresses perfect negative correlation. The correct statement for this is $G(y) = 1 - F(x)$. Since both $F(x)$ and $G(y)$ must be between 0 and 1, inclusive, one of them cannot be the negative of the other.

Feldblum Exercise 3. (Linear combination of copulas.) If C and C' are copulas and $0 \leq \lambda \leq 1$, show that $D = \lambda C + (1-\lambda)C'$ is also a copula.

Solution to Feldblum Exercise 3.

We show that property (1) is satisfied.
Since C and C' are copulas, it is the case that $C(u, 0) = 0$ and $C'(u, 0) = 0$.

Thus, $D(u, 0) = \lambda C(u, 0) + (1-\lambda)C'(u, 0) = \lambda*0 + (1-\lambda)*0 = 0$.

Since C and C' are copulas, it is the case that $C(u, 1) = u$ and $C'(u, 1) = u$.

Thus, $D(u, 1) = \lambda C(u, 1) + (1-\lambda)C'(u, 1) = \lambda*u + (1-\lambda)*u = (1-\lambda+\lambda)*u = u$.

Thus, property (1) holds.

We show that property (2), the "rectangle inequality", is satisfied.

Since C and C' are copulas, it is the case that

$A = C(u_1, v_1) - C(u_0, v_1) - C(u_1, v_0) + C(u_0, v_0) \geq 0$ and

$B = C'(u_1, v_1) - C'(u_0, v_1) - C'(u_1, v_0) + C'(u_0, v_0) \geq 0$

For D, each term on the left-hand side of the rectangle inequality is simply a linear combination of the two corresponding terms of C and C'. For instance,

$D(u_1, v_1) = \lambda*C(u_1, v_1) + (1-\lambda)*C'(u_1, v_1)$, implying that the entire left-hand side of the rectangle inequality for D is just $\lambda*A + (1-\lambda)*B$. Since $A \geq 0$ and $B \geq 0$ and $0 \leq \lambda \leq 1$, and products of nonnegative numbers are themselves nonnegative, it must be the case that $\lambda*A + (1-\lambda)*B \geq 0$, and the rectangle inequality is satisfied. **Q. E. D.**

Feldblum Exercise 4. Show that the weighted geometric mean of the maximum copula and the product copula is itself a copula. This copula is $C_\varphi(u, v) = \min(u, v)^\varphi*(uv)^{1-\varphi}$.

$C_\varphi(u, v)$ = uv1-φ if $u \leq v$ and u1-φv if $u > v$.

Solution to Feldblum Exercise 4.

We show that property (1) is satisfied.

$C_\varphi(u, 0) \rightarrow u \geq v \rightarrow C_\varphi(u, 0) = u^{1-\varphi}*0 = 0$.

$C_\varphi(u, 1) \rightarrow u \leq v \rightarrow C_\varphi(u, 1) = u*1^{1-\varphi} = u*1 = u$.

Thus, property (1) holds.

We show that property (2), the "rectangle inequality", is satisfied.

We need to show that $C_\varphi(u_1, v_1) - C_\varphi(u_0, v_1) - C_\varphi(u_1, v_0) + C_\varphi(u_0, v_0) \geq 0$.

There are five possibilities:

(i) $u_0 \geq v_1$
(ii) $u_1 \geq v_1$ and $u_0 \leq v_1$ and $u_0 \geq v_0$
(iii) $u_1 \leq v_1$ and $u_0 \leq v_1$ and $u_0 \geq v_0$
(iv) $u_1 \leq v_1$ and $u_0 \leq v_1$ and $u_0 \leq v_0$ but $u_1 \geq v_0$

(v) $v_0 \geq u_1$

Possibility (i):

$C_\varphi(u_1, v_1) - C_\varphi(u_0, v_1) - C_\varphi(u_1, v_0) + C_\varphi(u_0, v_0) =$

$u_1^{1-\varphi}v_1 - u_0^{1-\varphi}v_1 - u_1^{1-\varphi}v_0 + u_0^{1-\varphi}v_0 =$

