# The Actuary's Free Study Guide for Exam 3F / Exam MFE

## Second Edition

G. Stolyarov II,
ASA, ACAS, MAAA, CPCU, ARe, ARC, API, AIS, AIE, AIAF

**First Edition Published in February-May 2008**
**Second Edition Published in July 2014**

# Table of Contents

# Study Methods for Actuarial Exam 3F / Exam MFE

To accompany *The Actuary's Free Study Guide for Exam 3F / Exam MFE*, I offer a list of general studying methods, techniques, and insights that have guided my own preparation for the financial economics half of the third actuarial exam. While my methods may not be suited to every type of actuarial student - and it is ultimately your decision to embrace them or to reject them, based on your estimation of your abilities and ways in which you learn most efficaciously - I have found them tremendously helpful in making sense out of an immense exam syllabus. I will first discuss general study approaches and then address ideas to keep in mind for this exam in particular.

**General Studying Approaches**

**1. Begin studying early and study regularly.** The actuarial exams, as you are likely well aware, are not comparable to final exams in college, to which you might allot a few hours of study and get an A as a result. These exams require *months* of preparation in order to adequately learn and apply the material. I recommend starting at least 2.5 months ahead of the exam date and reading, solving practice problems, and even writing practice problems yourself every day.

**2.Set daily goals and develop a system to quantify your studying.** To make sure that you are putting in the necessary effort every day, it is not enough to have a subjective feeling that you have worked sufficiently. The exam material is quite difficult and, in my personal experience, after doing *any* work, one feels like one has done plenty. It is much wiser to set an objective goal in advance for each day and attempt to meet it. Of course, goals need not be rigid and can respond to any unforeseen challenges posed by the course material. Setting up a point system that needs to be met every day rather than insisting on highly specific and unalterable objectives.

Here, I will outline the point system I have used to prepare for Exam 3F/MFE as well as for exams 1/P and 2/FM. I require myself to accumulate at least 100 points per day. Here is how the points may be accumulated.

I receive 5 points for every page of exam-relevant text that I read.

I receive 10 points for every exam-relevant problem that I solve.

I receive 20 points for every exam-relevant problem I formally write myself and then solve.

I am, of course, allowed and encouraged to go over my 100-point target. I keep a running total of all the points I have accumulated during the course of studying as well as a current arithmetic average of points for all days thus far. My average acts like a grade in a course, and so I have an incentive to strive to keep my grade in Actuarial Studying high - above 100, if at all possible. I also sometimes amuse myself by trying to deliberately raise my overall "course" average on a particular day by doing extra exam preparation.

A point system helps one establish a common denominator by which to measure one's efforts at tasks that can often vary considerably. Just like money provides a convenient way to measure economic value while dealing with millions of goods, so does a point system allow one to compare disparate studying approaches and have some rough measure of accomplishment on a particular day. Like money, any point system is also highly imperfect at measuring the desired objectives - effort and learning. Develop a point system that you consider to be the most accurate and the most relevant to the approaches that work best for you.

The incentives provided by my point system have shown to be effective in my own experience. In preparing for Exam 1/P, I maintained an average of 114.95 points per day over 98 days. In preparing for Exam 2/FM, I maintained an average of 128.24 points per day over 105 days. I am doing even better thus far in preparing for Exam 3F/MFE; I have a current average of 143.35 over 79 days. My studying for the prior two exams seems to have been sufficient; I passed both on the first try, with scores of 10/10 and 9/10, respectively. And I still have detailed records of the studying I did on every day that I prepared for each exam.

**3. Do as many practice problems as possible. This includes writing your own!** The way to truly learn each concept relevant for the exam is to immerse oneself in it and repeatedly expose oneself to ways in which it might be applied. For me personally, a mathematical formula or idea does not stay in my mind if I just read about it. If I write down the relevant formulas, my memory functions somewhat better, but it is in the *use* of the formulas that I truly memorize them; after I have used a formula five or six times, I no longer have any difficulty recalling it.

I am a highly frugal individual and pride myself on only spending money when it is absolutely necessary for the improvement of my well-being. When I began to study for this exam, I noticed that there was a dearth of freely available practice problems for it - problems that would help students to make sense of the theoretical concepts described in McDonald's *Derivatives Markets* and to see how those concepts apply in situations relevant to them - i.e., in exam situations. Granted, some admirable efforts have been made by parties like Dr. Ewa Kubicka, Dr. Sam Broverman, Dr. Bill Cross, Dr. Abraham Weishaus, and Actuarial Brew to offer free practice problems - but even those combined were not enough to enable students to learn the entire syllabus from them alone. This is why I wrote *The Actuary's Free Study Guide for Exam 3F / Exam MFE*, which, in May 2008, accounted for 418 of the 625 free Exam 3F/MFE practice problems with free available solutions of which I was aware.

I encourage you to write any problems you think might be helpful in teaching actuarial students important exam-related concepts or skills. It is even possible to earn some small passive income from these problems if you render them available online on sites like Associated Content, which will pay you $1.50 per 1000 page views to your articles. While this is not a living, it might earn you more than the interest in your bank account and may even compensate for a significant fraction of your exam fees. If you choose to publish your practice problems via Associated Content, submit them under the "Display-Only" terms, which will enable you to edit your material, albeit with substantial time delays of about a week for the edits to get implemented. As is virtually inevitable with problems of this level of difficulty, any author will sometimes make substantial errors or oversights - and indeed, virtually all textbooks and practice manuals for this exam are accompanied by substantial sections of errata. The advantage of publishing problems online is that user feedback regarding any errors is extremely fast and contributes to an ultimately better product. Freely available online practice problems and solutions can be assured to have high quality in a manner similar to open-source software, which also gets continually improved on the basis of user feedback.

**Exam-Specific Approaches**

**4. Assure yourself some points by mastering the easier material.** Exam 3F/MFE is comprised of material of highly varying difficulty and open-endedness. Fortunately, several major topics on the syllabus - mostly represented in the earlier chapters of McDonald's *Derivatives Markets* - are quite systematic and straightforward in their application. Answering questions regarding them properly only requires you to thoroughly memorize a few formulas and problem-solving techniques. The easier topics on this exam include **put-call parity, binomial option pricing, and the Black-Scholes formula.** Questions related to these three topics combined have in the past accounted for over 50% of both the Casualty Actuarial Society's and the Society of Actuaries' exams - and there is no reason why you should get any of these questions wrong.

**5. Learn as much as possible about the intermediate-level material.** Approximately another 25% of the exam should be composed of material pertaining to **delta-hedging** and **exotic options.** All of this material is also manageable, although the variety of problems that could be asked is somewhat greater.

Much of your success at hedging questions will be assured if you memorize the formula for the **delta-gamma-theta approximation**: $C_{new} = C_{old} + \epsilon\Delta + (1/2)\epsilon^2\Gamma + h\theta$. The delta-gamma approximation is just the delta-gamma-theta approximation without the $h\theta$ term. You should also be able to answer questions regarding what it takes to **delta- and gamma-neutralize a portfolio**. In those questions, you will be given delta and gamma values for different kinds of options. Always work with gamma-neutralization first, because once you have rendered a portfolio's gamma zero, you can always compensate for the leftover delta by adding or subtracting shares of stock (as each stock has a delta of 1). The mathematics behind these problems is simple arithmetic, and again, there is no reason to get any of them wrong. Sometimes, you might be given a delta in disguise; remember that $\Delta = e^{-\delta T}N(d_1)$. If you are given $\delta$, T, and $N(d_1)$, you can find $\Delta$ using this formula.

Also remember the formulas for option elasticity ($\Omega = S\Delta/C$) and option volatility ($\sigma_{option} = \sigma_{stock} * |\Omega|$). These are easy to memorize and apply.

Less likely but quite possible are questions regarding the return and variance of the return to a delta-hedged market-maker. To be able to answer those questions, memorize the formulas

$R_{h,i} = (1/2)S^2\sigma^2\Gamma(x_i^2-1)h$ for the return and $Var(R_{h,i}) = (1/2)(S^2\sigma^2\Gamma h)^2$ for the variance of the return. Also remember that you can model the stock price movement in any time period h as $\sigma S_t\sqrt{(h)}$. You might be asked to determine said stock price movement, given h, $S_t$, and some function of $\sigma$ - probably $N(d_1)$ - from which you will be able to figure out $\sigma$.

It also would not hurt to memorize the Black-Scholes partial differential equation: $rC(S_t) = (1/2)\sigma^2 S_t^2 \Gamma_t + rS_t\Delta_t + \theta$.

For exotic options, each type of option is not difficult to understand conceptually, but there are a lot of types to keep in mind. For **Asian options**, make sure you remember that they are path-dependent and that you know how to take a geometric average. Pay attention to the difference between geometric average *price* options and geometric average *strike* options. For **barrier options**, remember the parity relationship Knock-in option + Knock-out option = Ordinary option. Know about rebate options as well; the concept is not difficult. For **compound options**, remember the parity relationship CallOnCall - PutOnCall + $xe^{-rt\_1}$ = BSCall. **Gap options** are priced just like ordinary options via the Black-Scholes formula, with the exception that the *trigger price* rather than the strike price is used in the formula for $d_1$, while the strike price is still used in the formula for the overall option price. For pricing **exchange options**, the Black-Scholes formula holds as well, with two significant differences. Sigma in the formula for exchange options is $\sigma = \sqrt{[\sigma_S^2 + \sigma_K^2 - 2\rho\sigma_S\sigma_K]}$; you will be given the individual volatilities of the underlying asset and the strike asset, as well as their correlation $\rho$. Also, S is your underlying asset price, K is your strike asset price, and you will be given the dividend yields $\delta_S$ and $\delta_K$ pertaining to these assets. Then your Black-Scholes exchange call price will be

$C= Se^{-(\delta\_S)T}N(d_1) - Ke^{-(\delta\_K)T}N(d_2)$.

**6. Memorize shortcut approaches to Brownian motion and interest rate models.** The most difficult - and most open-ended - topics on the exam will be based on Chapters 20 and 24 of McDonald's *Derivative Markets* - dealing respectively with Brownian motion and interest rate models. The best way to approach these topics is *not* from the vantage point of theory or derivation from first principles. Rather, you will save yourself a lot of time and stress by memorizing the general form of results obtained by doing specific kinds of problems.

You *do* need to know what arithmetic, geometric, and mean-reversion (Ornstein-Uhlenbeck) Brownian processes look like. You should also be familiar with terminology pertaining to them, including *drift*, *volatility*, and *martingale* (a martingale is a process for which $E[Z(t+s) \mid Z(t)] = Z(t)$).

**Ito's Lemma** can be easily memorized in the form

$dC(S, t) = C_S dS + (1/2)C_{SS}(dS)^2 + C_t dt$, which is applicable to geometric Brownian motion. It is doubtful that you will be asked to apply Ito's Lemma to non-geometric kinds of Brownian motion. When you apply Ito's Lemma, remember the multiplication rules, which state that $(dZ)^2 = dt$, and any other product of multiple dZ and dt is 0. If you have two correlated Brownian motions Z and Z', then $dZ * dZ' = \rho$.

If you memorize some results derived using Ito's Lemma, then you will not have to go through the derivation on the test. For example, if $dX(t)/X(t) = \alpha dt + \sigma dZ(t)$ (i.e., X follows a geometric Brownian motion, then $d[\ln(X)] = (\alpha - 0.5\sigma^2)dt + \sigma dZ(t)$ (i.e., ln(X) follows an arithmetic Brownian motion *with the exact same volatility*.) Just memorizing the latter formula can make several currently known prior exam questions extremely easy to solve.

Do not forget about the formula for the Sharpe ratio ($\varphi = (\alpha - r)/\sigma$) and the fact that the Sharpe ratios of two perfectly correlated Brownian motions are equal.

To figure out *whether a particular Brownian motion has zero drift* (or what kind of drift factor it has), you will need to apply Ito's Lemma to the given Brownian motion and see what the resultant dt term is.

Also recall that the **quadratic variation** of a Brownian process is expressible as $_{n\to\infty}\lim_{i=1}{}^{n}\Sigma(Z[ih] - Z[(i-1)h])^2 = T$ from time 0 to time T. Here, Z is the Standard Brownian Motion. In X is some other Brownian motion with volatility factor σ, then

$_{n\to\infty}\lim_{i=1}{}^{n}\Sigma(Y[ih] - Y[(i-1)h])^2 = \sigma^2T$.

Other useful facts to know are as follows.

$\mathrm{Var}[\sigma dZ(t) \mid Z(t)] = \sigma^2\mathrm{Var}[dZ(t) \mid Z(t)] = \sigma^2dt$.

For any arithmetic Brownian motion X(t), the random variable [X(t + h) - X(t)] is normally distributed for all $t \geq 0$, $h > 0$, and has a mean of $X(t) + \alpha h$ and a variance of $\sigma^2h$.

The Black-Scholes option pricing framework is based on the assumption that the underlying asset follows a geometric Brownian motion:
$dS(t)/S(t) = \alpha dt + \sigma dZ(t)$.

When X(t) follows an arithmetic Brownian motion, the following equation holds:
$X(t) = X(a) + \alpha(t - a) + \sigma\sqrt{(t-a)}\xi$.

When X(t) follows a geometric Brownian motion, the following equation holds:
$X(t) = X(a)\exp[(\alpha - 0.5\sigma^2)(t - a) + \sigma\sqrt{(t-a)}\xi]$.

If given either of those two equations, you should be able to recognize arithmetic and geometric Brownian motion.

Be sure to know how to value claims on *power derivatives* (of the form$S(T)^a$). This is a newly added topic and is thus likely to be tested on at least one question. Memorize the following formulas.

$d(S^a)/S^a = (a(\alpha - \delta) + 0.5a(a-1)\sigma^2)dt + a\sigma dZ(t)$

$\gamma = a(\alpha - r) + r$, where r is the annual continuously compounded risk-free interest rate.

$\delta^* = r - a(r - \delta) - 0.5a(a-1)\sigma^2$
$F_{0,T}[S(T)^a] = S(0)^a\exp((a(r - \delta) + 0.5a(a-1)\sigma^2)T)$
$F^P{}_{0,T}[S(T)^a] = e^{-rT}S(0)^a\exp((a(r - \delta) + 0.5a(a-1)\sigma^2)T)$

While many actuarial students might think that Brownian motion is the most difficult topic on the exam, I believe that the **interest-rate models** covered in Chapter 24 are in fact harder, because a virtually endless variety of practically impossible questions can be asked regarding them. I hope that the SOA and CAS will be reasonable in what they choose to test. For instance, asking students to use the explicit bond-price formulas for the Vasicek and Cox-Ingersoll-Ross (CIR) interest rate models would be uncalled for, as these formulas take tremendous effort to memorize. The past exam questions I have seen have not asked for these formulas. Instead, they tended to involve various "auxiliary" formulas for these models.

Of course, it is essential to know the Brownian motions associated with the Vasicek and CIR models. For the Vasicek model, $dr = a(b - r)dt + \sigma dZ$. For the CIR model, $dr = a(b - r)dt + \sigma\sqrt{(r)}dZ$. Note that the only difference is a $\sqrt{(r)}$ factor in the dZ term. This has important implications, however, as discussed in Section 72 of my study guide. Make sure you read this section to find out about the essential similarities and differences between these models as well as why the time-zero yield curve for either of these models cannot be exogenously prescribed (a learning objective on the exam syllabus).

For both the Vasicek model, the following "auxiliary" equations are important to know, as questions involving them have appeared on prior exams.

$P[t, T, r(t)] = \exp(-[\alpha(T - t) + \beta(T - t)r])$, where $\alpha(T - t)$ and $\beta(T - t)$ are constants that stay the same whenever the difference between T and t is the same, even if the values of T and t are different.

Now say you have a Vasicek model where

$dP[r(t), t, T]/P[r(t), t, T] = \alpha[r(t), t, T]dt + q[r(t), t, T]dZ(t)$. You are given some $\alpha(r_1, t_1, T_1)$ and are asked to find $\alpha(r_2, t_2, T_2)$. All you need to do is to solve the following equation:
$\mathbf{[\alpha(r_1, t_1, T_1) - r_1]/(1 - \exp[-a(T_1 - t_1)]) = [\alpha(r_2, t_2, T_2) - r_2]/(1 - \exp[-a(T_2 - t_2)])}$

The derivation of this formula is quite involved and is provided in Section 71. You would be well advised to simply memorize the end result (and read the derivation, if you are interested in the reasoning, but do not repeat the derivation each time unless you enjoy pain and suffering).

(It would not hurt to also memorize that $q[r(t), t, T] = -\sigma B(t, T)$ in the case above. A question using this formula is not likely to be asked, and if it is, I hope that σ and B(t, T) will be given.)

The Black-Derman-Toy (BDT) interest rate model has easy and difficult applications. The easy applications involve pricing caplets and interest rate caps, provided that you have had practice with the BDT discounting procedure. Remember that the risk-neutral probability for the BDT model is always 0.5 and the interest rates at the nodes in any given time period differ by a constant multiple of $\exp[2\sigma_i\sqrt{(h)}]$. To discount the expected value of a payment at any node, divide it by 1 plus the interest rate at that node. Remember that, for interest rate caps, you will need to account for payments made *each* time period when the interest rate exceeds the cap strike rate.

The difficult parts of using the BDT model involve finding yield volatilities and actually constructing BDT binomial trees. I am aware of a scant few exam-style questions on the former and of none on the latter - which I hope will be indicative of the composition of future MFE exams. There is no way around memorizing the formula for yield volatility in the BDT model:

Yield volatility = $0.5\ln(y[h, T, r_u]/y[h, T, r_d]) = 0.5\ln([P[h, T, r_u]^{-1/(T-h)} - 1]/[P[h, T, r_d]^{-1/(T-h)} - 1])$

You will likely be given a multi-period BDT tree and asked to calculate the volatility in time period 1 of the bond used in constructing the tree. I am hard-pressed to see how the formula above might be used in finding the volatilities for time periods greater than 1, since the formula relies on the presence of two and only two nodes in the binomial tree.

For constructing binomial trees, I hope that, if a question is asked, it will involve constructing a tree for a one-period model. To do this, learn the following formulas:

If $P_h$ is the price of an h-year bond, where h is one time period in the BDT model, then $P_h = 1/(1 + r_0)$ and thus $r_0 = 1/P_h - 1$. If $P_{2h}$ is the price of an 2h-year bond, then $R_h = r_d$ and $\sigma_h$ meet the following conditions: $r_0 = \sigma_h$and $P_{2h} = 0.5P_h(1/(1 + R_h\exp[2r_0]) + 1/(1 + R_h))$. Anything beyond this is intense algebraic busy work, and I hope that the SOA and CAS are kind enough not to embroil students in it.

Also remember to review the Black formula for pricing options on futures contracts (Section 37) and caplets (Section 75).

**7. Do not forget the "miscellaneous" topics!** I can think of two "miscellaneous" topics that appear seldom or not at all in McDonald's text but are highly likely to be tested. One of these is computing **historical volatility.** Fortunately, this is a straightforward and systematic procedure, and you should have no difficulty with it after working through Section 81. The other topic, **equity-linked insurance contracts**, can involve applications of virtually anything else on the syllabus. Nonetheless, you are likely to be asked to use one or both of the following formulas to discover option payoffs in disguise:

**Formula 80.1:** max(AB, C) = B*max(A, C/B)
**Formula 80.2:** max(A, B) = A + max(0, B - A)

Section 80 gives some equity-linked insurance problems that you might find beneficial.

**8. Take practice tests under exam conditions several days before the actual exam.** Taking practice exams will give you an idea of your current knowledge level as well as areas to which you might need to devote additional attention. If you have studied well and you do well on the practice exams, this will give you a lot of confidence coming into the actual test. I expect the passing threshold for future sessions of Exam 3F/MFE to be around 13 or 14 out of 20, since the Fall 2007 exam - a particularly difficult test - had a passing threshold of 12. You can compile many practice exams from the exam-style questions offered in various sections of my study guide.

This is as much essential studying advice as immediately comes to mind. Remember, however, that the more you know, the better your chances are on this exam, and that the information I have provided here does not substitute for comprehensive months-long preparation. Use all the resources at your disposal and learn everything you can. I hope that the free resources provided in this study guide will aid you in passing the exam and achieving a high mark.

# Section 1

# Put-Call Parity

Put-call parity for European options with the same strike price and time to expiration is

Call - put = present value of (forward price - strike price)

Equation for put-call parity:
$C(K, T) - P(K, T) = PV_{0,T}(F_{0,T} - K) = e^{-rT}(F_{0,T} - K)$

Meaning of variables:

K = strike price of the options

T = time to expiration of the options

C(K, T) = price of a European call with strike price K and time to expiration T.

P(K, T) = price of a European put with strike price K and time to expiration T.

$F_{0,T}$ = forward price for the underlying asset.

$PV_{0,T}$ = the present value over the life of the options.

$e^{-rT}*F_{0,T}$ = prepaid forward price for the asset.

$e^{-rT}*K$= prepaid forward price for the strike.

r = the continuously compounded interest rate.

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 9, p. 282.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem PCP1.** The European call option on Asset Q that expires in one year has strike price 32 and option price 4. The forward price of Asset Q in one year is 36. The annual continuously compounded interest rate is 0.08. Find the price of the put option on Asset Q with strike price of 32.

**Solution PCP1.** We have

$C(K, T) - P(K, T) = e^{-rT}(F_{0,T} - K)$

We have C(32, 1) = 4, $F_{0,T}$ = 36 and K = 32.

Thus, 4 - P(K, T) = $e^{-0.08}$(36 - 32)

4 - $e^{-0.08}$(4) = **P(K, T) = 0.3075346145**

**Problem PCP2.** The price for a prepaid forward contract for widgets expiring in one year is 9500. A European call option for widgets expiring in one year and with a strike price of 9432 has a price of 283, while a European put option for widgets expiring in one year and with a strike price of 9432 has a price of 125. Find the annual continuously compounded interest rate.

**Solution PCP2.** C(K, T) - P(K, T) = $e^{-rT}$($F_{0,T}$ - K), i.e.,

C(K, T) - P(K, T) = $e^{-rT}F_{0,T}$ - $e^{-rT}$K

We have C(9432, 1) = 283 and P(9432, 1) = 125.

Furthermore, we have $e^{-rT}F_{0,T}$ = 9500 and T = 1.

Thus, 283 - 125 = 9500 - $e^{-r}$*9432

158 = 9500 - $e^{-r}$*9432

9342 = $e^{-r}$*9432

$e^{-r}$ =0.9904580153

r = -ln(0.9904580153) = **r = 0.0095878012 = 0.95878012%**

**Problem PCP3.** The price for a forward contract on actuarial textbooks expiring in four months is 700. At a certain strike price, the price of a European call option expiring in four months for actuarial textbooks in 129, while the price of a European put option is 567. The annual continuously compounded interest rate is 0.023. Find the strike price of the call and put options.

**Solution PCP3.** We note that T here is equal to 4/12 = 1/3.

C(K, T) - P(K, T) = $e^{-rT}$($F_{0,T}$ - K). We want to find K.

C(K, 1/3) = 129; P(K, 1/3) = 567; r = 0.023

129 - 567 = $e^{-0.023/3}$(700 - K)

-438 = $e^{-0.023/3}$(700 - K)

-441.3709053 = 700 - K

700 + 441.3709053 = **K = 1141.370905**. (Prices like this one are the reason why you might want to use free study materials like this guide!)

**Problem PCP4.** A European call option on impossible-to-open medicine bottles expiring in 17 months has price 45. A European put option on impossible-to-open medicine bottles with the same strike price and expiration date has price 93. Both options have a strike price of 420. The annual continuously compounded interest rate is 0.3445. Find the 17-month forward price of impossible-to-open medicine bottles.

**Solution PCP4.** We note that T here is equal to 17/12.

$C(K, T) - P(K, T) = e^{-rT}(F_{0,T} - K)$. We want to find $F_{0,T}$.

$C(420, 17/12) = 45$; $P(420, 17/12) = 93$; $r = 0.3445$; $K = 420$

Thus, $45 - 93 = e^{-(17/12)0.3445}(F_{0,T} - 420)$

$-48 = 0.6138272964(F_{0,T} - 420)$

$-78.19789098 = F_{0,T} - 420$

$F_{0,T} = 420 - 78.19789098 =$ $\mathbf{F_{0,T} = 341.802109.}$

**Problem PCP5.** The prepaid forward price of pies is 3.1415926535. The contracts expire in pi half-months. A European call option on pies expiring in pi half-months has a price of √(2), while a European put option with the identical strike price has a price of e. During this time of war, hyperinflation, and radicalism, the annual effective interest rate is 0.7071067812. Find the strike price of the call and put options.

**Solution PCP5.** We note that T here is equal to (3.1415926535/2)/12 = 0.1308996939

$C(K, T) - P(K, T) = PV_{0,T}(F_{0,T} - K)$. We want to find K.

We also note that $PV_{0,T} = 1.7071067812^{-0.1308996939} = 0.9323890126$

$C(K, 0.1308996939) = \sqrt{(2)}$; $P(K, 0.1308996939) = e$.

Thus, $\sqrt{(2)} - e = 3.1415926535 - 0.9323890126*K$

$-1.304068266 = 3.1415926535 - 0.9323890126*K$

$0.9323890126*K = 4.44566092$

**K = 4.768032291**

# Section 2

# Parity of Options on Stocks

The equation expressing put-call parity for European options on stocks is

$$\mathbf{C(K, T) - P(K, T) = [S_0 - PV_{0,T}(Div)] - e^{-rT}K}$$

With respect to the forward contract on the stock, the following relationship holds between forward price and stock price:

$$\mathbf{e^{-rT}F_{0,T} = [S_0 - PV_{0,T}(Div)]}$$

When dividends are paid on the basis of a continuously compounded rate d, then

$$\mathbf{S_0 - PV_{0,T}(Div) = S_0e^{-dT}}$$

and

$$\mathbf{C(K, T) - P(K, T) = S_0e^{-dT} - PV_{0,T}(K)}$$

**Explanation of Variables:**

K = strike price of the options.

T = time to expiration of the options.

C(K, T) = price of a European call with strike price K and time to expiration T.

P(K, T) = price of a European put with strike price K and time to expiration T.

$PV_{0,T}$ = the present value over the life of the options.

Div = the stream of dividends paid on the stock.

$e^{-rT}*F_{0,T}$ = prepaid forward price for the asset.

$e^{-rT}*K$= prepaid forward price for the strike.

r = the continuously compounded interest rate.

$S_0$ = the current stock price.

$F_{0,T}$ = forward price for the underlying asset (in this case, the stock).

d = the continuously compounded dividend yield.

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 9, p. 283.

**Note:** McDonald uses the equation form $C(K, T) = P(K, T) + [S_0 - PV_{0,T}(Div)] - e^{-rT}K$. I prefer subtracting the put price from the call price so that the equation can be treated analogously to the formula for put-call parity. McDonald also uses "delta" in place for d, but symbolic constraints do not permit me to do so here.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem POS1.** The stock price of Carl Menger, Inc., is 98 per share. The company will pay two dividends of 3 per share, one six months from now and the other one year from now. A call option on Carl Menger, Inc., stock expiring one year from now with a certain strike price has a price of 23. A put option with the same strike price and expiration date has a price of 12. The annual continuously compounded rate of interest is 0.056. Find the strike price of the options.

**Solution POS1.** $C(K, T) - P(K, T) = [S_0 - PV_{0,T}(Div)] - e^{-rT}K$

Here, $C(K, 1) = 23$; $P(K, 1) = 12$; $S_0 = 98$; $r = 0.056$. We want to find K.

We note that $PV_{0,T}(Div) = 3{*}e^{-0.056/2} + 3{*}e^{-0.056} = 5.753782508$

Thus, $23 - 12 = [98 - 5.753782508] - e^{-0.056}K$

$11 = 92.24621749 - e^{-0.056}K$

$81.24621749 = e^{-0.056}K$

$81.24621749 = 0.9455391359K$

**K = 85.92581143**

**Problem POS2.** Hayekian LLC pays dividends on its stock on a continuously compounded basis with annual dividend yield of 0.02. One share of Hayekian LLC stock is currently worth 43. The monthly effective interest rate is 0.003. A put option on Hayekian LLC stock with strike price 45 currently has price 5. Find the price of the call option on Hayekian LLC stock with strike price 45. Both the call and the put option expire in 15 months.

**Solution POS2.** Since we want to find C(K, T), it is in this case more useful to employ McDonald's form of the parity equation:

$C(K, T) = P(K, T) + S_0e^{-dT} - PV_{0,T}(K)$

Here, $T = 15/12 = 1.25$

We also note that the present value discount factor for a time of 15 months is $1.003^{-15}$ = 0.9560618851.

From our given information, P(45, 1.25) = 5; $S_0$ = 43; K = 45; d = 0.02

Thus, C(45, 1.25) = 5 + $43e^{-0.02*1.25}$ - 0.9560618851*45

**C(45, 1.25) = 3.915541388**

**Problem POS3.** A one-year forward contract on the stock of Mises, Rothbard, & Associates has price 132. The annual effective interest rate is 0.054. Mises, Rothbard, & Associates pays monthly dividends of 2 per share, starting one month from now. Find the current price of a share of stock of Mises, Rothbard, & Associates.

**Solution POS3.** We use the equation $e^{-rT}F_{0,T} = [S_0 - PV_{0,T}(Div)]$, which we can transform into

$S_0 = e^{-rT}F_{0,T} + PV_{0,T}(Div)$ and solve for $S_0$.

We were not given a continuously compounded interest rate, but we note that $e^{-rT}$ is identical to $(1+i)^{-T}$, where i is the annual effective interest rate. Here, i = 0.054.

To find the present value of the dividend stream, we first need to find the *monthly* effective interest rate m, which is $1.054^{1/12}$ - 1 = 0.0043923223.

Recalling all that fun financial mathematics from Exam 2 / FM, we can treat this dividend stream as an annuity-immediate for 12 time periods, paying 2 per time period, with effective interest rate per time period equal to m = 0.0043923223.

Thus, $PV_{0,1}(Div)$ = $2(1-1.0043923223^{-12})/0.0043923223$ = 23.32861453.

From the given information, $F_{0,1}$ = 132.

Hence, $1.054^{-1}*132 + 23.32861453 = S_0 =$ **148.5658062**

**Problem POS4.** The stocks of two corporations - Reliable, Inc., and Equally Reliable, Inc. - have identical prices of 934 per share today, but different dividend structures. Reliable, Inc. pays dividends of 4 every 2 months starting 2 months from now at an annual effective interest rate of 0.032. Equally Reliable, Inc., pays dividends on a continuously compounded basis. Find the annual continuously compounded dividend yield for the stock of Equally Reliable, Inc.

**Solution POS4.** Since the two stock prices are identical, we can use the equation

$S_0 - PV_{0,T}(Div) = S_0e^{-dT}$, with $S_0$ = 934.

We can let T equal anything we want, so - to avoid annuity calculations - we can let T = 1/6 and only involve one of Reliable's dividend payments in our present value calculations.

Thus, $PV_{0,1/6}(Div) = 1.032^{-1/6}*4 = 3.979055913$

Hence, $934 - 3.979055913 = 934e^{-d/6}$

Thus, $930.0209441/934 = e^{-d/6} = 0.9957397688$

$-6*\ln(0.9957397688) =$ **$d = 0.0256159909$**

**Problem POS5.** Hazlitt Enterprises has stock price of 275 per share. A call option on Hazlitt Enterprises expiring in one year and with strike price of 325 has price 59. The price of a call option on Bastiat Corp. stock having strike price of 23 and expiring in one year is 6. One put option on Bastiat Corp. stock with the same strike price and expiration date has exactly one-tenth the price of one put option on Hazlitt Enterprises. Hazlitt Enterprises pays no dividends, while Bastiat Corp. pays a single dividend of 38 at the end of the year (this company believes in returning the vast majority of its profits to its shareholders as soon as they accrue). The continuously compounded rate of interest is 0.17. Find the current price of one share of Bastiat Corp. stock.

**Solution POS5.** This is a multi-step problem; first we want to find the price of a Hazlitt Enterprises put option; then we will use this to determine the price of Bastiat Corp. stock.

For Hazlitt, we use the equation $C(K, T) - P(K, T) = [S_0 - PV_{0,T}(Div)] - e^{-rT}K$ and rearrange it, taking into consideration the fact that Hazlitt pays no dividends.

$P(K, T) = C(K, T) - S_0 + e^{-rT}K$

**For Hazlitt,** we have $T = 1$; $K = 325$; $C(325, 1) = 59$; $S_0 = 275$; $r = 0.17$.

Thus, $P(325, 1) = 59 - 275 + e^{-0.17}325 = 58.19106539$

A put option on Bastiat Corp. stock costs one-tenth of the price of a put option on Hazlitt Enterprises. Thus, **for Bastiat**, we have $P(23, 1) = 5.819106539$; $C(23, 1) = 6$; $K = 23$; $Div = 38$; $r = 0.17$.

We will rearrange the relevant equation to make it more convenient to find $S_0$:

$S_0 = C(K, T) - P(K, T) + PV_{0,T}(Div) + e^{-rT}K$

Since we are dealing with a continuously compounded rate of interest, $PV_{0,T}(Div) = e^{-0.17}38 = 32.05926303$. Moreover, $e^{-rT}K = e^{-0.17}23 = 19.40429078$

Thus, $S_0 = 6 - 5.819106539 + 32.05926303 + 19.40429078 =$ **$S_0 = 51.64444727$**.

# Section 3

# Conversions and Reverse Conversions

In this section, we will explore conversions and reverse conversions - first through conceptual multiple-choice questions and then by means of practice problems that combine these concepts with the formulas for put-call parity discussed in Section 2.

**Problem CRC1.** What does a **conversion** involve?

(a) Selling a call, selling a put, buying the stock

(b) Selling a call, buying a put, short-selling the stock

(c) Buying a call, buying a put, short-selling the stock

(d) Selling a call, buying a put, buying the stock

(e) Buying a call, selling a put, short-selling the stock

(f) Buying a call, selling a put, buying the stock

**Solution CRC1.** A **conversion** is a transaction that involves selling a call, buying a put, and buying the stock. So the answer is **(d).** A good way to concisely write this down and memorize it is via an ordered listing of two-letter abbreviations, where the first letter denotes the action and the second denotes the asset being bought or sold. In the first place, "B" stands for "buying" and "S" stands for "selling." In the second place, "C" stands for "call," "P" stands for "put," and "S" stands for "stock." So the way to describe a conversion using this notation is (SC, BP, BS). This is my original notation for it - meant to assist with memorization. Feel free to use it or any other device that helps you.

**Problem CRC2.** What does a **reverse conversion** involve?

(a) Selling a call, selling a put, buying the stock

(b) Selling a call, buying a put, short-selling the stock

(c) Buying a call, buying a put, short-selling the stock

(d) Selling a call, buying a put, buying the stock

(e) Buying a call, selling a put, short-selling the stock

(f) Buying a call, selling a put, buying the stock

**Solution CRC2.** A **reverse conversion** is, quite intuitively enough, the exact reverse of a conversion and involves buying a call, selling a put, and short-selling the stock. So the answer is **(e).** Using the notation defined above, a reverse conversion can be described as (BC, SP, SS).

**Problem CRC3.** What kind of financial vehicle does a conversion synthetically create?

(a) A share of stock

(b) A forward contract

(c) A prepaid forward contract

(d) A T-bill

(e) A call option

(f) A put option

(g) A swap

(h) A certificate of deposit

(i) A futures contract

(j) A Ponzi scheme

**Solution CRC3.** A conversion synthetically creates a **T-Bill.** T-Bills require investment but have (practically) no risk involved in the position. By means of selling a call, buying a put, and buying the stock, a conversion hedges all the risk and enables the position's owner to earn the equivalent of risk-free interest income due to the time value of money. So the answer is **(d).**

**Problem CRC4.** Amon-Ra wishes to purchase shares of and options contracts on Mythological Industries in order to create a synthetic T-Bill. At a strike price of 23, he sells a call option for 5.43, buys a put for 2.35, and buys the stock for 20. The options expire in one year, and Amon-Ra holds them until expiration, at which time one of them is exercised. Mythological Industries pays no dividends on its stocks. What is the annual continuously compounded rate of return that Amon-Ra earns on his investment?

**Solution CRC4.** The simplified formula for put-call parity in this situation is

$C(K, T) - P(K, T) = S_0 - e^{-rT}K$. We want to find the value of r, so we rearrange the formula as follows:
$e^{-rT}K = S_0 - C(K, T) + P(K, T)$.

Here, $C(K, T) = 5.43$; $P(K, T) = 2.35$; $S_0 = 20$; $T = 1$, and $K = 23$.

Thus, $e^{-r}23 = 20 - 5.43 + 2.35$

$e^{-r}23 = 16.92$

$e^{-r} = 0.7356521739$

$-\ln(0.7356521739) =$ **r = 0.3069978618 = 30.69978618%** (I want that as a risk-free rate of return; don't you?)

**Problem CRC5.** Quetzalcoatl undertakes a reverse conversion using shares of and options on Pyramids and Monuments, Inc. He pays an annual effective rate of interest of 0.03 for this transaction - in which he stays for 5 months. Pyramids and Monuments, Inc. pays no dividends on its stock. Quetzalcoatl short-sells one share of stock at a price of 99 per share. He buys a call option at 6.57. How much money does he get from selling the put option? Both options have a strike price of 96 and expire 5 months from now.

**Solution CRC5.** The simplified formula for put-call parity in this situation is

$C(K, T) - P(K, T) = S_0 - e^{-rT}K$. We want to find $P(K, T)$, so we rearrange the formula as follows: $P(K, T) = C(K, T) - S_0 + e^{-rT}K$. Furthermore, we have $T = 5/12$ and our interest rate expressed as an annual effective rate rather than a continuously compounded rate. Thus, our present-value factor for K is $(1.03)^{-5/12} = 0.987759366$. Furthermore, $C(K, T) = 6.57$, $S_0 = 99$, and $K = 96$.

Thus, $P(K, T) = 6.57 - 99 + 0.987759366*96 =$ **P(K, T) = 2.394899134**.

# Section 4

# Parity of Options on Currencies

The formula for parity of options on currencies is

$C(K, T) - P(K, T) = x_0e^{-uT} - Ke^{-rT}$

**Explanation of Variables:**

K = strike price of the options.

T = time to expiration of the options.

C(K, T) = price of a European call with strike price K and time to expiration T.

P(K, T) = price of a European put with strike price K and time to expiration T.

$e^{-rT}*K$= prepaid forward price for the strike.

r = the continuously compounded interest rate, denominated in dollars or the currency of one's choice

u = the continuously compounded interest rate, denominated in euros or some other currency of one's choice

$x_0$ = the current exchange rate, denominated in dollars/euro or in [currency 1]/[currency 2].

$F_{0,T}$ = forward price for the underlying asset.

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 9, p. 286.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem POC1.** One euro currently trades for 1.4831 U.S. dollars. The annual dollar-denominated interest rate is 0.01% (due to the Federal Reserve's incessant artificial credit expansions and manipulations of the money supply), while the annual euro-denominated interest rate is 3%. Both interest rates are on a continuously compounded basis. A euro call option with strike price $1.34 has price $0.34. What is the price a euro put option with the same strike price? Both options expire in one year.

**Solution POC1.** We use the equation $C(K, T) - P(K, T) = x_0e^{-uT} - Ke^{-rT}$

and rearrange it as follows: $P(K, T) = C(K, T) - x_0e^{-uT} + Ke^{-rT}$

Here, $C(K, T) = 0.34$, $K = 1.34$, $r = 0.0001$, $u = 0.03$, $x_0 = 1.4831$, $T = 1$.

Thus, $P(K, T) = 0.34 - 1.4831*e^{-0.03} + 1.34e^{-0.0001}$ = **P(K, T) = 0.2405982359**

**Problem POC2.** One Terlefian nivand (TNV) currently trades for 89 Roblanian dufords (RD) (Yes, the names are randomly generated.). A call option on nivands with strike price of 100 RD is currently worth 8.93 RD, while a TNV put option sells for 4.35 RD. The options expire in six months. The annual continuously compounded TNV-denominated interest rate is 0.044. What is the annual continuously compounded RD-denominated interest rate?