$v_1*(u_1^{1-\varphi} - u_0^{1-\varphi}) + v_0*(u_0^{1-\varphi} - u_1^{1-\varphi}) = (v_1 - v_0)* (u_1^{1-\varphi} - u_0^{1-\varphi})$

Since, by definition, $u_0 \leq u_1$ and $v_0 \leq v_1$, the quantities $(v_1 - v_0)$and $(u_1^{1-\varphi} - u_0^{1-\varphi})$ is nonnegative, so

$C_\varphi(u_1, v_1) - C_\varphi(u_0, v_1) - C_\varphi(u_1, v_0) + C_\varphi(u_0, v_0) = (v_1 - v_0)* (u_1^{1-\varphi} - u_0^{1-\varphi}) \geq 0$.

This same reasoning can demonstrate the truth of Possibility (v), with "v" substituted for "u" and vice versa.

Possibility (ii)

$C_\varphi(u_1, v_1) - C_\varphi(u_0, v_1) - C_\varphi(u_1, v_0) + C_\varphi(u_0, v_0) =$

$u_1^{1-\varphi}v_1 - u_0v_1^{1-\varphi} - u_1^{1-\varphi}v_0 + u_0^{1-\varphi}v_0 =$

$v_0*(u_0^{1-\varphi} - u_1^{1-\varphi}) + v_1*(u_1^{1-\varphi} - u_0v_1^{-\varphi}) =$

$v_1*(u_1^{1-\varphi} - u_0v_1^{-\varphi}) - v_0*(u_1^{1-\varphi} - u_0^{1-\varphi})$.

It is given that $v_0 \leq v_1$.

Also, since $u_0 \leq v_1$, and $\varphi \geq 0$, $v_1^{-\varphi} \leq u_0^{-\varphi}$. Thus, $u_0*v_1^{-\varphi} \leq u_0*u_0^{-\varphi} \rightarrow u_0v_1^{-\varphi} \leq u_0^{1-\varphi}$.

Therefore, $(u_1^{1-\varphi} - u_0v_1^{-\varphi}) \geq (u_1^{1-\varphi} - u_0^{1-\varphi})$.

Since $v_1 \geq v_0$ and $(u_1^{1-\varphi} - u_0v_1^{-\varphi}) \geq (u_1^{1-\varphi} - u_0^{1-\varphi})$, it follows that

$v_1*(u_1^{1-\varphi} - u_0v_1^{-\varphi}) \geq v_0*(u_1^{1-\varphi} - u_0^{1-\varphi})$ and

$v_1*(u_1 1\text{-}\varphi - u_0v_1\text{-}\varphi) - v_0*(u_1 1\text{-}\varphi - u_0 1\text{-}\varphi) \geq 0$. The inequality holds for Possibility (ii).

Possibility (iii)

$C_\varphi(u_1, v_1) - C_\varphi(u_0, v_1) - C_\varphi(u_1, v_0) + C_\varphi(u_0, v_0) =$

$u_1v_1^{1-\varphi} – u_0v_1^{1-\varphi} – u_1^{1-\varphi}v_0 + u_0^{1-\varphi}v_0 =$

$u_1*(v_1^{1-\varphi} – v_0*u_1^{-\varphi}) – u_0*(v_1^{1-\varphi} - v_0*u_0^{-\varphi})$.

It is given that $u_0 \le u_1$. From this and the fact that $\varphi \ge 0$, it can be seen that $u_1^{-\varphi} \le u_0^{-\varphi}$ and $v_0*u_1^{-\varphi} \le v_0* u_0^{-\varphi}$ and thus

$(v_1^{1-\varphi} – v_0*u_1^{-\varphi}) \ge (v_1^{1-\varphi} - v_0*u_0^{-\varphi})$.

Since $u_1 \ge u_0$ and $(v_1^{1-\varphi} – v_0*u_1^{-\varphi}) \ge (v_1^{1-\varphi} - v_0*u_0^{-\varphi})$, it follows that

$u_1*(v_1^{1-\varphi} – v_0*u_1^{-\varphi}) \ge u_0*(v_1^{1-\varphi} - v_0*u_0^{-\varphi})$ and

$u_1*(v_1^{1-\varphi} – v_0*u_1^{-\varphi}) – u_0*(v_1^{1-\varphi} - v_0*u_0^{-\varphi}) \ge 0$. The inequality holds for Possibility (iii).

Possibility (iv)

$C_\varphi(u_1, v_1) - C_\varphi(u_0, v_1) - C_\varphi(u_1, v_0) + C_\varphi(u_0, v_0) =$

$u_1v_1^{1-\varphi} – u_0v_1^{1-\varphi} – u_1^{1-\varphi}v_0 + u_0v_0^{1-\varphi} =$

$u_1*(v_1^{1-\varphi} - v_0*u_1^{-\varphi}) - u_0*(v_1^{1-\varphi} - v_0^{1-\varphi})$

Since $u_1 \ge v_0$ and $\varphi \ge 0$, $v_0^{-\varphi} \ge u_1^{-\varphi} \rightarrow v_0*v_0^{-\varphi} \ge v_0*u_1^{-\varphi} \rightarrow v_0^{1-\varphi} \ge v_0*u_1^{-\varphi} \rightarrow$

$(v_1^{1-\varphi} - v_0*u_1^{-\varphi}) \ge (v_1^{1-\varphi} - v_0^{1-\varphi})$

Thus, since $u_1 \ge u_0$ and $(v_1^{1-\varphi} - v_0*u_1^{-\varphi}) \ge (v_1^{1-\varphi} - v_0^{1-\varphi})$, it follows that

$u_1*(v_1^{1-\varphi} - v_0*u_1^{-\varphi}) \ge u_0*(v_1^{1-\varphi} - v_0^{1-\varphi})$ and

$u_1*(v_1^{1-\varphi} - v_0*u_1^{-\varphi}) - u_0*(v_1^{1-\varphi} - v_0^{1-\varphi}) \ge 0$. The inequality holds for Possibility (iv).

Q. E. D.

Feldblum Exercise 5. (FGM Copula)

Show that the function $C_\varphi(u, v) = uv(1 + \varphi(1-u)(1-v))$ is a copula, where $0 \le \varphi \le 1$. φ is a dependence parameter. If $\varphi = 0$, the FGM copula is the independence copula. If $\varphi > 0$, the distributions of u and v are correlated.

Solution to Feldblum Exercise 5.

We show that property (1) is satisfied.

$C_\varphi(u, 0) = u*0(1 + \varphi(1-u)(1-0)) = 0$.

$C_\varphi(u, 1) \rightarrow u*1(1 + \varphi(1-u)(1-1)) = u(1 + 0) = u$.

Thus, property (1) holds.

We show that property (2), the "rectangle inequality", is satisfied.

We need to show that $C_\varphi(u_1, v_1) - C_\varphi(u_0, v_1) - C_\varphi(u_1, v_0) + C_\varphi(u_0, v_0) \geq 0$.

$C_\varphi(u_1, v_1) - C_\varphi(u_0, v_1) - C_\varphi(u_1, v_0) + C_\varphi(u_0, v_0) =$

$u_1v_1(1 + \varphi(1-u_1)(1-v_1)) - u_0v_1(1 + \varphi(1-u_0)(1-v_1)) - u_1v_0(1 + \varphi(1-u_1)(1-v_0)) +$

$u_0v_0(1 + \varphi(1-u_0)(1-v_0)) =$

$(u_1v_1 - u_0v_1 - u_1v_0 + u_0v_0) + \varphi(1-u_1)(1-v_1)*u_1v_1 - \varphi(1-u_0)(1-v_1)*u_0v_1 - \varphi(1-u_1)(1-v_0)*u_1v_0 + \varphi(1-u_0)(1-v_0)*u_0v_0$.