**Solution POC2.** One of the purposes of this problem is to show that the currencies in question need not be dollars or euros; the formula still applies, but we need to be careful about identifying which rate is u and which rate is r. Here, the TNV is the asset behind the call and options, whose prices are expressed in RD. So, RD is the currency analogous to the dollar. Thus, *r* is the rate we seek. We know that $u = 0.044$, $C(K, T) = 8.93$, $P(K, T) = 4.35$, $x_0 = 89$, $K = 100$.

We use the equation $C(K, T) - P(K, T) = x_0e^{-uT} - Ke^{-rT}$

and rearrange it as follows: $P(K, T) - C(K, T) + x_0e^{-uT} = Ke^{-rT}$

Thus, $4.35 - 8.93 + 89e^{-0.044/2} = 82.48338092 = Ke^{-rT}$.

$82.48338092 = 100e^{-r/2}$

$0.8248338092 = e^{-r/2}$

$-2\ln(0.8248338092)$ = **r = 0.3851467127** (either the Roblanian government is hyperinflating the duford, or the Roblanians tend to have a *really* high rate of time preference. In either case, you would probably be better off holding Terlefian nivands.)

**Problem POC3.** In the aftermath of Robert Mugabe's government-induced hyperinflation, One U. S. dollar (USD) currently trades for 30000 Zimbabwe dollars (ZWD). An oppressed Zimbabwean citizen wishes to get rid of his Zimbabwe dollars and trade them for the safer U. S. dollars. However, he does not yet know the annual continuously compounded USD-denominated interest rate. The annual effective ZWD interest rate is 1004. (No, this is not a typo. The *inflation rate alone in Zimbabwe has exceeded 100000%* – the highest in history!) Help the oppressed Zimbabwean figure out the annual continuously compounded USD-denominated interest rate using the following information. The price of a dollar call with a strike price of 834000 ZWD is 28000 ZWD, while the price of a dollar put with the same strike price is 2 ZWD. The options expire in one year.

**Solution POC3.** This is partly a trick question. Even though the rate we want to find is US-dollar-denominated, it is actually *u* in our formula, since the *USD* is the asset behind the options.

We use the equation $C(K, T) - P(K, T) = x_0e^{-uT} - Ke^{-rT}$ and rearrange it as follows:

$x_0e^{-uT} = C(K, T) - P(K, T) + K/(1+i)$, where i = 1004 is the annual effective ZWD-denominated interest rate.

Here, $C(K, T) = 28000$ and $P(K, T) = 2$. $K = 834000$ and $x_0 = 30000$. $T = 1$.

Thus, $30000e^{-u} = 28000 - 2 + 834000/1005$.

$30000e^{-u} = 28828.85075$

$e^{-u} = 0.9609616915$

$-\ln(0.9609616915) =$ **u = 0.0398207339**

**Problem POC4.** You want to exchange fonbats for lomteds, but you do not know the exchange rate between them. You do know that a lomted call costs 34 fonbats, and a lomted put costs 35 fonbats. Both options have a strike price of 650 and expire three months from now. The lomted-denominated annual effective interest rate is 0.04, while the fonbat-denominated annual continuously compounded interest is 0.06. What is the exchange rate in fonbats per lomted?

**Solution POC4.**

In this case, we are given the continuously compounded interest rate r, but instead of u, we have an annual effective interest rate i.

We use the equation $C(K, T) - P(K, T) = x_0e^{-uT} - Ke^{-rT}$ and rearrange it as follows:
$x_0(1+i)^{-T} = C(K, T) - P(K, T) + Ke^{-rT}$
Here, $C(K, T) = 34$; $P(K, T) = 35$; $K = 650$; $T = ¼$; $r = 0.06$; $i = 0.04$. Thus,
$x_0(1.04)^{-1/4} = 34 - 35 + 650e^{-0.06/4}$
$x_0(1.04)^{-1/4} = 639.3227607$
$639.3227607/(1.04)^{-1/4} =$ **$x_0$ = 645.6222678 fonbats/lomted**

**Problem POC5.** Tabmids are exchanged for fillomtams at a rate of 0.03 tabmids per fillomtam. The prepaid forward price for the strike on fillomtam options is 0.02. The semiannual effective fillomtam-denominated interest rate is 0.34. A fillomtam put expiring 45 months from now with the strike price in question has price 0.02. Find the price of a fillomtam call with the same strike price and time to expiration.

**Solution POC5.** We use the equation $C(K, T) - P(K, T) = x_0e^{-uT} - Ke^{-rT}$ and rearrange it as follows: $C(K, T) = x_0e^{-uT} - Ke^{-rT} + P(K, T)$.

Here, $Ke^{-rT} = 0.02$; $P(K, T) = 0.02$; $T = 45/12$; $x_0 = 0.03$. We multiply $x_0$ by $(1.34)^{2(-45/12)}$ to get the appropriate present value factor.

Thus, $C(K, T) = (1.34)^{2(-45/12)}0.03 - 0.02 + 0.02 =$ **C(K, T) = 0.003340683**

# Section 5

# Parity of Options on Bonds

The formula for parity of options on bonds is

$C(K, T) - P(K, T) = [B_0 - PV_{0,T}(\text{Coupons})] - PV_{0,T}(K)$

**Explanation of Variables:**

K = strike price of the options.

T = time to expiration of the options.

C(K, T) = price of a European call with strike price K and time to expiration T.

P(K, T) = price of a European put with strike price K and time to expiration T.

$B_0$ = bond price.

$PV_{0,T}$ (Coupons) = present value of the bond's coupons.

$PV_{0,T}$ (K) = present value of the strike price.

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 9, p. 286.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem POB1.** A zero-coupon bond issued by Indestructible Co. currently sells for \$67. The annual effective interest rate is 0.04. A call on the bond with a strike price of \$80 expiring in 9 years sells for \$11.56. Find the price of a put option on the bond with the same strike price and time to maturity.

**Solution POB1.** We use the formula $C(K, T) - P(K, T) = [B_0 - PV_{0,T}(\text{Coupons})] - PV_{0,T}(K)$

and rearrange it as follows, taking into account the absence of coupons:
$C(K, T) - B_0 + PV_{0,T}(K) = P(K, T)$.

Here, $C(K, T) = 11.56$, $B_0 = 67$, $T = 9$, $K = 80$, and $PV_{0,T}(K) = 1.04^{-9} \cdot 80 = 56.20693885$.

Thus, $11.56 - 67 + 56.20693885 =$ $\mathbf{P(K, T) = 0.7669388463}$

**Problem POB2.** A certain bond issued by Volatile Industries pays annual coupons of $10 for 10 years. The annual effective interest rate is 0.03. A call on the bond with a strike price of $200 expiring in 10 years sells for $20. A put option on the same bond with the same strike price and time to maturity sells for $3. Find the price of the bond.

**Solution POB2.** We use the formula $C(K, T) - P(K, T) = [B_0 - PV_{0,T}(\text{Coupons})] - PV_{0,T}(K)$

and rearrange it as follows:

$B_0 = C(K, T) - P(K, T) + PV_{0,T}(\text{Coupons})] + PV_{0,T}(K)$

We are given that C(K, T) = 20, P(K, T) = 3, K = 200.

We calculate $PV_{0,T}(K) = 1.03^{-10}*200 = 148.818783$

We calculate $PV_{0,T}(\text{Coupons})$, which we consider as the present value of an annuity paying $10 per period for ten periods. Thus, $PV_{0,T}(\text{Coupons}) = 10(1-1.03^{-10})/0.03 = 85.30202837$

Thus, $C(K, T) - P(K, T) + PV_{0,T}(\text{Coupons})] + PV_{0,T}(K) = 20 - 3 + 85.30202837 + 148.818783 =$

$\mathbf{B_0 = 251.1208114}$

**Problem POB3.** Irregular LLC issues coupon bonds which pay an annual coupon of X for 76 years. The annual continuously compounded interest rate is 0.05. The bond currently sells for 87. A call option on the bond with strike price 200 expiring in 76 years has price 5. A put option with the same strike price and time to expiration has price 2. Find X.

**Solution POB3.** We use the formula $C(K, T) - P(K, T) = [B_0 - PV_{0,T}(\text{Coupons})] - PV_{0,T}(K)$

and rearrange it as follows:

$PV_{0,T}(\text{Coupons}) = B_0 - PV_{0,T}(K) - C(K, T) + P(K, T)$

$K = 200$; $T = 76$; $PV_{0,T}(K) = e^{-76*0.05}*200 = 4.474154371$

$C(K, T) = 5$; $P(K, T) = 2$; $B_0 = 87$.

Thus, $PV_{0,T}(\text{Coupons}) = 87 - 4.474154371 - 5 + 2 = 79.52584563$

$79.52584563 = X(1 - e^{-0.05*76})/(e^{0.05} - 1)$

Thus, $79.52584563 = 19.06784323X$

So $\mathbf{X = 4.170678596}$

**Problem POB4.** Amorphous Industries issues a bond with price 100 and annual coupons of 2, paid for 23 years. The annual effective interest rate is 0.04. A put option with a certain strike price and expiring in 23 years has price 5, whereas a call option with the same strike price and time to expiration has price 3. Find the strike price of both options.

**Solution POB4.** We use the formula $C(K, T) - P(K, T) = [B_0 - PV_{0,T}(\text{Coupons})] - PV_{0,T}(K)$

and rearrange it as follows:

$PV_{0,T}(K) = B_0 - PV_{0,T}(\text{Coupons}) - C(K, T) + P(K, T)$

Here, $PV_{0,T}(\text{Coupons}) = 2(1-1.04^{-23})/0.04 = 29.71368333$.

$C(K, T) = 3$; $P(K, T) = 5$; $T = 23$; $B_0 = 100$.

$PV_{0,T}(K) = 100 - 29.71368333 - 3 + 5 = 72.28631667$

Thus, $72.28631667 = (1.04^{-23})K$

$72.28631667/(1.04^{-23}) =$ $\mathbf{K = 178.1652082}$

**Problem POB5.** Multinational Corp. issues a bond that pays annual coupons of 5 and has a price of 90. A call option on the bond expiring in one year and with a strike price of 100 has price 6, while a put option with the same strike price and time to expiration has price 7. Find the annual effective interest rate.

**Solution POB5.** This problem would be notoriously messy if we had to consider more than one coupon payment period. But, fortunately, the coupons are annual and the time to expiration is one year. Thus, $T = 1$ and $PV_{0,T}(\text{Coupons}) = 5(1+i)^{-1}$, where i is the desired interest rate.

Furthermore, $PV_{0,T}(K) = 100(1+i)^{-1}$. Thus, $PV_{0,T}(\text{Coupons}) + PV_{0,T}(K) = 105(1+i)^{-1}$, which is a single term we can isolate on one side of our equation, which becomes

$PV_{0,T}(\text{Coupons}) + PV_{0,T}(K) = B_0 - C(K, T) + P(K, T)$

Here, $B_0 = 90$; $C(K, T) = 6$; $P(K, T) =7$.

Thus, $105(1+i)^{-1} = 90 - 6 + 7$

Thus, $105(1+i)^{-1} = 91$ and $91(1+i) = 105$

Hence, $i = 105/91 - 1 =$ $\mathbf{i = 0.1538461538}$

# Section 6

# Generalized Put-Call Parity

It is not necessary for options on an asset to be paid for with cash at expiration. Rather, the strike asset (the asset which is being offered in exchange for the asset on which the option is written) can be something else - a different stock, bond, widget, or hippopotamus. The generalized put-call parity equation expresses the relationship between puts and calls where the underlying asset and the strike asset can possibly be anything.

The equation for generalized put-call parity is

$C(S_t, Q_t, T-t) - P(S_t, Q_t, T-t) = F^P_{t,T}(S) - F^P_{t,T}(Q)$

Here, the assets under our consideration are

Asset A - the underlying asset - the asset on which the options are written and

Asset B - the strike asset - which we give up in return for the underlying asset.

Explanation of variables:
$F^P_{t,T}(S)$ = the time t price of a prepaid forward on Asset A, paying $S_T$ at time T.

$F^P_{t,T}(Q)$ = the time t price of a prepaid forward on Asset B, paying $Q_T$ at time T. Note that this is the analog of $PV_{0,T}(K)$ in our prior put-call parity formula, where K was the strike price of the underlying asset and the strike asset was cash.

$C(S_t, Q_t, T-t)$ = price of a European call option with underlying asset A, strike asset B, and time to expiration T-t.

$P(S_t, Q_t, T-t)$ = price of a European put option with underlying asset A, strike asset B, and time to expiration T-t.

At time T, the call payoff is $C(S_T, Q_T, 0) = \max(0, S_T - Q_T)$.

At time T, the put payoff is $P(S_T, Q_T, 0) = \max(0, Q_T - S_T)$.

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 9, p. 287.

**Problem GPCP1.** Bard Co. and Drab Co. have publicly traded stock, and option contracts are written on the stock of Bard Co. with the stock of Drab Co. as the strike asset. (One share of Drab Co. is the strike for an option written on one share of Bard Co.) Neither companies' stock pays any dividends. Bard Co. trades at $34 per share, and Drab Co. trades at $56 per share. The price of this kind of put option on Bard Co. is currently \$25. Find the price of a call option on Bard Co. Both options expire in 2 years.

**Solution GPCP1.** Neither companies' stock pays any dividends, so the stocks' prepaid forward prices equal the stock prices themselves.

Thus, we can use the formula $C(S_t, Q_t, T\text{-}t) - P(S_t, Q_t, T\text{-}t) = F^P_{t,T}(S) - F^P_{t,T}(Q)$,

where $F^P_{t,T}(S)$ is the price of Bard Co. stock, i.e., 34, and $F^P_{t,T}(Q)$ is the price of Drab Co. stock, i.e., 56, while $P(S_t, Q_t, T\text{-}t) = 25$. We note that *it does not matter when the options expire in this case*, because the stocks do not pay dividends, so their prepaid forward prices will always equal the stock prices.

Hence, we rearrange the equation as follows:
$C(S_t, Q_t, T\text{-}t) = F^P_{t,T}(S) - F^P_{t,T}(Q) + P(S_t, Q_t, T\text{-}t) = 34 - 56 + 25 =$ $\mathbf{C(S_t, Q_t, T\text{-}t) = 3}$

**Problem GPCP2.** The stock of Tram Law, Inc., currently trades at \$356 per share. One hippopotamus can be purchased for \$1000. Options on Tram Law stock are written, with hippopotami as the strike asset. (I told you it could be done!) The contracts stipulate that, at expiration, one hippopotamus can be exchanged for *three* shares of Tram Law, Inc. Neither Tram Law, Inc. nor a hippopotamus pays any dividends. A call option of this kind has time to expiration of 5 years and costs \$95. What is the price of a put option of this kind?

**Solution GPCP2.** Likewise, because neither the underlying asset nor the strike asset pay any dividends, the assets' prepaid forward prices equal their current prices. Furthermore, for this reason, time to expiration of these options does not matter - unless you are the person having to hold on to hippopotami for 5 years. We note, however, that the underlying asset is actually *three* shares of Tram Law, Inc., and this asset has a price of $356*3 = 1068 = F^P_{t,T}(S)$. $F^P_{t,T}(Q)$ is 1000, the price of one hippopotamus. $C(S_t, Q_t, T\text{-}t) = 95$.

We arrange the generalized put-call parity equation as follows.

$C(S_t, Q_t, T\text{-}t) - F^P_{t,T}(S) + F^P_{t,T}(Q) = P(S_t, Q_t, T\text{-}t)$

Thus, $95 - 1068 + 1000 =$ $\mathbf{P(S_t, Q_t, T\text{-}t) = 27}$

**Problem GPCP3.** Profligate, Inc., pays dividends on its stock; its continuously compounded dividend yield is 0.02. The price of a share of Profligate, Inc., stock is currently \$67. Misers and Associates pays no dividends on its stock; its current stock currently trades for \$95/share. A call option on Profligate, Inc., stock with Misers and Associates stock as the strike asset expires 13 months from now. The price of this call option is \$45. Find the price of a put option on Profligate, Inc., stock with the same time to expiration. (One share of Misers and Associates is the strike for an option written on one share of Profligate, Inc.)

**Solution GPCP3.** Since Profligate, Inc., pays dividends on its stock, its prepaid forward price will differ from the stock price. Namely, since time to expiration is 13/12 years, $F^P_{t,T}(S) = e^-$

$^{0.02*13/12}*67 = 65.56394676$. $F^P_{t,T}(Q)$ is still 95, since Misers and Associates pays no dividends. We are given that $C(S_t, Q_t, T\text{-}t) = 45$.

Thus, we can use the formula $C(S_t, Q_t, T\text{-}t) - P(S_t, Q_t, T\text{-}t) = F^P_{t,T}(S) - F^P_{t,T}(Q)$,

rearranging it as follows:
$P(S_t, Q_t, T\text{-}t) = C(S_t, Q_t, T\text{-}t) - F^P_{t,T}(S) + F^P_{t,T}(Q)$

Thus, $45 - 65.56394676 + 95 =$ $\mathbf{P(S_t, Q_t, T\text{-}t) = 74.43605324}$ (Profligate, Inc., stock is quite a volatile asset, indeed!)

**Problem GPCP4.** Hippopotami multiply at an annual continuously compounded rate of 0.1, whereas ostriches multiply at an annual continuously compounded rate of 0.25. One hippopotamus currently costs $2000, whereas one ostrich currently costs X. Call and put options are written on hippopotami, with one ostrich as the strike asset per one hippopotamus option. The call option currently costs $543, whereas the put option currently costs $324. Both options expire in 12 years. How much does one ostrich currently cost? (That is, find X).

**Solution GPCP4.** Did you think that hippopotami and ostriches could not pay dividends? It turns out that they can; here, their rate of multiplication is analogous to a continuously compounded dividend yield.

We can use the formula $C(S_t, Q_t, T\text{-}t) - P(S_t, Q_t, T\text{-}t) = F^P_{t,T}(S) - F^P_{t,T}(Q)$, noting that

$C(S_t, Q_t, T\text{-}t) = 543$ and $P(S_t, Q_t, T\text{-}t) = 324$; thus, $C(S_t, Q_t, T\text{-}t) - P(S_t, Q_t, T\text{-}t) = 219$.

Hippopotami are the underlying asset, so $F^P_{t,T}(S) = e^{-0.1*12}*2000 = 602.3884238$.

Ostriches are the strike asset, so $F^P_{t,T}(Q) = e^{-0.25*12}*X = 0.0497870684X$

Thus, $219 = 602.3884238 - 0.0497870684X$

So $602.3884238 - 219 = 383.3884238 = 0.0497870684X$

$383.3884238/0.0497870684 =$ $\mathbf{X = 7700.562337}$

**Problem GPCP5.** Superwidgets spontaneously replicate themselves at an annual effective rate of 0.05. One superwidget costs $356. Gold does not spontaneously replicate itself and currently costs $567 per ounce. A call option on superwidgets with gold as the strike asset currently costs $34. A put option on superwidgets with the same time to expiration costs $290. One ounce of gold is the strike for an option written on one superwidget. Determine the time to expiration of the call and put options.

**Solution GPCP5.** We can use the formula

$C(S_t, Q_t, T\text{-}t) - P(S_t, Q_t, T\text{-}t) = F^P_{t,T}(S) - F^P_{t,T}(Q)$, noting that

$C(S_t, Q_t, T-t) = 34$ and $P(S_t, Q_t, T-t) = 290$; so $C(S_t, Q_t, T-t) - P(S_t, Q_t, T-t) = -256$

Gold is the strike asset and pays no dividends, so the prepaid forward price is equal to the asset price. Thus, $F^P_{t,T}(Q) = 567$.

Superwidgets are the underlying asset, with time Y used in the present value factor. This is our time to expiration. $F^P_{t,T}(S) = (1.05)^{-Y}*356$

Thus, $-256 = (1.05)^{-Y}*356 - 567$

$311 = (1.05)^{-Y}*356$

$(1.05)^Y = 356/311 = 1.144694534$

$Y = \ln(1.144694534)/\ln(1.05) =$ **Y = 2.769775855 years**

# Section 7

# Classification of Calls and Puts

This is the generalized formula for currency options - relating calls denominated in one currency (say, the dollar or any other currency A of our choice) to puts denominated in another currency (say, the euro or any other currency B of our choice). We will use this formula in our exploration of how puts and calls are classified with relation to our choice of the underlying asset and the strike asset.

$C_A(x_0, K, T) = x_0 K P_B(1/x_0, 1/K, T)$

**Explanation of variables:**

K = strike price, denominated in currency A.

T = time to expiration of the options.

$x_0$ = exchange rate in terms of [currency A/currency B].

$C_A(x_0, K, T)$ = price of A-denominated call option on B (in units of currency A).

$P_B(1/x_0, 1/K, T)$ = price of B-denominated put option on A (in units of currency B).

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 9, p. 292.

**Problem CCP1.** Bard Co. stock and Drab Co. stock are used as underlying or strike assets in writing options contracts. Geoffrey owns an option contract with the following payoff: max(0, $S_{Bard} - S_{Drab}$). Which of the following are correct ways of describing this contract? More than one answer may be correct.

(a) A call option on Bard stock with Drab stock as the strike asset.

(b) A call option on Drab stock with Bard stock as the strike asset.

(c) A put option on Bard stock with Drab stock as the strike asset.

(d) A put option on Drab stock with Bard stock as the strike asset.

(e) A call option on both Bard and Drab stock.

(f) A put option on both Bard and Drab stock.

**Solution CCP1.**

For Geoffrey, if Bard is the underlying asset, then Geoffrey can gain at expiration if the price of Bard exceeds the price of Drab - so this option functions as a Bard call.

Someone aiming for the price of Drab to decline relative to the price of Bard can gain at expiration if the price of Drab is less than the price of Bard. So this option can also be interpreted as a put option on Drab stock. All that differs between the interpretations is the labeling of assets as underlying assets or strike assets. So **(a)** and **(d)** are both correct answers.

**Problem CCP2.** In this list, some two of the descriptions are equivalent, as are the remaining two descriptions. Find the pairs of equivalent descriptions.

(a) A put option on dollars, with XYQ stock as the strike asset.

(b) A call option on dollars, with XYQ stock as the strike asset.

(c) A call option on XYQ stock, with dollars as the strike asset.

(d) A put option on XYQ stock, with dollars as the strike asset.

**Solution CCP2.**

Both **(b) and (d)** refer to transactions in which you buy X dollars in exchange for one share of XYQ stock. So **(b) and (d)** are the same.

Both **(a) and (c)** refer to transactions in which you sell X dollars in exchange for one share of XYQ stock. So **(a) and (c)** are the same.

**Problem CCP3.** A 1-year dollar-denominated call option on euros with a strike price of $1.50 has a payoff of max(0, R - 1.50), where R is the exchange rate in dollars per euro one year from now. Determine the payoff of a 1-year euro-denominated put option on dollars.

**Solution CCP3.** Conceptually, a 1-year euro-denominated put option on dollars is identical to a 1-year dollar-denominated call option on euros; the only difference is the *scale* of the payoff for one contract in terms of the euro. The exchange rate one year from now in terms of euros per dollar will be 1/R, and the strike price will be 1/1.50 euros = 2/3 euros. Furthermore, on the put option, we gain whenever the exchange rate in terms of euros per dollar is *less* than the strike price.

Thus, the 1-year euro-denominated put option on dollars has a payoff of

**max(0, 2/3 - 1/R)**

**Problem CCP4.** A 1-year dollar-denominated call option on euros with a strike price of $1.50 has a payoff of max(0, R - 1.50), where R is the exchange rate in dollars per euro one year from now. The current exchange rate is $1.52 per euro. The premium on one dollar-denominated call option on euros is $0.04. Calculate the premium in euros on one euro-denominated put option on dollars.

**Solution CCP4.** We first determine the cost of buying 1/1.5 1-year dollar-denominated call options, which is (1/1.5)*0.04 = $0.02666666666667. We need to convert this to euros - and the result will be the cost of buying one 1-year euro-denominated put options.

$0.02666666666667*(1/1.52)euros per dollar = the premium in euros on one euro-denominated put option on dollars = **0.0175438596 euros**

**Problem CCP5.** A 1-year call option on Terlefian nivands (TNV), denominated in Roblanian dufords (RD), costs 34 RD. This option has a strike price of 356 RD. The exchange rate between the currencies is 350 RD/TNV. Find the price in terms of TNV of one 1-year put option on RD, denominated in TNV.

**Solution CCP5.** We use the formula $C_A(x_0, K, T) = x_0\ K\ P_B(1/x_0, 1/K, T)$, with A = RD and B = TNV. Here, $C_A(x_0, K, T) = 34$, $x_0 = 350$, and K = 356.

So $34 = 350*356*\ P_B(1/x_0, 1/K, T)$ and $P_B(1/x_0, 1/K, T) = 34/(350*356) =$ **0.0002728731942 TNV**

# Section 8

# Maximum and Minimum Option Prices

Constraints on prices of American and European call options:

$S \geq C_{Amer}(S, K, T) \geq C_{Eur}(S, K, T) \geq \max[0, PV_{0,T}(F_{0,T}) - PV_{0,T}(K)]$

Constraints on prices of American and European put options:

$K \geq P_{Amer}(S, K, T) \geq P_{Eur}(S, K, T) \geq \max[0, PV_{0,T}(K) - PV_{0,T}(F_{0,T})]$

**Meaning of variables:**
K = strike price.

T = time to expiration.

S = price of the stock.

$C_{Amer}(S, K, T)$ = price of American call.

$C_{Eur}(S, K, T)$ = price of European call.

$P_{Amer}(S, K, T)$ = price of American put.

$P_{Eur}(S, K, T)$ = price of European put.

$PV_{0,T}(K)$ = present value of the strike price.

$PV_{0,T}(F_{0,T})$ = prepaid forward price for the stock.

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 9, p. 293-294.

**Problem MMOP1.** Sagacious Co. stock currently sells for $562 per share. Both American and European call options are written on Sagacious Co. stock for a strike price of $550. The options expire one year from now. The annual effective interest rate is 0.09. The prepaid forward price for Sagacious Co. stock is $546 (the forward contract expires one year from now). Which of these are possible prices for the American and European call options? More than one correct answer is possible.

(a) $C_{Amer}(S, K, T)$ = $256, $C_{Eur}(S, K, T)$ = $295
(b) $C_{Amer}(S, K, T)$ = $56, $C_{Eur}(S, K, T)$ = $35
(c) $C_{Amer}(S, K, T)$ = $45, $C_{Eur}(S, K, T)$ = $43

(d) $C_{Amer}(S, K, T)$ = \$990, $C_{Eur}(S, K, T)$ = \$270
(e) $C_{Amer}(S, K, T)$ = \$41.5, $C_{Eur}(S, K, T)$ = \$43

**Solution MMOP1.** We note that T = 1 and $PV_{0,T}(K) = 550*1.09^{-1} = 504.587156$.

So $PV_{0,T}(F_{0,T}) - PV_{0,T}(K) = 546 - 504.587156 = 41.41284404$.

Thus, 41.41284404 is the lower bound on both call prices.

This rules out answer (b), since there the European call is priced at \$35.

Furthermore, in (a) and in (e), the European call is more expensive than the American call, which is impossible.

In (d), the American call's price exceeds S = 562, which is impossible.

Answer (c) has both the American and European call price within our upper and lower bounds and has the American call priced greater than the European call. So **(c)** is indeed a possible case.

**Problem MMOP2.** Imperious LLC stock currently sells for \$95 per share. Both American and European put options are written on Imperious LLC stock for a strike price of \$102. The options expire 17 months from now. The annual continuously compounded interest rate is 0.06. The 17-month forward price of Imperious LLC stock is \$104. Which of these are possible prices for the American and European put options? More than one correct answer is possible.

(a) $P_{Amer}(S, K, T) = 103$, $P_{Eur}(S, K, T) = 102$
(b) $P_{Amer}(S, K, T) = 0.00005$, $P_{Eur}(S, K, T) = 0.00003$
(c) $P_{Amer}(S, K, T) = 86$, $P_{Eur}(S, K, T) = 87$
(d) $P_{Amer}(S, K, T) = 102$, $P_{Eur}(S, K, T) = 102$
(e) $P_{Amer}(S, K, T) = 30$, $P_{Eur}(S, K, T) = 34$

**Solution MMOP2.** The strike price - 102 - is the upper bound on our put option values.

Now we determine the lower bound: $PV_{0,T}(K) = e^{(-17/12)0.06}102 = 93.68825301$

$PV_{0,T}(F_{0,T}) = e^{(-17/12)0.06}104 = 95.52527758$

So $PV_{0,T}(K) - PV_{0,T}(F_{0,T}) = 93.68825301 - 95.52527758 < 0$, so

$\max[0, PV_{0,T}(K) - PV_{0,T}(F_{0,T})] = 0$ and the lower bound is 0.

For (a), the American put price exceeds the strike price, which is impossible.

For (c) and (e), the European put price exceeds the American put price, which is impossible.

But with (b), because the lower bound on the put price is 0, even these very low put prices are possible - since the American put price exceeds the European put price. Furthermore, we recall that our inequality looks as follows:

$K \geq P_{Amer}(S, K, T) \geq P_{Eur}(S, K, T) \geq \max[0, PV_{0,T}(K) - PV_{0,T}(F_{0,T})]$

So it is permissible for the strike price to be *equal to* the American put price and the European put price, and (d) is a possibility. So both **(b) and (d)** are correct answers.

**Problem MMOP3.** Devious Co. stock currently sells for $43 per share. Both American and European call and put options are written on Devious Co. stock for a strike price of $56. The options expire 1 year from now. The annual continuously compounded interest rate is 0.10. The one-year forward price of Devious Co. stock is $50. Which of these are possible prices for the American call and European put options? More than one correct answer is possible.

(a) $C_{Amer}(S, K, T) = 47$; $P_{Eur}(S, K, T) = 45$
(b) $C_{Amer}(S, K, T) = 42$; $P_{Eur}(S, K, T) = 2$
(c) $C_{Amer}(S, K, T) = 9$; $P_{Eur}(S, K, T) = 60$
(d) $C_{Amer}(S, K, T) = 3$; $P_{Eur}(S, K, T) = 5.5$
(e) $C_{Amer}(S, K, T) = 32$; $P_{Eur}(S, K, T) = 53$

**Solution MMOP3.** Here, we use both inequalities:

$S \geq C_{Amer}(S, K, T) \geq C_{Eur}(S, K, T) \geq \max[0, PV_{0,T}(F_{0,T}) - PV_{0,T}(K)]$

$K \geq P_{Amer}(S, K, T) \geq P_{Eur}(S, K, T) \geq \max[0, PV_{0,T}(K) - PV_{0,T}(F_{0,T})]$

For the call options, the upper bound is the stock price 43.

For the put options, the upper bound is the strike price 56.

$PV_{0,T}(F_{0,T}) = e^{-0.10}*50 = 45.2418709$

$PV_{0,T}(K) = e^{-0.10}*56 = 50.67089541$

Here, $PV_{0,T}(K) > PV_{0,T}(F_{0,T})$, so the lower bound on the call price is 0, while the lower bound on the put price is 50.67089541 - 45.2418709 = 5.42902451

(a) is impossible because the call price exceeds the upper bound of 43.

(b) is impossible because the put price is less than the lower bound of 5.42902451.

(c) is impossible because the put price exceeds the upper bound of 56.

(d) has the call price between 0 and 43 and the put price between 5.42902451 and 56, so (d) is possible.

(e) has the call price between 0 and 43 and the put price between 5.42902451 and 56, so (e) is possible.

So **(d) and (e)** are correct answers.

Note: It is possible for an American *call* to be less expensive than a European *put,* and vice versa, as this problem illustrates.

**Problem MMOP4.** Highly Predictable, Inc. pays no dividends on its stock. The stock currently sells for $1231 per share. You know that a certain put option on Highly Predictable, Inc., stock cannot have a price less than $32. The option expires in 43 years, and the annual effective interest rate is 0.02. What is the strike price of this put option?

**Solution MMOP4.** By our inequality

$K \geq P_{Amer}(S, K, T) \geq P_{Eur}(S, K, T) \geq \max[0, PV_{0,T}(K) - PV_{0,T}(F_{0,T})]$, we know that our lower bound $\max[0, PV_{0,T}(K) - PV_{0,T}(F_{0,T})] = 32$. Moreover, 0 is not 32, so

$PV_{0,T}(K) - PV_{0,T}(F_{0,T}) = 32$.

Since Highly Predictable, Inc. pays no dividends, the prepaid forward price is equal to the stock price. Thus, $PV_{0,T}(F_{0,T}) = 1231$. Thus, $PV_{0,T}(K) = 32 + 1231 = 1263$.

So $1263 = 1.02^{-43}*K$ and $1.02^{43}*1263 =$ **K = $2959.448156**

**Problem MMOP5.** A European put option on Transparency, Ltd., costs the same as an American call option on this firm. We know that the European put is 5 times more expensive than its lowest possible value, which is nonzero. We know that the American call is $23 more expensive than the European call, and that the European call is one-seventh of the stock price. All options expire in one year and have a strike price of $80. The annual continuously compounded interest rate is 0.04. A prepaid 1-year forward contract on one share of Transparency, Ltd., stock costs $70. Find the price of one share of Transparency, Ltd., stock.

**Solution MMOP5.** The lowest possible value of the European put option is

$PV_{0,T}(K) - PV_{0,T}(F_{0,T}) = e^{-0.04}*80 - 70 = 6.863155132$.

The price of the European put and American call is thus 5*6.863155132 = 34.31577566.

The price of the European call is 34.31577566 - 23 = 11.31577566.

So the price of one share of stock is 11.31577566*7 = **$79.21042962**

# Section 9

# Early Exercise on American Options

Early exercise is never optimal for American call options on non-dividend-paying stock. But it can be optimal for a dividend-paying stock just before dividends are paid.

Early exercise for American calls is not optimal at any time where $K - PV_{t,T}(K) > PV_{t,T}(Div)$.

But early exercise *can be* optimal if $PV_{t,T}(Div) \geq K - PV_{t,T}(K)$. The best time to exercise in this case is just before the ex-dividend date so as to have the benefit of receiving all the interest accumulated prior to that time.

Early exercise for American puts can be optimal even on non-dividend-paying stock. If the interest rate is positive, exercising early will enable one to receive the strike price K now, whereas the value of holding on to the put is only $PV_{t,T}(K) < K$.

But an American put will *never* be exercised if $P > K - S$. If $P \leq K - S$, early exercise is possible. The possibility of early exercise does not *necessarily* imply that doing so will be optimal whenever $P \leq K - S$ is the case.

**Explanation of Variables:**
P = American put price.
K = strike price.
t = the present time, from our viewpoint.
T = time at expiration.
$PV_{t,T}(Div)$ = present value of dividends.
$PV_{t,T}(K)$ = present value of strike price.
S = stock price.

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 9, p. 294-297.

**Problem EEAO1.** Oblivious Co. pays monthly dividends of $4 per share, starting one month from now. American call options on Oblivious Co. are issued for a strike price of $23. The annual effective interest rate is 0.03. Is it ever optimal to exercise such American call options early? Demonstrate why or why not using the appropriate inequality.

**Solution EEAO1.** We compare $K - PV_{t,T}(K)$ with $PV_{t,T}(Div)$. Here, the *monthly* effective interest rate is $1.03^{1/12} - 1 = 0.00246627$, so $PV_{t,T}(Div) = 4(1 - 1.00246627^{-12})/0.00246627 = 47.239298667$.

On the other hand, $PV_{t,T}(K) = 23/1.03 = 22.33009709$.

So K - $PV_{t,T}$(K) = 23 - 22.33009709 = 0.6699029126.

0.6699029126 < 47.239298667, so K - $PV_{t,T}$(K) < $PV_{t,T}$(Div). Thus, **it may be optimal to exercise such options early.**

**Problem EEAO2.** Delirious, Inc., stock currently trades for $96 per share. Put options are written for a strike price of $100. What is the maximum price of the put options at which early exercise *might* be optimal?

**Solution EEAO2.** If P ≤ K - S, early exercise is possible. Here, K = 100 and S = 96. So if P ≤ 100 - 96, i.e., if P ≤ 4, early exercise is possible. Thus, the maximum put price at which early exercise might be optimal is **$4.**

**Problem EEAO3.** Insidious LLC pays annual dividends of $5 per share of stock, starting one year from now. For which of these strike prices of American call options on Insidious LLC stocks might early exercise be optimal? All of the call options expire in 5 years. More than one answer may be correct.

(a) $756
(b) $554
(c) $396
(d) $256
(e) $43
(f) $10

**Solution EEAO3.** We will compare K - $PV_{t,T}$(K) with $PV_{t,T}$(Div). We calculate that $PV_{t,T}$(Div) = $5(1-1.03^{-5})/0.03$ = 22.89853594. Whenever K - $PV_{t,T}$(K) < 22.89853594, early exercise may be optimal.

For (a), K - $PV_{t,T}$(K) = 756 -756/$1.03^5$ = 103.867759 > 22.89853594, so early exercise is never optimal.

For (b), K - $PV_{t,T}$(K) = 554 -554/$1.03^5$ = 76.11473345 > 22.89853594, so early exercise is never optimal.

For (c), K - $PV_{t,T}$(K) = 396 -396/$1.03^5$ = 54.40692138 > 22.89853594, so early exercise is never optimal.

For (d), K - $PV_{t,T}$(K) = 256 -256/$1.03^5$ = 35.1721512 > 22.89853594, so early exercise is never optimal.

For (e), K - $PV_{t,T}$(K) = 43 -43/$1.03^5$ = 5.907822271< 22.89853594, so early exercise is possible.

For (f), K = 10, so K - $PV_{t,T}$(K) < 10 < 22.89853594, so early exercise is possible.

So **(e) and (f)** are correct answers.

**Problem EEAO4.** Insidious LLC pays annual dividends of $5 per share of stock, starting one year from now and ending five years from now. Thereafter, Insidious LLC will never again pay dividends. For which of these strike prices and times to expiration of American call options on Insidious LLC stocks might early exercise be optimal? More than one answer may be correct.

(a) $756, 2 months
(b) $554, 4 months
(c) $396, 1 year
(d) $256, 2 years
(e) $43, 19 years
(f) $10, 123 years

**Solution EEAO4.** We will compare K - $PV_{t,T}$(K) with $PV_{t,T}$(Div). Just as in problem EEAO3, re calculate that $PV_{t,T}$(Div) = $5(1-1.03^{-5})/0.03$ = 22.89853594. Whenever K - $PV_{t,T}$(K) < 22.89853594, early exercise may be optimal.

For (a), K - $PV_{t,T}$(K) = $756 - 756/1.03^{1/6}$ = 3.71525004 < 22.89853594, so early exercise is possible.
For (b), K - $PV_{t,T}$(K) = $554 - 554/1.03^{1/3}$ = 5.431722337 < 22.89853594, so early exercise is possible.
For (c), K - $PV_{t,T}$(K) = 396 -396/1.03 = 11.53398058 < 22.89853594, so early exercise is possible.
For (d), K - $PV_{t,T}$(K) = $256 - 256/1.03^2$ = 14.69544726 < 22.89853594, so early exercise is possible.
For (e), K - $PV_{t,T}$(K) = $43 - 43/1.03^{19}$ = 18.47770085 < 22.89853594, so early exercise is possible.
For (f), K - $PV_{t,T}$(K) = $10 - 10/1.03^{123}$ = 9.736353894 < 22.89853594, so early exercise is possible.

So for all of these cases, early exercise is at least conceivable. Therefore, **all answers are correct.**

**Problem EEAO5.** American put options on Ingenious Co. currently cost $56 per option. We know that it is never optimal to exercise these options early. Which of these are possible values of the strike price K on these options and the stock price S of Ingenious Co. stock? More than one answer may be correct.

(a) S = 123, K = 124
(b) S = 430, K = 234
(c) S = 234, K = 430
(d) S = 1234, K = 1275
(e) S = 500, K = 600
(f) S = 850, K = 800

**Solution EEAO5.** Since early exercise is never optimal, we know that P > K - S, i.e., 56 > K - S.

For (a), K - S = 1 < 56, so (a) is a possible answer.