Let $(u_1v_1 - u_0v_1 - u_1v_0 + u_0v_0) = A$. This is the left-hand side of the rectangle inequality for the independence copula, so we know that $A \geq 0$. We can simplify the expression above to

$A + \varphi(1-u_1)(1-v_1)*u_1v_1 - \varphi(1-u_0)(1-v_1)*u_0v_1 - \varphi(1-u_1)(1-v_0)*u_1v_0 + \varphi(1-u_0)(1-v_0)*u_0v_0 =$

$A + \varphi v_1*(1-v_1)*((1-u_1)*u_1 - (1-u_0)*u_0) - \varphi v_0*(1-v_0)*((1-u_1)*u_1 - (1-u_0)*u_0) =$

$A + \varphi*((1-u_1)*u_1 - (1-u_0)*u_0)*((1-v_1)*v_1 - (1-v_0)*v_0)$.

Now we consider that $A = (u_1v_1 - u_0v_1 - u_1v_0 + u_0v_0) = (u_1 - u_0)*(v_1 - v_0)$. Thus, the above expression can be written as

$(u_1 - u_0)*(v_1 - v_0) + \varphi*((1-u_1)*u_1 - (1-u_0)*u_0)*((1-v_1)*v_1 - (1-v_0)*v_0) =$

$(u_1 - u_0)*(v_1 - v_0) + \varphi*((u_1 - u_0) + (u_0^2 - u_1^2))*((v_1 - v_0) + (v_0^2 - v_1^2)) =$

$(u_1 - u_0)*(v_1 - v_0) + \varphi*(u_1 - u_0)*(v_1 - v_0) + \varphi*(u_1 - u_0)*(v_0^2 - v_1^2) + \varphi*(v_1 - v_0)*(u_0^2 - u_1^2) + \varphi*(u_0^2 - u_1^2)*(v_0^2 - v_1^2) =$

$(1 + \varphi)*(u_1 - u_0)*(v_1 - v_0) + \varphi*(u_1 - u_0)*(v_0^2 - v_1^2) + \varphi*(v_1 - v_0)*(u_0^2 - u_1^2) + \varphi*(u_0^2 - u_1^2)*(v_0^2 - v_1^2) =$

$(1 + \varphi)*A + \varphi*(u_1 - u_0)*(v_0 + v_1)*(v_0 - v_1) + \varphi*(v_1 - v_0)*(u_0 + u_1)*(u_0 - u_1) + \varphi*(u_0 + u_1)*(u_0 - u_1)*(v_0 + v_1)*(v_0 - v_1) =$

$(1 + \varphi)*A + \varphi*(v_0 + v_1)*(-A) + \varphi*(u_0 + u_1)*(-A) + \varphi*A*(u_0 + u_1)*(v_0 + v_1) =$

$A*(1 + \varphi - \varphi*(v_0 + v_1) - \varphi*(u_0 + u_1) + \varphi*(u_0 + u_1)(v_0 + v_1)) =$

$A*(1 + \varphi*(1 + (u_0 + u_1)(v_0 + v_1)) - \varphi*(u_0 + u_1 + v_0 + v_1)) =$

$A*(1 + \varphi*(1 + (u_0 + u_1)(v_0 + v_1) – (u_0 + u_1 + v_0 + v_1)).$

If it can be shown that $\varphi*(1 + (u_0 + u_1)(v_0 + v_1) – (u_0 + u_1 + v_0 + v_1)) \geq -1$, then this would demonstrate that the rectangle inequality holds for the FGM copula.

An intermediate step would be to show that
$1 + (u_0 + u_1)(v_0 + v_1) – (u_0 + u_1 + v_0 + v_1) \geq -1.$

$1 + (u_0 + u_1)(v_0 + v_1) – (u_0 + u_1 + v_0 + v_1) =$

$1 + u_0v_0 + u_1v_0 + u_0v_1 + u_1v_1 – u_0 – u_1 – v_0 – v_1 =$

$(v_1 + v_0 - 1)(u_1 + u_0 - 1).$

Since $v_1 + v_0 \geq 0$ and $u_1 + u_0 \geq 0$, it follows that $v_1 + v_0 – 1 \geq -1$ and $u_1 + u_0 – 1 \geq -1$.