For (b), K - S = 234 - 430 < 0 < 56, so (b) is a possible answer.

For (c), K - S = 430 - 234 = 196 > 56, so (c) is not a possible answer.

For (d), K - S = 1275 - 1234 = 41 < 56, so (d) is a possible answer.

For (e), K - S = 600 - 500 = 100 > 56, so (e) is not a possible answer.

For (f), K - S = 800 - 850 = - 50 < 56, so (f) is a possible answer.

Thus, **(a), (b), (d) and (f)** are correct answers.

# Section 10

# Option Prices and Time to Expiration

**Principles:**

"An American call with more time to expiration is at least as valuable as an otherwise identical call with less time to expiration." Let t and T be times to expiration and $t < T$.

Then $C_{Amer}(K, T) \geq C_{Amer}(K, t)$.

"A longer-lived American put is always worth at least as much as an otherwise equivalent American put." Let t and T be times to expiration and $t < T$. Then $P_{Amer}(K, T) \geq P_{Amer}(K, t)$.

"A European call on a non-dividend-paying stock will be at least as valuable as an otherwise identical call with a shorter time to expiration." But with a dividend-paying stock, it may be possible for a shorter-lived European option to be worth more than a long-lived European option.

So if the stock does not pay dividends and t and T are times to expiration such that $t < T$, then $C_{Eur}(K, T) \geq C_{Eur}(K, t)$ and $P_{Eur}(K, T) \geq P_{Eur}(K, t)$. But if the stock pays dividends, this need not be the case.

**Problem OPTE1.** Which of these statements about call and put options on Representative Co. is always true? More than one answer may be correct.

(a) An American call with a strike price of $43 expiring in 2 years is worth at least as much as an American call with a strike price of $52 expiring in 1 year.
(b) An American call with a strike price of $43 expiring in 2 years is worth at least as much as an American call with a strike price of $43 expiring in 1 year.
(c) An American call with a strike price of $43 expiring in 2 years is worth at least as much as a European call with a strike price of $43 expiring in 2 years.
(d) An American put option with a strike price of $56 expiring in 3 years is worth at least as much as an American put option with a strike price of $56 expiring in 4 years.
(e) A European put option with a strike price of $56 expiring in 3 years is worth at least as much as a European put option with a strike price of $56 expiring in 2 years.

**Solution OPTE1.**

(b) is true, because $C_{Amer}(43, 2) \geq C_{Amer}(43, 1)$.

It is also the case that, of two call options with the same time to expiration, the option with the lower strike price is worth more - as the difference between the current price of the stock and the strike price of the option would be more positive for any current price of the stock.

Thus, $C_{Amer}(43, 1) \geq C_{Amer}(52, 1)$, and so $C_{Amer}(43, 2) \geq C_{Amer}(52, 1)$. This means that (a) is true.

(c) is true because $C_{Amer}(43, 2) \geq C_{Eur}(43, 2)$, from Section 8.

(d) is false; indeed, $P_{Amer}(56, 4) \geq P_{Amer}(56, 3)$.

(e) is false, because it may be the case that Representative Co. is bankrupt, in which case the put options will be worth the present value of the strike price; with the same strike price and longer time to expiration, the longer-term European put option will have a smaller present value factor by which the strike price is multiplied.

Thus, **(a),(b) and (c)** are correct answers.

**Problem OPTE2.** European put options are traded on the stock of Chronic Bankruptcy, Inc. Chronic Bankruptcy, Inc., is bankrupt and currently has a stock price of $0 per share. What is the price of a put option that has strike price $92 and expires 2 years from now? The annual continuously compounded interest rate is 0.21.

**Solution OPTE2.**

Since Chronic Bankruptcy, Inc., is bankrupt, the put option has price equal to the present value of the strike price. Thus, $PV_{0,T}(K) = 92e^{-0.21*2}$ = **P = 60.44830743**.

**Problem OPTE3.** European put options are traded on the stock of Chronic Bankruptcy, Inc. Chronic Bankruptcy, Inc., is bankrupt and currently has a stock price of $0 per share. The annual continuously compounded interest rate is 0.21. Which of these put options has the highest price? The strike price K is given for each option, as is the time to expiration T.
(a) K = 23; T = 0.3
(b) K = 95; T = 23

(c) K = 64; T = 2
(d) K = 42; T = 0.65
(e) K = 1000; T = 100
(f) K = 95; T = 21

**Solution OPTE3.**

(b) and (f) have the same strike price, but (f) expires sooner, so (f) has a higher price and we know that (b) is not the correct answer.

Since Chronic Bankruptcy, Inc., is bankrupt, the put option has price equal to the present value of the strike price.

For (a), $P = PV_{0,T}(K) = 23e^{-0.21*0.3} = 21.59569989$

For (c), $P = PV_{0,T}(K) = 64e^{-0.21*2} = 42.05099647$

For (d), $P = PV_{0,T}(K) = 42e^{-0.21*0.65} = 36.64106545$

For (e), $P = PV_{0,T}(K) = 1000e^{-0.21*100} = 7.582560428*10^{-7}$

For (f), $P = PV_{0,T}(K) = 95e^{-0.21*21} = 1.154741941$.

Of these values, 42.05099647 is the largest. So **(c)** is the correct answer.

**Problem OPTE4.** Imprudent Industries plans to pay a liquidating dividend of $20 in one year. Currently, European calls and puts on Imprudent Industries are traded with the following strike prices (K) and times to expiration (T). Which of the following European options will have the highest value? (Assume that, for T = 1, the options can be exercised just before the dividend is paid.) The annual effective interest rate is 0.03.

(a) Call; K = 3; T = 1.1
(b) Call; K = 19; T = 1
(c) Call; K = 19; T = 1.1
(d) Put; K = 10; T = 2
(e) Put; K = 10; T = 0.4

**Solution OPTE4.** Both (a) and (c) are worthless, because these calls can only be exercised once the company has liquidated itself and its stock price is 0. (b) can only be exercised immediately prior to liquidation, when the stock price will be equal to 20 (right before the $20 dividend is paid out). So profit on the option in one year will be $1 and the current price of (b) is $PV_{0,1}(1)$.

The price of (e) will exceed the price of (d), because both are present values of 10, except (e) involves a shorter time period and thus a larger present value factor.

The price of (e) is $10*1.03^{-0.4} = 9.882461023 > 1 > PV_{0,1}(1)$. So we know the (e) exceeds both (d) and (b). Thus, **(e)** is the correct answer.

**Problem OPTE5.** Gloom and Doom, Inc. will be bankrupt in 2 years, at which time it will pay a liquidating dividend of $30 per share. You own a European call option on Gloom and Doom, Inc., stock expiring in 2 years with a strike price of $20. The annual continuously compounded interest rate is 0.02. How much is the option currently worth? (Assume that, for T = 2, the option can be exercised just before the dividend is paid.)

**Solution OPTE5.** In 2 years, you will be able to exercise the option right before the dividend is paid. Then, the stock will be worth $30 and you will buy it for $20 and sell it for $30 right away, getting a $10 profit. This profit is a certainty. So the call price is the present value of $10, which is $10*e^{-0.02*2}$ = $**9.607894392**.

# Section 11

# Option Prices for Different Strike Prices

When the strike price of an option grows at the same rate as the interest rate, the premiums on European calls and puts on a non-dividend-paying stock increase as time to maturity increases.

Let T > t and let both T and t be times to expiration.

For call and put options where x is some time (and the present time is time 0),

$K_x = Ke^{rx}$, where r is the annual continuously compounded interest rate. That is, the option's strike price grows at the interest rate.

Then it is always the case that P(T) > P(t) and C(T) > C(t).

Given, two strike prices $K_1 < K_2$, the following relationships always hold with regard to European and American call and put prices $C(K_1)$, $C(K_2)$, $P(K_1)$, and $P(K_2)$, all other things being equal.

$C(K_1) \geq C(K_2)$

$P(K_2) \geq P(K_1)$

$C(K_1) - C(K_2) \leq K_2 - K_1$

$P(K_2) - P(K_1) \leq K_2 - K_1$

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 9, p. 298-300.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem OPDSP1.** Call options on Oblivious Co. are issued with a certain strike price that grows with the interest rate. Which of these times to maturity will result in the highest call premium? (A higher number indicates later time to maturity.)

(a) 2 months
(b) 4 months
(c) 1 year
(d) 2 years
(e) 2 weeks
(f) 366 days

**Solution OPDSP1.** The greatest time to maturity among these choices is given by (d): 2 years. Because the strike price grows with the interest rate, it is always the case that C(T) > C(t) when T > t. Thus, **(d)** is the correct answer.

**Problem OPDSP2.** Obsequious Co. stock options trade on the market, largely at two strike prices: 77 and 89. Which of these statements are true about the following American call and put options? More than one answer may be correct.

(a) $P(77) \leq P(89)$
(b) $P(89) - P(77) \geq 12$
(c) $C(77) \geq C(89)$
(d) $P(77) \geq P(89)$
(e) $C(89) - C(77) \geq 12$
(f) $C(77) - C(89) \geq 12$

**Solution OPDSP2.** Here, $K_1 = 77$ and $K_2 = 89$. (a) accords with the inequality $P(K_2) \geq P(K_1)$, so (a) is true. (b) violates the inequality $P(K_2) - P(K_1) \leq K_2 - K_1$ and so is false. (c) accords with the inequality $C(K_1) \geq C(K_2)$ and so is true. (d) violates the inequality $P(K_2) \geq P(K_1)$ and so is false. (e) violates the inequality $C(K_1) \geq C(K_2)$, since if $C(77) \geq C(89)$, then $C(89) - C(77)$ can be at most 0 and cannot be $\geq 12$. (f) violates with the inequality $C(K_1) - C(K_2) \leq K_2 - K_1$ and so is false. So **(a) and (c)** are correct answers.

**Problem OPDSP3.** The stock of Impervious LLC trades on the market. Call options on Impervious LLC stock with a strike price of \$54 have a premium of \$19. Which of these are possible values for call options on Impervious LLC stock with a strike price of \$32 and the same time to expiration? More than one answer may be correct.

(a) 16
(b) 20
(c) 34
(d) 25
(e) 45

(f) It is impossible to have call options for this stock with a strike price of \$32.

**Solution OPDSP3.** Here, we consider two constraints, knowing that $K_1 = 32$ and $K_2 = 54$. Since $C(K_1) \geq C(K_2)$, we know that $C(32) \geq 19$. Furthermore, $C(K_1) - C(K_2) \leq K_2 - K_1$

Implies that $C(32) - 19 \geq 54 - 32$, which is equivalent to $C(32) \leq 41$. So, answers (b), (c), and (d) are all correct, as they are between 19 and 41. Furthermore, (f) is incorrect, because while a put option's price cannot exceed its strike price, there is no such restriction on call option prices. So **(b), (c), and (d)** are the correct answers.

**Problem OPDSP4.** The stock of Impervious LLC trades on the market. Put options on Impervious LLC stock with a strike price of \$54 have a premium of \$19. Which of these are possible values for put options on Impervious LLC stock with a strike price of \$32 and the same time to expiration? More than one answer may be correct.

(a) 16
(b) 20
(c) 34
(d) 25
(e) 45
(f) It is impossible to have put options for this stock with a strike price of \$32.

**Solution OPDSP4.** Here, we consider two constraints, knowing that $K_1 = 32$ and $K_2 = 54$. We are given that $P(K_2) = 19$. Since $P(K_2) \geq P(K_1)$, we know that $19 \geq P(32)$.

Furthermore, $P(K_2) - P(K_1) \leq K_2 - K_1$ implies that $19 - P(32) \leq 54 – 32 \rightarrow 19 - P(32) \leq 22$, which is equivalent to $P(32) \geq 19 – 22 \rightarrow P(32) \geq -3$, and, in practice, $P(32) \geq 0$ (since option prices cannot be negative). Only (a) with a price of 16 fulfills these constraints. (f) is incorrect, because a price of 16 is well below the strike price of 32 and so is not inconsistent with any other boundaries on the possible put price. So **(a)** is the correct answer.

**Problem OPDSP5.** Call options on Loquacious, Inc., stock with a strike price of \$74 are currently twice as expensive as put options on Loquacious, Inc., stock with a strike price of \$74. Put options on Loquacious, Inc., stock with a strike price of \$54 are currently worth \$21. Which of these are possible prices for call options on Loquacious, Inc., stock with a strike price of \$54? More than one answer may be correct.

(a) 51
(b) 23
(c) 103
(d) 34
(e) It is impossible to have a consistent answer given the information in this problem.

**Solution OPDSP5.** Here, $K_1 = 54$ and $K_2 = 74$, and $K_2 - K_1 = 20$. Thus, we know that

$C(54) \geq C(74) = 2P(74) \geq 2P(54) = 42$. So $C(54) \geq 42$ and choices (b) and (d) are ruled out by this constraint. Moreover, we know that $P(K_2) - P(K_1) \leq K_2 - K_1$ , which implies that $P(74) - 21 \leq 20$ and so $P(74) \leq 41$ and so $C(74) \leq 82$. We also know that

$C(K_1) - C(K_2) \leq K_2 - K_1$, which implies that $C(54) - C(74) \leq 20$, so $C(54) \leq 20 + C(74) \leq 20 + 82$. Thus, $C(54) \leq 102$, ruling out choice (c). Only choice (a) = 51 is within the constrains $42 \leq C(54) \leq 102$. Thus, only **(a)** is the correct answer.

# Section 12

# Strike-Price Convexity

For strike prices $K_1 < K_2 < K_3$, the following inequalities express the **convexity** of the option price with regard to the strike price:

$[C(K_1) - C(K_2)]/[K_2 - K_1] \geq [C(K_2) - C(K_3)]/[K_3 - K_2]$

$[P(K_2) - P(K_1)]/[K_2 - K_1] \leq [P(K_3) - P(K_2)]/[K_3 - K_2]$

Where $C(K_x)$ and $P(K_x)$ are prices of American or European options with strike price $K_x$.

It is also possible to define $\lambda = [K_3 - K_2]/[K_3 - K_1]$ and rewrite the inequality for convexity with calls as

$C(K_2) \leq \lambda C(K_1) + (1- \lambda)C(K_3)$

A similar relationship holds for puts:

$P(K_2) \leq \lambda P(K_1) + (1- \lambda)P(K_3)$

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem SPC1.** Vacuous Co. stock options trade for three strike prices: \$46, \$32, and \$90. Determine $\lambda$ in the inequality for strike price convexity.

**Solution SPC1.** $\lambda = [K_3 - K_2]/[K_3 - K_1]$ Here, $K_1 = 32$, $K_2 = 46$, and $K_3 = 90$.

Thus, $\lambda = [90- 46]/[90- 32] =$ $\boldsymbol{\lambda = 0.7586296897}$

**Problem SPC2.** Call options on Voracious, Inc., stock trade for three strike prices: \$43, \$102, and \$231. The price of the \$43-strike option is \$56. The price of the \$231-strike option is \$23. What is the maximum possible price of the \$102-strike option? All options have the same time to expiration.

**Solution SPC2.** First, we find $\lambda = [K_3 - K_2]/[K_3 - K_1]$. Here, $K_1 = 43$, $K_2 = 102$, and $K_3 = 231$. Thus, $\lambda = [231- 102]/[231- 43] = 0.6861702128$.

We are given that $C(K_1) = 56$ and $C(K_3) = 23$. Thus, by the inequality

$C(K_2) \leq \lambda C(K_1) + (1- \lambda)C(K_3)$, $C(K_2) \leq 0.6861702128*56 + (1- 0.6861702128)*23 =$ 45.64361702. So the maximum price of the \$102-strike option is **\$45.64361702**.

**Problem SPC3.** Call options on Specious LLC stock trade for three strike prices, \$32, \$34, and \$23. The \$32-strike call currently costs \$10, while the \$34-strike call costs \$7. The \$23-strike call costs at least \$X. Find X.

**Solution SPC3.** First, we find $\lambda = [K_3 - K_2]/[K_3 - K_1]$. Here, $K_1 = 23$, $K_2 = 32$, and $K_3 = 34$. Thus, $\lambda = [34- 32]/[34- 23] = 0.1818181818$. We rearrange the inequality

$C(K_2) \leq \lambda C(K_1) + (1- \lambda)C(K_3)$ to look as follows:

$C(K_2) - (1- \lambda)C(K_3) \leq \lambda C(K_1)$

$[C(K_2) - (1- \lambda)C(K_3)]/\lambda \leq C(K_1)$

Thus, $C(K_1) \geq [10 - (1- 0.1818181818)7]/0.1818181818 = 23.5$. Thus, the \$23-strike call costs at least **\$23.5**.

**Problem SPC4.** Put options on Meritorious Co. trade for three strike prices: \$102, \$105, and X, which is the highest. λ is equal to 0.5. The \$102-strike put is worth \$20, the \$105 strike put is worth \$22, and the \$X-strike put is worth \$24. Find the value of X.

**Solution SPC4.** This problem contains a lot of excess information.

$\lambda = [K_3 - K_2]/[K_3 - K_1]$, so all we need to know is that $\lambda = 0.5$, $K_1 = 102$, and $K_2 = 105$.

Thus, $0.5 = (X - 105)/(X-102)$. Hence, $0.5X - 51 = X - 105$ and $0.5X = 54$.

Therefore, **X = \$108**.

**Problem SPC5.** Put options on Meritorious Co. trade for three strike prices: \$102, \$105, and \$112. The \$102-strike put is worth \$20, the \$105 strike put is worth \$22, and the \$112-strike put is worth at least \$F. Find the value of F.

**Solution SPC5.** Here, $K_1 = 102$, $K_2 = 105$, and $K_3 = 112$.

$\lambda = [K_3 - K_2]/[K_3 - K_1] = (112-105)/(112-102) = 0.7$

We rearrange the inequality $P(K_2) \leq \lambda P(K_1) + (1- \lambda)P(K_3)$ to look as follows:

$P(K_2) - \lambda P(K_1) \leq (1- \lambda)P(K_3)$

$[P(K_2) - \lambda P(K_1)]/(1- \lambda) \leq P(K_3)$

$[22 - 0.7*20]/0.3 = 26.66666667 \leq P(K_3)$. So **F = 26.66666667.**

# Section 13

# Exam-Style Questions on Put-Call Parity and Arbitrage

The problems in this section were designed to be similar to problems from past versions of Exam 3F / Exam MFE. They use original exam questions as their inspiration – and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

**Problem ESQPCP1.**

**Similar to Question 1 from the Society of Actuaries' Sample MFE Questions and Solutions:**

Subterranean, Inc., pays no dividends on its stock, which currently trades for $123 per share. You know that the European call option on Subterranean, Inc., stock is $3 more expensive than the European put option with the same strike price and time to expiration. The strike price of the options is $145. The options expire in 9 years. Calculate the annual continuously compounded risk-free interest rate.

Note: The test will give you five multiple-choice possibilities for answers to this kind of problem. See if you can do without them. I will only include multiple-choice possibilities when they are necessary to answering the question or would make doing so *more* difficult rather than less.

**Solution ESQPCP1.** We use the put-call parity formula

$C(K, T) - P(K, T) = [S_0 - PV_{0,T}(Div)] - e^{-rT}K$, which, with no dividends, simplifies to

$C(K, T) - P(K, T) = S_0 - e^{-rT}K$.

Here, $T = 9$, $C(K, T) - P(K, T) = 3$, $S_0 = 123$, $K = 145$, and we want to find r.

So $3 = 123 - 145e^{-9r}$

Thus, $145e^{-9r} = 120$ and $-9r = \ln(120/145)$.

Hence $-\ln(120/145)/9 =$ **$r = 0.0210268888$**

Now try the corresponding test question, if you have not done so already. Do this after you do each of the problems here.

**Problem ESQPCP2.**

**Similar to Question 13 from the Casualty Actuarial Society's Fall 2007 Exam 3:**

You are given the price of an underlying asset (currently \$300), the strike prices (K) of various American and European call options on this asset, their time to expiration (T), and their type. Given that none of them has the same price as any of the others, which of these options can be expected to have the highest price today?