Two values that are both greater than -1 cannot have a product that is less than -1, when the sum of the two values is less than 2. Since each value of u and v is by itself at most 1, the sum of any two values of u or any two values of v cannot be more than 2. Thus,

$(v_1 + v_0 - 1)(u_1 + u_0 - 1) \geq -1$, and so

$\varphi*(v_1 + v_0 - 1)(u_1 + u_0 - 1) \geq -1. \rightarrow$

$A*(1 + \varphi*(v_1 + v_0 - 1)(u_1 + u_0 - 1)) \geq A*(1-1) = 0$

Thus, $A*(1 + \varphi*(v_1 + v_0 - 1)(u_1 + u_0 - 1)) \geq 0$. This proves that the rectangle inequality holds. **Q. E. D.**

Feldblum Exercise 6. (FGM Copula)

Let C_φ be the FGM copula with parameter φ: $C_\varphi(u, v) = uv(1 + \varphi(1-u)(1-v))$, where $0 \leq \varphi \leq 1$.

(a) What is the density function $c_\varphi(u, v)$ for this copula? [Take the second partial derivative with respect to u and v.]
(b) What values of u and v give density equal to 1?
(c) Which regions of the unit square have the highest density values?
(d) Which regions of the unit square have the lowest density values?

Solution to Feldblum Exercise 6.

(a) $C_\varphi(u, v) = uv(1 + \varphi(1-u)(1-v)) = uv + \varphi*uv*(1-u)(1-v) =$

$uv + \varphi*uv*(1 - u - v + uv) =$

$uv + \varphi*uv - \varphi*u^2v - \varphi*uv^2 + \varphi*u^2v^2$.

$\partial C_\varphi(u, v)/(\partial u) = v + \varphi*v - 2\varphi*uv - \varphi v^2 + 2\varphi*uv^2$.

$\partial^2 C_\varphi(u, v)/(\partial u*\partial v) =$ $\mathbf{c_\varphi(u, v) = 1 + \varphi - 2\varphi*u - 2\varphi v + 4\varphi*uv = 1 + \varphi*(1-2u)*(1-2v).}$

(b) We set $1 = 1 + \varphi*(1-2u)*(1-2v) \rightarrow 0 = (1-2u)*(1-2v)$. If either **u = 1/2 or v = 1/2**, then the density is equal to 1.

(c) Highest density values are found by maximizing $c_\varphi(u, v) = 1 + \varphi*(1-2u)*(1-2v)$. The most $c_\varphi(u, v)$ can be is $1 + \varphi$, which occurs when **(u = 0 and v = 0)** and **(u = 1 and v = 1)**.

(d) Lowest density values are found by minimizing $c_\varphi(u, v) = 1 + \varphi*(1-2u)*(1-2v)$. The values of u and v cannot exceed 1, and so (1-2u) and (1-2v) are each between -1 and 1. If one of these quantities were -1 and the other were 1, then $c_\varphi(u, v)$ would equal $1 - \varphi$, which is the least it can be. This occurs when either **(u = 0 and v = 1)** or **(u = 1 and v = 0)**.

Feldblum Exercise 7. (Bivariate Exponential Copula).

Let $H_\theta(y, z)$ by the following bivariate distribution:
$H_\theta(y, z) = 1 - e^{-y} - e^{-z} - e^{-(y+z+\theta yz)}$, when $y \geq 0$ and $z \geq 0$.

$H_\theta(y, z) = 0$ otherwise. $\theta \in [0, 1]$ is a parameter.

(a) What are the marginal distributions of Y and Z? Do these marginals look familiar?
(b) What is the copula function associated with this bivariate distribution?