(a) American, K = 326, T = 19.6 years
(b) European, K = 324, T = 17.3 years
(c) American, K = 324, T = 19.6 years
(d) European, K = 324, T = 19.6 years
(e) American, K = 324, T = 17.3 years

**Solution ESQPCP2.** We recall that American options are always at least as valuable as European options with the same strike price and time to expiration. So we know that (c) > (d) and (e) > (b). Furthermore, we recall that American options with longer time to expiration are always worth more than otherwise equivalent American options with shorter time to expiration. Thus, (c) > (e) > (b). Furthermore, call options with a smaller strike price are always worth more than otherwise equivalent call options with a larger strike price. Thus, (c) > (a) and the price of **(c)** is the largest one.

**Problem ESQPCP3.**

**Similar to Question 14 from the Casualty Actuarial Society's Fall 2007 Exam 3:**

Pernicious Co. will pay a \$99.23 dividend on each share of its stock in three years. Currently, Pernicious Co. stock is worth \$23000 per share. European call options on Pernicious Co. stock have the following properties: call price = \$234, strike price = \$95600, time to expiration = 9 years. The annual continuously compounded risk-free interest rate is 14%. Find the price of a European put option on Pernicious Co. stock with the same strike price and time to expiration.

**Solution ESQPCP3.**

We use the put-call parity equation $C(K, T) - P(K, T) = [S_0 - PV_{0,T}(Div)] - e^{-rT}K$, rearranging it as follows:
$P(K, T) = C(K, T) - S_0 + PV_{0,T}(Div) + e^{-rT}K$

Here, $r = 0.14$ and $PV_{0,T}(Div) = 99.23e^{-0.14*3} = 65.19875593$

$T = 9$, so $e^{-rT}K = 95600e^{-0.14*9} = 27117.32493$

Since $C(K, T) = 234$ and $S_0 = 23000$, it follows that

$P(K, T) = 234 - 23000 + 65.19875593 + 27117.32493 =$ $\mathbf{P(K, T) = \$4416.523689}$

**Problem ESQPCP4.**

**Similar to Question 15 from the Casualty Actuarial Society's [Fall 2007 Exam 3](#):**

Terlefian nivands (TNV) trade for Roblanian dufords (RD) at an exchange rate of 6 TNV/RD. A call option denominated in TNV expiring 4 years from now and having a strike price of 9 TNV currently trades for 1 TNV. A TNV-denominated put option with the same strike price and time to expiration currently trades for 3 TNV. The annual continuously-compounded risk-free rate on Terlefian nivands is 13%. Find the annual continuously-compounded risk-free rate on Roblanian dufords.

**Solution ESQPCP4.**

The formula for parity of options on currencies is

$C(K, T) - P(K, T) = x_0e^{-uT} - Ke^{-rT}$

We rearrange this equation: $x_0e^{-uT} = C(K, T) - P(K, T) + Ke^{-rT}$.

Here, T = 4, C(K, T) = 1, P(K, T) = 3, $x_0$ = 6, r = 0.13, and K = 9. We want to find u.

Thus, $6e^{-4u} = 1 - 3 + 9e^{-0.13*4}$

$6e^{-4u} = 3.350684932$

Hence, $e^{-4u} = 0.5584474886$

Thus, u = -ln(0.5584474886)/4 = **u = 0.1456486718**

**Problem ESQPCP5.**

**Similar to Question 2 from the Society of Actuaries' Sample MFE Questions and Solutions:**

For six European options on Invincible Co. stock, all expiring T years from now, you have the following information:

For strike price \$100, call price = \$21 and put price = \$5.

For strike price \$120, call price = \$14 and put price = \$7.

For strike price \$130, call price = \$12 and put price = \$10.

Carus thinks that an arbitrage opportunity is possible through purchasing one \$100-strike call option, selling 5 \$120-strike call options, and purchasing a certain number of \$130-strike call options – along with lending \$1.

Timdon thinks that an arbitrage opportunity is possible through purchasing one \$100-strike call and selling one \$100-strike put option, purchasing 6 \$120-strike puts and selling 6 \$120-strike calls, and purchasing some number of \$130-strike calls while selling the same number of \$130-strike puts – along with lending \$2.

Hingab believes that Carus and Timdon are sadly mistaken in thinking that their proposals would result in arbitrage profit.

Who is correct?

(a) Nobody is correct.

(b) Only Hingab is correct.

(c) Only Timdon is correct.

(d) Only Carus is correct.

(e) Both Timdon and Carus are correct.

**Solution ESQPCP5.**

One of the requirements for an arbitrage position is that it costs nothing on net to enter into. The second requirement is that it will make the owner a profit, irrespective of future price movements.

How might it be possible for Carus's series of transactions to cost nothing to enter into?

Purchasing one \$100-strike call costs \$21, while selling 5 \$120-strike call options gives him \$14*5 = \$70. He also buys X \$130-strike calls at a price of \$12 each and lends out \$1. So his net cost is -21 + 70 - 1 - 12X or 48 - 12X. In order for 48 - 12X to be 0, X must equal 4.

At expiration time T, given interest rate r, Carus will have the following gains.

If the stock price $S < 100$, all the calls are worthless and Carus only gets $e^{rT} > 0$ from lending the \$1.

If $100 \leq S < 120$, Carus gets $S – 100$ from the \$100-strike call, along with the $e^{rT}$ from lending the \$1. His profit is $S – 100 + e^{rT} > 0$.

If $120 \leq S < 130$, Carus gets $S – 100$ from the \$100-strike call, along with the $e^{rT}$ from lending the \$1. But he must also pay $5(S - 120)$ for his 5 sold \$120-strike calls. His net profit is $S – 100 + e^{rT} – 5S + 600 = 500 – 4S + e^{rT}$. Note that if r is small so that $e^{rT}$ is approximately 1, then it is possible (if, for instance, $S = 129$) for $4S$ to be greater than $500 + e^{rT}$ and for

$500 - 4S + e^{rT} < 0$. Hence, for some of the higher stock prices in the range $120 \leq S < 130$, Carus could actually lose money at expiration. Thus, Carus's idea of an arbitrage opportunity is not valid.

Now we consider Timdon's proposal:

How might it be possible for Timdon's series of transactions to cost nothing to enter into?

Purchasing one \$100-strike call and selling one \$100-strike put option costs 21 - 5 = \$16.

Purchasing 6 \$120-strike puts and selling 6 \$120-strike calls brings in 6(14 - 7) = \$42.

Lending \$2 costs \$2 at time = 0. This leaves a cost 42 - 16 - 2 = 24 to be accounted for in purchasing X \$130-strike calls and selling X \$130-strike puts. Doing so costs X(12 - 10) = 2X. Thus, 2X = 24 and X =12.

At expiration time T, depending on the stock price S, Timdon will have a payoff of:

> $2e^{rT}$ from lending \$2.
>
> (S - 100) from purchasing one \$100-strike call and selling one \$100-strike put option.
>
> 6(120 - S) for purchasing 6 \$120-strike puts and selling 6 \$120-strike calls.
>
> 12(S - 130) for purchasing 12 \$130-strike calls and selling 12 \$130-strike puts.

Thus, his total payoff is $2e^{rT} + S - 100 + 720 - 6S + 12S - 1560 = 2e^{rT} + 7S - 740$.

We note that in order for 7S to be greater than 740, S must be greater than 105.7142857. If S goes substantially below 105.7142857, it is possible that even the accumulated value of the lent-out \$2 will not be enough to preserve a profit for Timdon. Thus, this position is only profitable for S above a certain value (which is slightly less than 105.7142857, depending on the interest rate) and Timdon's proposal is not a genuine arbitrage opportunity.

So neither Carus nor Timdon is correct, meaning that Hingab is right to think that they have not stumbled upon arbitrage opportunities. Thus, **(b)** is the correct answer.

# Section 14

# Exam-Style Questions on Put-Call Parity and Arbitrage – Part 2

The problems in this section were designed to be similar to problems from past versions of Exam 3F / Exam MFE. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

**Problem MESQPCPA1.**

**Similar to Question 16 from the Casualty Actuarial Society's Fall 2007 Exam 3:**

The stock of Precarious Co. does not pay any dividends. It currently trades at $565 per share, and the annual continuously compounded interest rate is 20%. European call and put options on Precarious Co. stock are available with strike price of $596, expiring in 3 years. There are no arbitrage opportunities in the pricing of these options. Digtammar decides to purchase 782 call options and sell 782 put options on Precarious Co. stock. What is the net cost of this transaction?

**Solution MESQPCPA1.** We use the put-call parity formula

$C(K, T) - P(K, T) = [S_0 - PV_{0,T}(Div)] - e^{-rT}K$, which, with no dividends, simplifies to

$C(K, T) - P(K, T) = S_0 - e^{-rT}K$.

We note that Digtammar will pay $782[C(K, T) - P(K, T)]$ for this transaction, which is the same as $782[S_0 - e^{-rT}K] = 782[565 - e^{-0.2*3}596] = 186044.2631$. So the net cost of the transaction to Digtammar is **$186,044.2631** (i.e., this is the amount he pays).

Now try the corresponding test question, if you have not done so already. Do this after you do each of the problems here.

**Problem MESQPCPA2.**

**Similar to Question 3 from the Casualty Actuarial Society's Spring 2007 Exam 3:**

Spurious, Inc., stock currently trades for $63 per share. Call and put options on Spurious, Inc., stock with strike price of $76 and time to expiration of 17 months are currently available. The call price is $10, and the put price is $19. The annual continuous risk-free rate is 12%, while the annual continuous dividend yield is 7%. Tadart notices that arbitrage profit is possible under these conditions. Calculate the amount of arbitrage profit per share.

**Solution MESQPCPA2.** We use the put-call parity formula

$C(K, T) - P(K, T) = S_0e^{-dT} - e^{-rT}K$ and notice that here the equality need not hold. We calculate each side of the equation separately and then calculate the positive difference between them, which will be the arbitrage profit per share.

$C(K, T) - P(K, T) = 10 - 19 = -9$

$S_0e^{-dT} - e^{-rT}K = 63e^{-0.07*17/12} - 76e^{-0.12*17/12} = -7.066244962$

The positive difference between these two values is -7.066244962 - (-9) = **$1.933755038,** which is the arbitrage profit per share.

**Problem MESQPCPA3.**

**Similar to Question 4 from the Casualty Actuarial Society's Spring 2007 Exam 3:**

Tedious LLC will pay three dividends of $4 on its stock - one in 5 years, another in 19 years, and a third in 32 years. The current price of Tedious LLC stock is $101 per share. All continuously compounded risk free interest rates are 1%. The price of a European call option on Tedious LLC stock with a strike price of $157 and expiring in 41 years is $20. Find the price of a European put option on Tedious LLC stock with the same strike price and time to expiration.

**Solution MESQPCPA3.**

We use the put-call parity formula

$C(K, T) - P(K, T) = [S_0 - PV_{0,T}(Div)] - e^{-rT}K$, which we rearrange thus:
$P(K, T) = C(K, T) - S_0 + PV_{0,T}(Div) + e^{-rT}K$

Here, $C(K, T) = 20$ and $S_0 = 101$, while $e^{-rT}K = e^{-0.01*41}157 = 104.1930893$

Now we find $PV_{0,T}(Div) = 4(e^{-0.01*5} + e^{-0.01*19} + e^{-0.01*32}) = 10.01735038$

So P(K, T) = 20 - 101 + 104.1930893 + 10.01735038 = **P(K, T) = 33.21043968**

**Problem MESQPCPA4.**

**Similar to Question 12 from the Casualty Actuarial Society's Spring 2007 Exam 3:**

Evaluate the truth or falsehood of each of these propositions:
(a) The premiums for an American put decrease when the stock price increases.
(b) The premiums for a European call decrease when the strike price increases.

(c) Premiums on American options are smaller than premiums for otherwise equivalent European options.

**Solution MESQPCPA4.**

**(a) is true**. The payoff on a put option is max[0, K - S]. The higher S is, the smaller K - S becomes, and so we should expect the premium to decline.

**(b) is true.** The payoff on a call option is max[0, S - K]. As K increases, S - K decreases, so we should expect the premium to decline.

**(c) is false.** An American option is always at least as expensive as an otherwise equivalent European option, because the American option also gives the holder the choice to exercise at any time before expiration.

**Problem MESQPCPA5.**

**Similar to Question 13 from the Casualty Actuarial Society's [Spring 2007 Exam 3]:**

Spacious Co. pays no dividends on its stock, which is currently worth $450 per share. For a strike price of $440 and time to expiration of 1 month, the price of a European call on Spacious Co. stock is $23, while the price of a European put is $11. A synthetic T-Bill can be created using these options. Find the annual risk-free continuously compounded rate on said T-Bill.

**Solution MESQPCPA5.**

This is a put-call parity problem phrased in slightly different terms. We seek to find r, given everything else. We use the put-call parity formula

$C(K, T) - P(K, T) = [S_0 - PV_{0,T}(Div)] - e^{-rT}K$, which, with no dividends, simplifies to

$C(K, T) - P(K, T) = S_0 - e^{-rT}K$. We can rearrange the formula thus:

$e^{-rT}K = S_0 - C(K, T) + P(K, T)$. We know that $T = 1/12$, $S_0 = 450$, $K = 440$, $C(K, T) = 23$, and $P(K, T) = 11$. Thus,

$e^{-r/12}440 = 450 - 23 + 11$.

Thus, $e^{-r/12}440 = 438$ and $e^{-r/12} = 438/440$

Hence, $r = -12\ln(438/440)$ = **r = 0.0546697984**

# Section 15

# One-Period Binomial Option Pricing

The binomial option pricing model makes the simplified assumption that within any given time period, a stock price can only move up by a discrete amount or down by a discrete amount. No other changes are permitted.

With the original stock price being S, the following "tree" shows that in the next time period, the stock price could either be equal to uS (u = 1 + rate of capital gain on stock) or to dS (d = 1 + rate of capital loss on stock).

S - - - uS
S - - - dS

Now we let $C_u$ be the call option price when the stock price increases and $C_d$ be the call option price when the stock price decreases. Let C be the original call option price. The one-period binomial option "tree" for call option prices is as follows:

C - - - $C_u$
C - - - $C_d$

There also exists some replicating portfolio which precisely duplicates the option payoff. According to the law of one price, in the absence of arbitrage opportunities, this replicating portfolio must have the same cost as the equivalent call option. The replicating portfolio consists of Δ (delta) shares and B in lending, expressible as follows:

$\Delta = e^{-\partial h}(C_u - C_d)/[S(u-d)]$, where $\partial$ is the annual continuously-compounded dividend yield and h is the time period in question.

$B = e^{-rh}(uC_d - dC_u)/(u-d)$, where r is the annual continuously-compounded interest rate.

So the cost of our option is expressible as follows:
$C = \Delta S + B$ or

$C = e^{-rh}\{C_u[(e^{(r-\partial)h} - d)/(u-d)] + C_d[(u - e^{(r-\partial)h})/(u-d)]\}$

You are well-advised to use the former of these two formulas unless it is absolutely impossible to do so.

**Source:** McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 10, pp. 313-317.

We consider the following practice problems that will enable us to use the binomial pricing model for one time period.

**Problem OPBOP1.** The stock of Predictable Co. is currently worth $100 per share. In one year, this price can either be $120 or $90. Predictable Co. stock does not pay dividends. The annual continuously compounded risk-free interest rate is 5%. The strike price of a European call option on Predictable Co. stock is $110. Using, the one-period binomial option pricing model, find the price today of one such call option on Predictable Co. stock.

**Solution OPBOP1.** First, we consider the call option price tree

C - - - $C_u$
C - - - $C_d$

In one year, if the stock is worth $120, the call option will be worth $C_u$ = 120 - 110 = 10.

If the stock is worth $90, the call option will be worth $C_d$ = 0 (the option cannot be exercised).

We are given $\partial = 0$, r = 0.05, S = 100, h = 1, u = 1.2, and d = 0.9.

Thus, $\Delta = e^{-\partial h}(C_u - C_d)/[S(u-d)] = 1(10 - 0)/(100*(1.2-0.9)) = 10/30 = \Delta = 1/3$

$B = e^{-0.05}(1.2*0 - 0.9*10)/(1.2-0.9) = -28.53688274$

C = ΔS + B = (1/3)100 - 28.53688274 = **C = $4.796450598**

**Problem OPBOP2.** Impeccable LLC currently has a stock price of $555 per share. A replicating portfolio for a particular call option on Impeccable LLC stock involves borrowing $56 and buying ¾ of one share. Calculate the price of the call option using the one-period binomial option pricing model.

**Solution OPBOP2.** We use the formula C = ΔS + B. Here, S = 555, B = -56, and Δ = ¾. (Note: B is *negative* if money is being borrowed.)

Thus, C = (3/4)555 - 56 = **C = $360.25**

**Problem OPBOP3.** A call option on Reliable, Inc., stock currently trades for $45. The stock itself is worth $900 per share. Using the one-period binomial option pricing model, a replicating portfolio for the call option is equal to buying (1/5) shares of stock and borrowing $X. Calculate X.

**Solution OPBOP3.** We use the formula C = ΔS + B and rearrange it as follows:

B = C - ΔS. Here, C = 45, S = 900, and Δ = 1/5. Thus, B = 45 - 900/5 = -135. So $135 is borrowed and **X = 135**.

**Problem OPBOP4.** Currently, the annual continuously-compounded interest rate is 0.11. Company Co. stock trades for $23 per share, and the annual continuously-compounded dividend yield on Company Co. stock is 0.05. In two months, Company Co. stock will trade for either $18 per share or $29 per share. The strike price of a European call option on Company Co. stock is $25. Using the one-period binomial option pricing model, find the price today of one such call option on Company Co. stock.

**Solution OPBOP4.**

First, we consider the call option price tree

C - - - $C_u$
C - - - $C_d$

In 2 months, if the stock is worth $29, the call option will be worth $C_u = 29 - 25 = 4$.
If the stock is worth $18, the call option will be worth $C_d = 0$ (the option cannot be exercised).
We are given $\partial = 0.05$, $r = 0.11$, $S = 23$, $h = 1/6$, $u = 29/23$, and $d = 18/23$.

Thus, $\Delta = e^{-\partial h}(C_u - C_d)/[S(u-d)] = e^{-0.05/6}(4 - 0)/[23(29/23-18/23)] = e^{-0.05/6}(4/11) = 0.3606186519$
$B = e^{-rh}(uC_d - dC_u)/(u-d) = e^{-0.11/6}(-(18/23)4)/(29/23-18/23)) = e^{-0.11/6}(-72/11) =$
$-6.426547854$

$C = \Delta S + B = 0.3606186519*23 - 6.426547854 =$ **$C = \$1.86768114$**

**Problem OPBOP5.** The stock of Tractable LLC pays dividends. It is currently worth $65; in one year, it will be worth either $45 or $85. $\Delta = 0.45$ for a replicating portfolio equivalent to one call option on Tractable LLC stock that has a strike price of $64. The option expires in one year. Using the one-period binomial option pricing model, what is Tractable LLC's annual continuously-compounded dividend yield?

**Solution OPBOP5.** First, we consider the call option price tree

C - - - $C_u$
C - - - $C_d$

In one year, if the stock is worth $85, $C_u = 85 - 64 = 21$. If the stock is worth $45, $C_d = 0$.

So $(C_u - C_d) = 21$. Here, $S = 65$, $h = 1$, and $(u - d) = 85/65 - 45/65 = 40/65 = 8/13$. We want to find $\partial$.

$0.45 = \Delta = e^{-\partial h}(C_u - C_d)/[S(u-d)]$
$0.45 = e^{-\partial}(21/[65(8/13)])$
$0.45 = 0.525e^{-\partial}$
$e^{-\partial} = 6/7$
$\partial = -\ln(6/7) =$ **$\partial = 0.1541506798$**

# Section 16

# Risk-Neutral Probability in Binomial Option Pricing

Here, we develop the one-period binomial option pricing model introduced in Section 15.

The **risk-neutral probability** of an increase in the stock price from S to uS in the next time period is p*, which can be expressed as follows:
$p^* = (e^{(r-\partial)h} - d)/(u - d)$

Then the price of a call option on the stock today using the one-period binomial option pricing model is
$C = e^{-rh}[p^*C_u + (1 - p^*)C_d]$

Furthermore, the expected undiscounted price of the stock today is
$e^{(r-\partial)h}S = (p^*)uS + (1 - p^*)dS = F_{t,\ t+h}$

So the one-period binomial model can be used to determine the price of the forward contract on shares of the stock in question. p* can be thought of as the probability that the expected stock price is the forward price.