Solution to Feldblum Exercise 7.

(a) Let the marginal distribution functions be F(y) and G(z).

$F(y) = H_\theta(y, \infty) = 1 - e^{-y} - e^{-\infty} - e^{-(y+\infty+\theta y\infty)} = 1 - e^{-y} - 0 - 0 = 1 - e^{-y}$.

$G(z) = H_\theta(\infty, z) = 1 - e^{-\infty} - e^{-z} - e^{-(y+\infty+\theta y\infty)} = 1 - 0 - e^{-z} - 0 = 1 - e^{-z}$.

These are both exponential distribution functions with mean 1.

(b) $C_\theta(u, v) = H_\theta(F^{-1}(y), G^{-1}(z))$.

We find $F^{-1}(y)$: $F(y) = 1 - e^{-y} \rightarrow e^{-y} = 1 - F(y) \rightarrow -y = \ln(1 - F(y)) \rightarrow y = -\ln(1 - F(y)) \rightarrow =$

$F^{-1}(y) = -\ln(1-y)$. By similar reasoning, $G^{-1}(z) = -\ln(1-z)$.

Thus, $C_\theta(u, v) = H_\theta(-\ln(1-y), -\ln(1-z)) =$

$1 - \exp(-\ln(1-y)) - \exp(-\ln(1-z)) - \exp\text{-}(-\ln(1-y) - \ln(1-z) + \theta*\ln(1-y)*\ln(1-z)) =$

$\mathbf{1 - 1/(1-y) - 1/(1-z) - (1/((1-y)(1-z)))*\exp(\theta*\ln(1-y)*\ln(1-z))}$.

Feldblum Exercise 8. (Fréchet Copulas).

Fréchet copulas are convex linear combinations of the copulas $\alpha M(u, v) + \beta W(u, v) + (1-\alpha-\beta)*\Pi(u,v)$.

Solution to Feldblum Exercise 8. A generic proof that linear combinations of copulas are themselves copulas was given in the solution to the Feldblum Exercise 3. This is a particular case of that general principle. Because Fréchet copulas are linear combinations of known copulas M(u, v), W(u, v), and Π(u,v), they are themselves copulas.

Feldblum Exercise 9. We have probability density functions $f(y) = 2y$ and $g(z) = 3z^2$ for $0 \le y, z \le 1$.

The marginal distributions are joined by a Fréchet copula with $\alpha = 1/3$ and $\beta = 1/3$.

(a) What are F(y) and G(z).

(b) Write the algebraic expression of $H_{1/3,1/3}(y,z)$ for any $0 \le y, z \le 1$.

(c) What is the Fréchet copula with $\alpha = 1/3$ and $\beta = 1/3$?

(d) For y = 1/2 and z = 1/2, what is $H_{1/3,1/3}(y,z)$?

Solution to Feldblum Exercise 9.

(a) F(y) is the integral of f(y) → $\mathbf{F(y) = y^2}$. G(z) is the integral of g(z) → $\mathbf{G(z) = z^3}$.

(b) $H_{1/3,1/3}(y,z) = (1/3)*M(F^{-1}(y), G^{-1}(z)) + (1/3)*W(F^{-1}(y), G^{-1}(z)) + \Pi(F^{-1}(y), G^{-1}(z))$.

(c) We find $F^{-1}(y) = y^{1/2}$ and $G^{-1}(z) = z^{1/3}$.

We now work with the formula in (b):

$(1/3)*M(F^{-1}(y), G^{-1}(z)) + (1/3)*W(F^{-1}(y), G^{-1}(z)) + \Pi(F^{-1}(y), G^{-1}(z)) =$

$H_{1/3,1/3}(y,z) = \mathbf{(1/3)*min(y^{1/2}, z^{1/3}) + (1/3)*max(0, y^{1/2} + z^{1/3}-1) + (1/3)*(y^{1/2}*z^{1/3})}$.

(d) We substitute y = 1/2 and z = 1/2:

$(1/3)*min(y^{1/2}, z^{1/3}) + (1/3)*max(0, y^{1/2} + z^{1/3}-1) + (1/3)*(y^{1/2}*z^{1/3}) =$

$(1/3)*min((1/2)^{1/2}, (1/2)^{1/3}) + (1/3)*max(0, (1/2)^{1/2} + (1/2)^{1/3}-1) + (1/3)*((1/2)^{1/2}*(1/2)^{1/3}) =$

$(1/3)*(1/2)^{1/2} + (1/3)*(0.5008073072) + 0.1870770081 =$ **0.5897150375.**

Feldblum Exercise 10. (Multivariate distributions and copulas)

Let $f(y) = 6y(1-y)$ for $0 \le y \le 1$ and $g(z) = \exp(-z/2)/2$ for $0 \le z$. Join the marginal distributions of Y and Z with a Fréchet copula with parameters $\alpha = 1/2$ and $\beta = 1/2$, where β is applied to the independence copula.

(a) What are F(y) and G(z)?

(b) What is the algebraic expression for the Fréchet copula?

(c) For y = 1/2 and z = 2, what is H(y, z)?

Solution to Feldblum Exercise 10.

(a) $f(y) = 6y - 6y^2 \rightarrow F(y)$ is the integral of $f(y)$: $\mathbf{F(y) = 3y^2 - 2y^3}$.

$g(z) = \exp(-z/2)/2 \rightarrow G(z)$ is the integral of $g(z)$: $\mathbf{G(z) = 1 - \exp(-z/2)}$.

(b) The Frechet copula is $(1/2)*M(F^{-1}(y), G^{-1}(z)) + (1/2)*\Pi(F^{-1}(y), G^{-1}(z)) =$

$\mathbf{(1/2)*\min(F^{-1}(y), G^{-1}(z)) + (1/2)*(F^{-1}(y))*(G^{-1}(z))}$.

We find $G^{-1}(z)$:

$G(z) = 1 - \exp(-z/2)$.

$\exp(-z/2) = 1 - G(z)$

$-z/2 = \ln(1 - G(z))$

$z = -2\ln(1 - G(z))$.

$G^{-1}(z) = -2\ln(1-z)$.

$F^{-1}(y)$ is too cumbersome to calculate. It is

$$\frac{1}{2}\sqrt[3]{2\sqrt{y^2-y}-2y+1}+\frac{1}{2\sqrt[3]{2\sqrt{y^2-y}-2y+1}}+\frac{1}{2}$$

For $y = 1/2$, $F^{-1}(y)$ does not exist. This means that we cannot find $H(y, z)$ for $y = 1/2$ and $z = 2$.

This problem appears to have been erroneously designed -- particularly in the choice of $F(y)$.

Feldblum Exercise 11. Consider the scatterplot in Figure 16. What does the corresponding χ-plot look like?

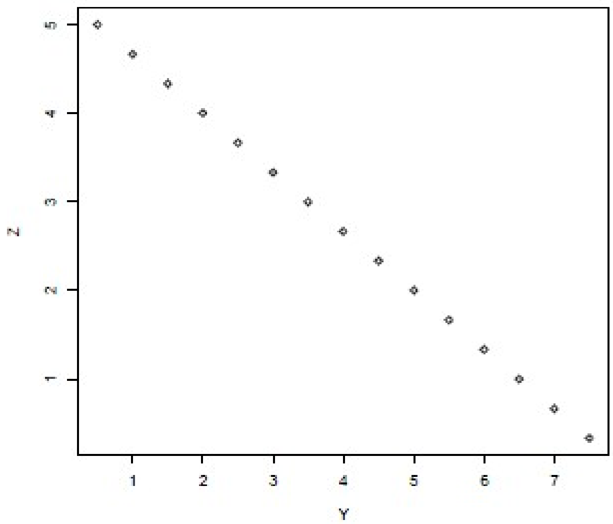


Figure 16: What is the χ-plot associated with this scatterplot?