**Meanings of variables:**

C = current call option price.
$C_u$ = the call option price if the stock price increases.
$C_d$ = the call option price if the stock price decreases.
S = current stock price.
u = 1 + rate of capital gain on stock if stock price increases.
d = 1 + rate of capital loss on stock if stock price decreases.
r = annual continuously-compounded risk-free interest rate.
$\partial$ = annual continuously-compounded dividend yield.
$F_{t,\ t+h}$ = price of forward contract made at time t and expiring at time t + h.
h = one time period in the binomial model.

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 10, pp. 320-321.

**Problem RNPBOP1.** At the end of 1 year, the stock price of Digital Co. can change by either a factor of 1.5 or by a factor of 0.7. The annual continuously-compounded risk-free interest rate is 0.12, and the stock's annual continuously-compounded dividend yield is 0.09. Using the one-period binomial option pricing model, calculate the risk-neutral probability of the increase in Digital Co.'s stock price.

**Solution RNPBOP1.** We use the formula $p^* = (e^{(r-\partial)h} - d)/(u - d)$, where h = 1, u = 1.5, d = 0.7, r = 0.12, and $\partial$ = 0.09. Thus, $p^* = (e^{(0.12-0.09)} - 0.7)/(1.5 - 0.7)$ = **p* = 0.4130681674**.

**Problem RNPBOP2.** Atypical, Inc., pays dividends on its stock at an annual continuously-compounded yield of 0.05. In two months, its stock price could either be twice what it is now or half what it is now. The risk-neutral probability of the increase in Atypical, Inc.'s stock price is 0.56. Using the one-period binomial option pricing model, what is the annual continuously-compounded risk-free interest rate?

**Solution RNPBOP2.** We use the formula $p^* = (e^{(r-\partial)h} - d)/(u - d)$ and rearrange it thus:
$p^*(u - d) + d = e^{(r-\partial)h}$. Here, h = 1/6, u = 2, d = 0.5, $\partial$ = 0.05, and p* = 0.56. So
$e^{(r-0.05)/6} = 0.56(2-0.5) + 0.5 = 1.34$
ln(1.34) = 0.292669614 = (r - 0.05)/6

r = 6*0.292669614 + 0.05 = **r = 1.806017684** (No, this is not a typo. This just means that the annual continuously compounded risk-free interest rate is about 180.6%. Atypical, I know.).

**Problem RNPBOP3.** The stock of Reputable LLC will sell for either \$130 or \$124 one year from now. The annual continuously compounded interest rate is 0.11. The risk-neutral probability of an increase in the stock price (to \$130) is 0.77. Using the one-period binomial option pricing model, find the current price of a call option on Reputable LLC stock with a strike price of \$122.

**Solution RNPBOP3.** We use the formula $C = e^{-rh}[(p^*)C_u + (1 - p^*)C_d]$. In one year, if the stock is at \$130, the call will be worth $C_u$ = 130 - 122 = \$8. If the stock is at \$124, the call will be worth $C_d$ = 124 - 122 = \$2. p* = 0.77, h = 1, r = 0.11. Thus,
$C = e^{-0.11}[0.77*8 + (1 - 0.77)2]$ = **C = \$5.930421976**.

**Problem RNPBOP4.** The probability that the stock of Respectable Co. will be \$555 one year from now is 0.6. The probability that the stock of Respectable Co. will be \$521 one year from now is 0.4. Using the one-period binomial option pricing model, what is the price today of a one-year forward contract on Respectable Co. stock?

**Solution RNPBOP4.** We use the formula $(p^*)uS + (1 - p^*)dS = F_{t, t+h}$, where uS = 555, dS = 521, and p* = 0.6. Thus, 0.6*555 + 0.4*521 = **$F_{t, t+1}$ = 541.4**

**Problem RNPBOP5.** The stock of Discrete Co. will either be \$43 - with probability 0.2 - or \$49 - with probability 0.8 - in two years. The annual continuously-compounded interest rate is 0.05, and the annual continuously-compounded dividend yield is 0.02. Using the one-period binomial option pricing model, find the expected price of Discrete Co. stock today.

**Solution RNPBOP5.** Here, h = 2, r = 0.05 and $\partial$ = 0.02 (so r - $\partial$ = 0.03), p* = 0.8, uS = 49, and dS = 43. We use the formula $e^{(r-\partial)h}S = (p^*)uS + (1 - p^*)dS$
Thus, $e^{0.03*2}S = 0.8*49 + 0.2*43 = 47.8$
$S = 47.8e^{-0.06}$ = **S = \$45.01634471**

# Section 17

# Constructing Binomial Trees for Option Prices

Here we explore a method of constructing binomial trees in the one-period binomial option pricing model. The formula for a forward price is

$F_{t,\, t+h} = e^{(r-\partial)h}S_t$ where

r = annual continuously-compounded risk-free interest rate.
$\partial$ = annual continuously-compounded dividend yield.
$F_{t,\, t+h}$ = price of forward contract made at time t and expiring at time t + h.
h = one time period in the binomial model.
$S_t$ = stock price at time t.

Furthermore, we let
u = 1 + rate of capital gain on stock if stock price increases,
d = 1 + rate of capital loss on stock if stock price decreases,
σ = the annualized standard deviation of the continuously compounded stock return.

Then the possible evolution of future stock prices can be modeled via the following formulas:

$uS_t = F_{t,\, t+h}e^{\sigma\sqrt{(h)}}$
$dS_t = F_{t,\, t+h}e^{-\sigma\sqrt{(h)}}$

The terms u and d can be found as follows:

$u = e^{(r-\partial)h + \sigma\sqrt{(h)}}$
$d = e^{(r-\partial)h - \sigma\sqrt{(h)}}$

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 10, pp. 321-322.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem CPTOP1.** The annualized standard deviation of the continuously compounded stock return for Malicious Co. is currently 0.90. The annual continuously compounded interest rate is 0.07, and Malicious Co. pays dividends on its stock at an annual continuously compounded yield of 0.05. Using the one-period binomial option pricing model, what is the factor by which the price of Malicious Co. might increase in 3 years?

**Solution CPTOP1.** We use the formula $u = e^{(r-\partial)h + \sigma\sqrt{(h)}}$, and we are given that h = 3, σ = 0.9, r = 0.07, ∂ = 0.05 (so r - ∂ = 0.02). Thus, $u = e^{0.02*3 + 0.9\sqrt{(3)}} = e^{1.618845727} =$
**u = 5.047261035**

**Problem CPTOP2.** The annualized standard deviation of the continuously compounded stock return for Malicious Co. is currently 0.90. The annual continuously compounded interest rate is 0.07, and Malicious Co. pays dividends on its stock at an annual continuously compounded yield of 0.05. Using the one-period binomial option pricing model, what is the factor by which the lower possible price of Malicious Co. might be multiplied in 5 years? (That is, find the possible ratio of Malicious Co.'s lower price in 5 years to its price today.)

**Solution CPTOP2.** We use the formula $d = e^{(r-\partial)h - \sigma\sqrt{(h)}}$, and we are given that h = 5, σ = 0.9, r = 0.07, ∂ = 0.05 (so r - ∂ = 0.02). Thus, $d = e^{0.02*5 - 0.9\sqrt{(5)}} = e^{-1.91246118} =$
**d = 0.1477163823**

**Problem CPTOP3.** The 10-year forward contract on Auspicious, Inc., stock is currently worth $567. The annualized standard deviation of the continuously compounded stock return for Auspicious, Inc., is currently 0.02. If the price of Auspicious, Inc., increases after 10 years, what will it be using the one-period binomial option pricing model?

**Solution CPTOP3.** We use the formula $uS_t = F_{t,\, t+h}e^{\sigma\sqrt{(h)}}$, and we are given that h = 10, σ = 0.02, $F_{t,\, t+h} = 567$. Thus, $uS_t = 567*e^{0.02\sqrt{(10)}} =$ **$uS_t$ = $604.0185183**.

**Problem CPTOP4.** You think that the price of Suspicious LLC stock will decline in 1 month. Currently, a 1-month forward contract on Suspicious LLC stock sells for $423. The annualized standard deviation of the continuously compounded stock return for Suspicious LLC stock is currently 0.56. Using the one-period binomial option pricing model, what might the lower price of Suspicious LLC stock be in 1 month?

**Solution CPTOP4.** We use the formula $dS_t = F_{t,\, t+h}e^{-\sigma\sqrt{(h)}}$, and we are given that h = 1/12, σ = 0.56, $F_{t,\, t+h} = 423$. Thus, $dS_t = 423*e^{-0.56\sqrt{(1/12)}} =$ **$dS_t$ = $359.8596534**

**Problem CPTOP5.** Vicious Co. stock may increase or decline in 1 year under the assumptions of the one-period binomial option pricing model. Vicious Co. pays no dividends on its stock, and the annualized standard deviation of the continuously compounded stock return for Vicious Co. stock is 0.81. A 1-year forward contract on Vicious Co. stock currently sells for $100. Vicious Co. stock currently sells for $90. What is the annual continuously compounded risk-free interest rate?

**Solution CPTOP5.** We first use the formula $dS_t = F_{t,\, t+h}e^{-\sigma\sqrt{(h)}}$, knowing that σ = 0.81, h = 1, $F_{t,\, t+h} = 100$, $S_t = 90$. Thus, $d = F_{t,\, t+h}e^{-\sigma\sqrt{(h)}}/S_t = (100/90)e^{-0.81} = 0.4942867402$

Furthermore, we apply the formula $d = e^{(r-\partial)h - \sigma\sqrt{(h)}}$. Thus, because the stock pays no dividends, $0.4942867402 = e^{r-0.81}$. Thus, r = ln(0.4942867402) + 0.81 = **r = 0.1053605157**

# Section 18

# Multi-Period Binomial Option Pricing with Recombining Trees

A **recombining** binomial option price tree is one in which an up move in the stock price for one period followed by a down move in the stock price in the next period is identical in its result ($S_{ud}$) to a down move in the first period followed by an up move in the next ($S_{du}$).

In the recombining tree, there are three possible stock prices after two time periods:
$S_{uu} = u^2S$

$S_{ud} = S_{du} = udS$

$S_{dd} = d^2S$

We recall from Section 17 that the terms u and d can be found as follows:

$u = e^{(r-\partial)h + \sigma\sqrt{(h)}}$

$d = e^{(r-\partial)h - \sigma\sqrt{(h)}}$

Definitions of variables:

r = annual continuously-compounded risk-free interest rate.

$\partial$ = annual continuously-compounded dividend yield.

$F_{t,\, t+h}$ = price of forward contract made at time t and expiring at time t + h.

h = one time period in the binomial model.

S= current stock price.

u = 1 + rate of capital gain on stock if stock price increases,

d = 1 + rate of capital loss on stock if stock price decreases,

σ = the annualized standard deviation of the continuously compounded stock return.

The way to approach multi-period binomial option pricing models is to figure out the stock and option prices in the *latest* period and work backward from there using any and all the formulas introduced in Sections 15 through 18.

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 10, pp. 323-328.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem MPBOPWRT.** Gregarious, Inc., stock is currently worth $56. Every year, it can change by a factor of 0.9 or 1.3. The stock pays no dividends, and the annual continuously-compounded risk-free interest rate is 0.04. Using a two-period binomial option pricing model, find the price today of one two-year European call option on Gregarious, Inc., stock with a strike price of $70.

**Solution MPBOPWRT1.**

In one year, the stock will either be worth $S_u = 1.3*56 = 72.8$, or it will be worth $S_d = 0.9*56 = 50.4$. In two years, the stock will either be worth

$S_{uu} = 1.3^2*56 = 94.64$ or $S_{ud} = S_{du} = 1.3*0.9*56 = 65.52$ or $S_{dd} = 0.9*0.9*56 = 45.36$.

At $S_{uu} = 94.64$, the call is worth $C_{uu} = 94.64 - 70 = 24.64$

At $S_{du} = 65.52$, the call is worth $C_{du} = 0$

At $S_{dd} = 45.36$, the call is worth $C_{dd} = 0$

Using $C_{uu} = 24.64$ and $C_{ud} = 0$, we calculate the call option value $C_u$ at the end of one year in the event of an up move in the stock price.

We recall our formula for the risk-neutral probability in the increase in the stock price over one time period:

$p^* = (e^{(r-\partial)h} - d)/(u - d)$. Here, for every time period, $p^* = (e^{0.04} - 0.9)/(1.3 - 0.9) = 0.3520269355$.

We also recall the formula $C = e^{-rh}[p^*C_u + (1 - p^*)C_d]$.

Thus, in year 1, $C_u = e^{-0.04}[0.3520269355*24.64 + (1 - 0.3520269355)0] =$

$C_u = 8.333833493$

Using $C_{du} = 0$ and $C_{dd} = 0$, we calculate the call option value $C_d$ at the end of one year in the event of an up move in the stock price. This is evidently $C_d = 0$.

Thus, we can calculate $C = e^{-rh}[p^*C_u + (1 - p^*)C_d] =$

$e^{-0.04}[0.3520269355*8.333833493 + (1 - 0.3520269355)0] =$ **$C = \$2.818700515$**

**Problem MPBOPWRT2.** Complicated, Inc., pays dividends on its stock at an annual continuously compounded yield of 0.06. The annual effective interest rate is 0.09. Complicated, Inc., stock is currently worth $100. Every two years, it can change by a factor of 0.7 or 1.5. Using a two-period binomial option pricing model, find the price today of one four-year European call option on Gregarious, Inc., stock with a strike price of $80.

**Solution MPBOPWRT2.** We are given that r = 0.09, $\partial$ = 0.06, and h = 2. Thus,

$(r-\partial)h = (0.09 - 0.06)*2 = 0.06$.

We find

$S_{uu} = 1.5^2*100 = 225$, which implies that $C_{uu} = 145$

$S_{ud} = S_{du} = 1.5*0.7*100 = 105$, which implies that $C_{ud} = 25$

$S_{dd} = 0.7^2*100 = 49$, which implies that $C_{dd} = 0$.

$p^* = (e^{(r-\partial)h} - d)/(u - d)$. Here, for every time period, $p^* = (e^{0.06} - 0.7)/(1.5 - 0.7) = p^* =$ 0.4522956832.

We now use the formula $C = e^{-rh}[p^*C_u + (1 - p^*)C_d]$.

Thus, $C_u = e^{-0.09*2}[0.4522956832*145 + (1 - 0.4522956832)25] = C_u = 66.21644859$.

$C_d = e^{-0.09*2}[0.4522956832*25 + (1 - 0.4522956832)0] = C_d = 9.444727773$

Thus, $C = e^{-0.09*2}[0.4522956832*66.21644859 + (1 - 0.4522956832)9.444727773] =$

**C = $29.3366377**.

**Problem MPBOPWRT3.**

The annualized standard deviation of the continuously compounded stock return on Prominent Co. is 0.23. The annual continuously compounded rate of interest is 0.12, and the annual continuously compounded dividend yield on Prominent Co. is 0.07. The current price of Prominent Co. stock is $35 per share. Using a two-period binomial model, find the price of Prominent Co. stock if it moves up twice over the course of 7 years.

**Solution MPBOPWRT3.**

First, we find $u = e^{(r-\partial)h + \sigma\sqrt{(h)}}$, where, if the two-period model is applied over 7 years, it follows that one period = h = 3.5 years. Also, σ = 0.23, r = 0.12, and $\partial$ = 0.07. Thus,

$(r-\partial)h = (0.12 - 0.07)*3.5 = 0.175$. Hence, $u = e^{0.175 + 0.23\sqrt{(3.5)}} = 1.831784447$.

We want to find $S_{uu} = u^2S = 1.831784447^2*35 =$ **$S_{uu}$ = 117.4401991**.

**Problem MPBOPWRT4.**

The annualized standard deviation of the continuously compounded stock return on Prominent Co. is 0.23. The annual continuously compounded rate of interest is 0.12, and the annual continuously compounded dividend yield on Prominent Co. is 0.07. The current price of Prominent Co. stock is $35 per share. Using a two-period binomial model, find the price of Prominent Co. stock if it moves up once and then down once over the course of 7 years.

**Solution MPBOPWRT4.**

By design, this problem has the same initial conditions as Problem MPBOPWRT3. We already know that u = 1.831784447.

It remains to find $d = e^{(r-\partial)h - \sigma\sqrt{(h)}} = e^{0.175 - 0.23\sqrt{(3.5)}} = d = 0.7746913403$

We then want to find $S_{ud} = udS = 1.831784447*0.7746913403*35 =$ **$S_{ud}$ = 49.6673642**

**Problem MPBOPWRT5.** The annualized standard deviation of the continuously compounded stock return on Prominent Co. is 0.23. The annual continuously compounded rate of interest is 0.12, and the annual continuously compounded dividend yield on Prominent Co. is 0.07. The current price of Prominent Co. stock is $35 per share. Find the price of one 7-year European call option on Prominent Co. stock with a strike price of $40.

**Solution MPBOPWRT5.**

By design, this problem has the same initial conditions as Problems MPBOPWRT3-4.

There, we determined that u = 1.831784447 and d = 0.7746913403.

Also, we know that
$S_{uu} = 117.4401991$, so $C_{uu} = 77.4401991$
$S_{ud} = 49.6673642$, so $C_{ud} = 9.6673642$

It is not hard to find $S_{dd} = d*dS = 0.7746913403^2*35 = 21.00513355$, in which case $C_{dd} = 0$.
$p^* = (e^{(r-\partial)h} - d)/(u - d)$. Here, for every time period,
$p^* = (e^{0.175} - 0.7746913403)/(1.831784447 - 0.7746913403) = p^* = 0.3940569412$.
We now use the formula $C = e^{-rh}[p^*C_u + (1 - p^*)C_d]$.
$C_u = e^{-0.12*3.5}[0.3940569412*77.4401991 + (1 - 0.3940569412)9.6673642] =$
$C_u = 23.89923719$
$C_d = e^{-0.12*3.5}[0.3940569412*9.6673642 + (1 - 0.3940569412)0] = C_d = 2.503014582$
$C = e^{-0.12*3.5}[0.3940569412*23.89923719 + (1 - 0.3940569412)2.503014582] =$
**C = $7.184376357**

# Section 19

# Binomial Option Pricing with Puts

Binomial option pricing with puts can be done using the exact same formulas and conceptual tools developed in Sections 15-18 except that calculating the put price at expiration uses the formula P = max(0, K - S) instead of C = max(0, S - K).

Here, we will use one and two-period binomial models for all practice problems, because the objective of this section is to establish the *conceptual* approach to binomial option pricing with puts. Students should be aware that this approach can translate to larger multi-period models as well, using the same essential procedure as the one illustrated here.

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 10, pp. 328-329.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem BOPWP1.** The stock of Predictable Co. is currently worth $100 per share. In one year, this price can either be $120 or $90. Predictable Co. stock does not pay dividends. The annual continuously compounded risk-free interest rate is 5%. The strike price of a European put option on Predictable Co. stock is $130. Using, the one-period binomial option pricing model, find the price today of one such put option on Predictable Co. stock.

**Solution BOPWP1.** First, we consider the put option price tree

P - - - $P_u$

P - - - $P_d$

In one year, if the stock is worth $120, the put option will be worth $P_u$ = 130 - 120 = 10.

If the stock is worth $90, the put option will be worth $P_d$ = 130 - 90 = 40.

We are given ∂ = 0, r = 0.05, S = 100, h = 1, u = 1.2, and d = 0.9.

We can still use the same formula for the risk-neutral probability of the stock price's *increase* next year:
$p^* = (e^{(r-\partial)h} - d)/(u - d) = (e^{0.05} - 0.9)/(1.2 - 0.9) = p^* = 0.5042369879$

We also note that
$P = e^{-rh}[p^*P_u + (1 - p^*)P_d] = e^{-0.05}[0.5042369879*10 + (1 - 0.5042369879)40] =$

**P = $23.65982519**

**Problem BOPWP2.** Currently, the annual continuously-compounded interest rate is 0.11. Company Co. stock trades for $23 per share, and the annual continuously-compounded dividend yield on Company Co. stock is 0.05. In two months, Company Co. stock will trade for either $18 per share or $29 per share. The strike price of a European put option on Company Co. stock is $30. Using the one-period binomial option pricing model, find the price today of one such put option on Company Co. stock.