Solution to Feldblum Exercise 11. For each point i, H_i, the fraction of points in the third, southwest quadrant, is zero. Because the formula for the χ-plot coordinate is

$\chi_i = (H_i - F_i*G_i)/\sqrt{(F_i*(1-F_i)*G_i*(1-G_i))}$, it follows that each χ_i is negative.

For each point i, the fraction of points in the first, northeast quadrant is also zero. Thus, it follows that all the points are either in the second quadrant or the fourth quadrant. Since F_i = (fraction of points in the second and third quadrants) and G_i = (fraction of points in the third and fourth quadrants), and there are no points in the third quadrant, it follows that F_i represents the second quadrant, and G_i represents the fourth quadrant. Thus, $F_i = 1 - G_i$ and $\chi_i = -(F_i*G_i)/\sqrt{(F_i*(1-F_i)*G_i*(1-G_i))}$ =

$-(F_i*G_i)/\sqrt{(F_i*G_i*G_i*F_i)} = -(F_i*G_i)/(F_i*G_i) = -1$. So each point on the χ-plot is -1.

Feldblum Exercise 12. Consider the scatterplot in Figure 4. How many points (and where) should we add to this scatterplot to change the χ-coordinate of point b from $1/\sqrt{(3)}$ to a value smaller than -1/4?

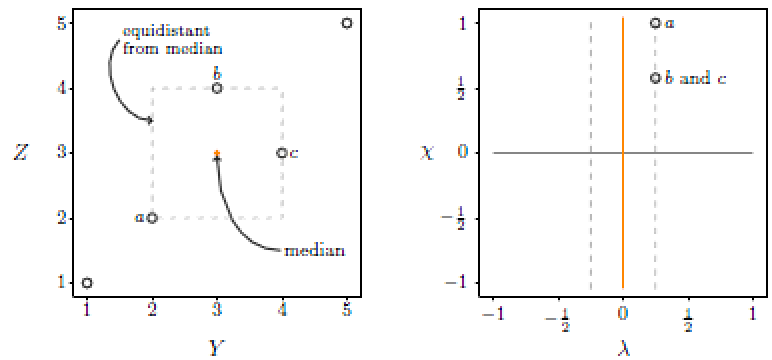


Figure 4: A simple scatterplot on the left-hand side and its corresponding χ-plot on the right-hand side

Solution to Feldblum Exercise 12.

We want it to be the case that $\chi_i = (H_i - F_i*G_i)/\sqrt{(F_i*(1-F_i)*G_i*(1-G_i))} < -1/4$.

Currently, for b, H = (1/2), F = (1/2), and G = (2/4) (per Feldblum, p. 10).

The more we increase F, the more negative the numerator will become. To increase F without increasing the other values, we can add points to the fourth (northwest) quadrant. Let x be the number of points added to the northwest quadrant.

Then H = 2/(4 + x), G = 3/(4 + x) and F = (2+x)/(4 + x).

Thus, $\chi_i = (2/(4 + x) - 3(2+x)/(4 + x)^2) /\sqrt{((2+x)/(4 + x)*(1-(2+x)/(4 + x))*3/(4 + x)*(1-3/(4 + x)))} =$

$(2(4 + x) - 3(2+x))/((4 + x)^2\sqrt{((2+x)*2*3*(1+x))/(4 + x)^4}) =$

$(8 + 2x - 6 - 3x)/\sqrt{(6(x+1)(x+2))} = (2-x)/\sqrt{(6(x+1)(x+2))} < -1/4$.

Let $K = (2-x)/\sqrt{(6(x+1)(x+2))}$. We can try various values of x.

If x = 1, K = 1/6. If x = 2, K = 0. If x = 5, K = -0.1889822365. If x = 7, K = -0.24056261216. If x = 8, K = -0.25819888975. So x = 8 is the first value that makes K < -1/4. Hence, **we add 8 points to the northwest quadrant.**