**Solution BOPWP2.**

First, we consider the put option price tree

P - - - $P_u$

P - - - $P_d$

In 2 months, if the stock is worth $29, the put option will be worth $P_u$ = 30 - 29 = 1.

If the stock is worth $18, the put option will be worth $P_d$ = 30 - 18 = 12.

We are given $\partial$ = 0.05, r = 0.11, S = 23, h = 1/6, u = 29/23, and d = 18/23.

$p^* = (e^{(r-\partial)h} - d)/(u - d) = (e^{(0.11-0.05)/6} - 18/23)/(29/23 - 18/23) = p^* = 0.4755594403$

$P = e^{-rh}[p^*P_u + (1 - p^*)P_d] = e^{-0.11/6}[0.4755594403*1 + (1 - 0.4755594403)12] =$

**P = $6.645881266**

**Problem BOPWP3.** The stock of Reputable LLC will sell for either $130 or $124 one year from now. The annual continuously compounded interest rate is 0.11. The risk-neutral probability of an increase in the stock price (to $130) is 0.77. Using the one-period binomial option pricing model, find the current price of a one-year European put option on Reputable LLC stock with a strike price of $160.

**Solution BOPWP3.**

We note that $P_u$ = 160 - 130 = $P_u$ = 30 and

$P_d$ = 160 - 124 = $P_d$ = 36.

Here, p* = 0.77, h = 1, and r = 0.11.

So $P = e^{-rh}[p^*P_u + (1 - p^*)P_d] = e^{-0.11}[0.77*30 + (1 - 0.77)36] =$ **P = $28.11127517**

**Problem BOPWP4.** Gregarious, Inc., stock is currently worth $56. Every year, it can change by a factor of 0.9 or 1.3. The stock pays no dividends, and the annual continuously-compounded risk-free interest rate is 0.04. Using a two-period binomial option pricing model, find the price today of one two-year European put option on Gregarious, Inc., stock with a strike price of $120.

**Solution BOPWP4.** In one year, the stock will either be worth $S_u$ = 1.3*56 = 72.8, or it will be worth $S_d$ = 0.9*56 = 50.4. In two years, the stock will either be worth

$S_{uu} = 1.3^2*56 = 94.64$ or $S_{ud} = S_{du} = 1.3*0.9*56 = 65.52$ or $S_{dd} = 0.9*0.9*56 = 45.36$.
At $S_{uu}$ = 94.64, the put is worth $P_{uu}$ = 120 - 94.64 = $P_{uu}$ = 25.36
At $S_{du}$ = 65.52, the put is worth $P_{du}$ = 120 - 65.52 = $P_{du}$ = 54.48
At $S_{dd}$ = 45.36, the call is worth $P_{dd}$ = 120 - 45.36 = $P_{dd}$ = 74.64

Now we calculate $p^* = (e^{(r-\partial)h} - d)/(u - d) = (e^{0.04} - 0.9)/(1.3 - 0.9) = 0.3520269355$.

We note that there can be a *direct* calculation of the put price today using the three possible put prices two periods from now using the binomial model. The formula might make intuitive sense to you if you consider the way a binomial probability distribution works:

$P = e^{-2rh}[(p^*)^2P_{uu} + 2(p^*)(1-p^*)P_{ud} + (1 - p^*)^2P_{dd}]$

$P = e^{-2*0.04}[(0.3520269355)^2 25.36 + 2(0.3520269355)(1- 0.3520269355)54.48 + (1 - 0.3520269355)^2 74.64]$ = **P = $54.77396157**.

This approach is much faster than finding the intermediate put prices. The identical kind of formula can be applied to call pricing using the two-period binomial model as well.

**Problem BOPWP5.** Complicated, Inc., pays dividends on its stock at an annual continuously compounded yield of 0.06. The annual effective interest rate is 0.09. Complicated, Inc., stock is currently worth $100. Every two years, it can change by a factor of 0.7 or 1.5. Using a two-period binomial option pricing model, find the price today of one four-year European put option on Gregarious, Inc., stock with a strike price of $130.

**Solution BOPWP5.** We are given that r = 0.09, $\partial$ = 0.06, and h = 2. Thus,
$(r-\partial)h = (0.09 - 0.06)*2 = 0.06$.

$p^* = (e^{(r-\partial)h} - d)/(u - d)$. Here, for every time period, $p^* = (e^{0.06} - 0.7)/(1.5 - 0.7) = p^* =$ 0.4522956832.

We find
$S_{uu} = 1.5^2*100 = 225$, which implies that $P_{uu} = 0$
$S_{ud} = S_{du} = 1.5*0.7*100 = 105$, which implies that $P_{ud} = 130 - 105 = P_{ud} = 25$
$S_{dd} = 0.7^2*100 = 49$, which implies that $P_{dd} = 130 - 49 = P_{dd} = 81$.

Now we use the great time-saving formula
$P = e^{-2rh}[(p^*)^2P_{uu} + 2(p^*)(1-p^*)P_{ud} + (1 - p^*)^2P_{dd}] =$
$e^{-2*0.09*2}[0 + 2(0.4522956832)(1-0.4522956832)25 + (1 - 0.4522956832)^2 81] =$
**P = $25.59397445**

# Section 20

# Binomial Option Pricing with American Options

With American options, it is possible for the option to be exercised early. Thus, to determine the option's price at any given "node" in a binomial tree, it is necessary to compare its value if it is held to expiration to the gain that could be realized upon immediate exercise. The higher of these is the American option price.

So for an American put,

$P(S, K, t) = \max(K - S, e^{-rh}[P(uS, K, t+h)p^* + P(dS, K, t+h)(1-p^*)])$, where

$p^* = (e^{(r-\partial)h} - d)/(u - d)$

For an American call,

$C(S, K, t) = \max(S - K, e^{-rh}[C(uS, K, t+h)p^* + C(dS, K, t+h)(1-p^*)])$,

Definitions of variables:

$r$ = annual continuously-compounded risk-free interest rate.

$\partial$ = annual continuously-compounded dividend yield.

$h$ = one time period in the binomial model.

$t$ = the time equivalent to some "node" in the binomial model.

$S$= stock price at time $t$

$K$ = option strike price

$u$ = 1 + rate of capital gain on stock if stock price increases,

$d$ = 1 + rate of capital loss on stock if stock price decreases,

$\sigma$ = the annualized standard deviation of the continuously compounded stock return.

$P(S, K, t)$ = price of an American put with strike price $K$ and underlying stock price $S$.

$C(S, K, t)$ = price of an American call with strike price $K$ and underlying stock price $S$.

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 10, p. 329.

**Problem BOPWAO1.** The stock of Predictable Co. is currently worth $100 per share. In one year, this price can either be $120 or $90. Predictable Co. stock does not pay dividends. The annual continuously compounded risk-free interest rate is 5%. The strike price of a one-year *American* put option on Predictable Co. stock is $130. Using, the one-period binomial option pricing model, find the price today of one such Americanput option on Predictable Co. stock.

**Solution BOPWAO1.** To find the American put price, we will take the maximum of (K - S) and the otherwise equivalent European put price using the binomial option pricing model.

We note that K - S = 130 - 100 = 30

Now we calculate the otherwise equivalent European put price:

First, we consider the put option price tree

$P$ - - - $P_u$

$P$ - - - $P_d$

In one year, if the stock is worth $120, the put option will be worth $P_u = 130 - 120 = 10$.

If the stock is worth $90, the put option will be worth $P_d = 130 - 90 = 40$.

We are given $\partial = 0$, $r = 0.05$, $S = 100$, $h = 1$, $u = 1.2$, and $d = 0.9$.

We can still use the same formula for the risk-neutral probability of the stock price's *increase* next year:

$p^* = (e^{(r-\partial)h} - d)/(u - d) = (e^{0.05} - 0.9)/(1.2 - 0.9) = p^* = 0.5042369879$

We also note that

$P = e^{-rh}[p^*P_u + (1 - p^*)P_d] = e^{-0.05}[0.5042369879*10 + (1 - 0.5042369879)40] =$

$P = \$23.65982519$

We note that 23.65982519 < 30, so our American put option price is **$30.**

**Problem BOPWAO2.** The stock of Reputable LLC will sell for either $130 or $124 one year from now. The annual continuously compounded interest rate is 0.11. The risk-neutral probability of the stock price being $130 in one year is 0.77. What is the current stock price for which the one-year American put option on Reputable LLC stock with a strike price of $160 will have the same value whether calculated by means of the binomial option pricing model or by taking the difference between the stock price and the strike price?

**Solution BOPWAO2.**

We are essentially looking for the stock price S where

$P = K - S = e^{-rh}[P(uS, K, t+h)p^* + P(dS, K, t+h)(1-p^*)])$

First, we find P:

We note that $P_u = 160 - 130 = P_u = 30$ and

$P_d = 160 - 124 = P_d = 36$.

Here, $p^* = 0.77$, $h = 1$, and $r = 0.11$.

So $P = e^{-rh}[p^*P_u + (1 - p^*)P_d] = e^{-0.11}[0.77*30 + (1 - 0.77)36] = P = \$28.11127517$

We know that K = 160, so

$S = K - P = 160 - 28.11127517 =$ **$S = \$131.8887248$**.

**Problem BOPWAO3.** Complicated, Inc., pays dividends on its stock at an annual continuously compounded yield of 0.06. The annual effective interest rate is 0.09. Complicated, Inc., stock is currently worth $100. Every two years, it can change by a factor of 0.7 or 1.5. Using a two-period binomial option pricing model, find the price two years from today of one four-year *American* call option on Gregarious, Inc., stock with a strike price of $80 in the event that the stock price increases two years from today.

**Solution BOPWAO3.** We are essentially asked to find $C_u$ and compare it to S - K if the stock price increases next year. If the stock price increases next year, the stock will be worth $150, so $S - K = 150 - 80 = 70$

Now we find $C_u$:

We are given that $r = 0.09$, $\partial = 0.06$, and $h = 2$. Thus,

$(r-\partial)h = (0.09 - 0.06)*2 = 0.06$.

We find

$S_{uu} = 1.5^2*100 = 225$, which implies that $C_{uu} = 145$

$S_{ud} = S_{du} = 1.5*0.7*100 = 105$, which implies that $C_{ud} = 25$

$p^* = (e^{(r-\partial)h} - d)/(u - d)$. Here, for every time period, $p^* = (e^{0.06} - 0.7)/(1.5 - 0.7) = p^* =$ 0.4522956832.

We now use the formula $C = e^{-rh}[p^*C_u + (1 - p^*)C_d]$.

Thus, $C_u = e^{-0.09*2}[0.4522956832*145 + (1 - 0.4522956832)25] = C_u = 66.21644859$.

We note that 66.21644859 < 70, so the American call price next year if the stock price goes up would be **$70.**

**Problem BOPWAO4.** Complicated, Inc., pays dividends on its stock at an annual continuously compounded yield of 0.06. The annual effective interest rate is 0.09. Complicated, Inc., stock is currently worth $100. Every two years, it can change by a factor of 0.7 or 1.5. Using a two-period binomial option pricing model, find the price two years from today of one four-year *American* call option on Gregarious, Inc., stock with a strike price of $80 in the event that the stock price *decreases* two years from today.

**Solution BOPWAO4.** We are essentially asked to find $C_d$ and compare it to S - K if the stock price decreases next year. If the stock price decreases next year, the stock will be worth $70, so S - K = 70 - 80 = -10

We are given that $r = 0.09$, $\partial = 0.06$, and $h = 2$. Thus,

$(r-\partial)h = (0.09 - 0.06)*2 = 0.06$.

We find

$S_{ud} = S_{du} = 1.5*0.7*100 = 105$, which implies that $C_{ud} = 25$

$S_{dd} = 0.7^2*100 = 49$, which implies that $C_{dd} = 0$.

$p^* = (e^{(r-\partial)h} - d)/(u - d)$. Here, for every time period, $p^* = (e^{0.06} - 0.7)/(1.5 - 0.7) = p^* =$ 0.4522956832.

We now use the formula $C = e^{-rh}[p^*C_u + (1 - p^*)C_d]$.

$C_d = e^{-0.09*2}[0.4522956832*25 + (1 - 0.4522956832)0] = C_d = 9.444727773$

Since 9.444727773 > -10, the American call price next year if the stock price decreases would be **$9.444727773.**

**Problem BOPWAO5.** Complicated, Inc., pays dividends on its stock at an annual continuously compounded yield of 0.06. The annual effective interest rate is 0.09. Complicated, Inc., stock is currently worth $100. Every two years, it can change by a factor of 0.7 or 1.5. Using a two-period binomial option pricing model, find the price today of one four-year *American* call option on Gregarious, Inc., stock with a strike price of $80.

**Solution BOPWAO5.** We use the formula

$C(S, K, t) = \max(S - K, e^{-rh}[C(uS, K, t+h)p^* + C(dS, K, t+h)(1-p^*)])$.

We note that today, S - K = 100 - 80 = 20

From Solution BOPWAO3, we get C(uS, K, t+h) = 70.

From Solution BOPWAO4, we get C(uS, K, t+h) = 9.444727773

From both solutions, we get p* = 0.4522956832.

Thus, $e^{-rh}[C(uS, K, t+h)p^* + C(dS, K, t+h)(1-p^*)] =$

$e^{-0.09*2}[0.4522956832*70 + (1-0.4522956832)* 9.444727773] = 30.766602222$

Since 30.766602222 > 20, the American call price today is **$30.766602222**.

# Section 21

# Binomial Pricing for Currency Options

A binomial model can be constructed to price options on currencies, using the following equations:

$F_{0,h} = x_0e^{(r-f)h}$

$ux = xe^{(r-f)h + \sigma\sqrt{(h)}}$

$dx = xe^{(r-f)h - \sigma\sqrt{(h)}}$

$p^* = (e^{(r-f)h} - d)/(u - d)$

$\Delta dxe^{fh} + Be^{rh} = C_d$

$\Delta uxe^{fh} + Be^{rh} = C_u$

Definitions of variables:

r = annual continuously-compounded risk-free interest rate for currency 1 (the "domestic" currency or the currency in terms of which the option prices are denominated).

f = annual continuously-compounded risk-free interest rate for currency 2 (the "foreign" currency).

$x_0$ = spot price of "foreign" currency in terms of "domestic" currency.

u = 1 + rate of capital gain on stock if "foreign" currency price increases.

d = 1 + rate of capital loss on stock if "foreign" currency price decreases.

σ = the annualized standard deviation of the continuously compounded return on the "foreign" currency.

p* = the risk-neutral probability of an increase in the "foreign" currency's price.

h = one time period in the binomial model.

$F_{0,h}$ = the time-h forward price for the currency.

Δ (delta) = the number of units of the "foreign" currency contained in the replicating portfolio for the option.

B = the number of dollars lent out in the replicating portfolio for the option.

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 10, p. 332.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem BPCO1.** One euro currently trades for $1.56. The dollar-denominated annual continuously-compounded risk-free interest rate is 0.02, and the euro-denominated annual continuously-compounded risk-free interest rate is 0.09. Calculate the price of a 10-year forward contract on euros, denominated in dollars.

**Solution BPCO1.** We use the formula $F_{0,h} = x_0e^{(r-f)h}$. We are given that $x_0 = 1.56$, h = 10, r = 0.02, and f = 0.09. Thus, $F_{0,10} = 1.56e^{(0.02-0.09)10}$ = $\mathbf{F_{0,10} = \$0.7746730739}$

**Problem BPCO2.** One euro currently trades for $1.56. The dollar-denominated annual continuously-compounded risk-free interest rate is 0.02, and the euro-denominated annual continuously-compounded risk-free interest rate is 0.09. The annualized standard deviation of the continuously compounded return on the euro is 0.54. Using a one-period binomial model, calculate what the euro price in dollars will be in two years if the euro's price increases.

**Solution BPCO2.** We are asked to find $ux = xe^{(r-f)h + \sigma\sqrt{(h)}}$, given that $x_0 = 1.56$, h = 2, σ = 0.54 r = 0.02, and f = 0.09. Hence, $ux = 1.56e^{(0.02-0.09)2 + 0.54\sqrt{(2)}}$ = **ux = $2.910605529**. (Can you imagine the euro costing so many dollars in two years?)

**Problem BPCO3.** One piece of stone from Yap (YPS) currently trades for 45 cowry shells (CS). The Yap-stone-denominated annual continuously-compounded risk-free interest rate is 0.13, while the cowry-shell-denominated annual continuously-compounded risk-free interest rate is 0.05. The annualized standard deviation of the continuously compounded return on Yap stone pieces is 0.81. Using the one-period binomial option pricing model, what is the risk-neutral probability that the price of a Yap stone will increase in two months?

**Solution BPCO3.** We first need to find $u = e^{(r-f)h + \sigma\sqrt{(h)}}$ and $d = e^{(r-f)h - \sigma\sqrt{(h)}}$. We are given that r = 0.05, f = 0.13, σ = 0.81, and h = 1/6. So (r - f) = -0.08

Thus, $u = e^{(-0.08/6) + 0.81\sqrt{(1/6)}}$ = u = 1.37348016

$d = e^{(-0.08/6) - 0.81\sqrt{(1/6)}}$ = d = 0.7089186851

Now we use the formula $p^* = (e^{(r-f)h} - d)/(u - d)$ =

$(e^{(-0.08/6)} - 0.7089186851)/(1.37348016 - 0.7089186851)$ = **p* = 0.4180749069**

**Problem BPCO4.** One piece of stone from Yap (YPS) currently trades for 45 cowry shells (CS). The Yap-stone-denominated annual continuously-compounded risk-free interest rate is 0.13, while the cowry-shell-denominated annual continuously-compounded risk-free interest rate is

0.05. The annualized standard deviation of the continuously compounded return on Yap stone pieces is 0.81. For a certain two-month European call option on one YPS, the replicating portfolio involves buying (3/4) Yap stone pieces and borrowing 23 cowrie shells. Using the one-period binomial option pricing model, what would the price of this call option be in two months if the YPS price decreased?

**Solution BPCO4.** We are trying to find $C_d = \Delta dxe^{fh} + Be^{rh}$. From Solution BPCO3, we are given that d = 0.7089186851. Also, B = -23, Δ = 0.75, r = 0.05, f = 0.13, h = 1/6, and x = 45.

So $C_d = 0.75*0.7089186851*45e^{0.13/6} - 23e^{0.05/6}$ = $\mathbf{C_d = \$1.257591655}$

**Problem BPCO5.** One piece of stone from Yap (YPS) currently trades for 45 cowry shells (CS). The Yap-stone-denominated annual continuously-compounded risk-free interest rate is 0.13, while the cowry-shell-denominated annual continuously-compounded risk-free interest rate is 0.05. The annualized standard deviation of the continuously compounded return on Yap stone pieces is 0.81. For a certain two-month European call option on one YPS, the replicating portfolio involves buying (3/4) Yap stone pieces and borrowing 23 cowrie shells. Using the one-period binomial option pricing model, what is the current price of this call option?

**Solution BPCO5.** In Solution BPCO3, we found that p* = 0.4180749069, and in Solution BPCO4, we found that $C_d$ = \$1.257591655. We can also find $\Delta uxe^{fh} + Be^{rh} = C_u$:

From Solution BPCO3, u = 1.37348016. Also B = -23, Δ = 0.75, r = 0.05, f = 0.13, h = 1/6, and x = 45.

Thus, $\Delta uxe^{fh} + Be^{rh} = 0.75*1.37348016*45e^{0.13/6} - 23e^{0.05/6} = C_u = 24.17780481$

Now we use the formula $C = e^{-rh}[p*C_u + (1 - p*)C_d]$ =

$e^{-0.05/6}[0.4180749069*24.17780481 + (1 - 0.4180749069)1.257591655]$ = **C = \$10.75**

# Section 22

# Binomial Pricing for Options on Futures Contracts

This use of the binomial model to price options on futures contracts assumes that the futures price is equal to the forward price. If this is the case, then the following formulas apply.

$u = e^{\sigma\sqrt{(h)}}$

$d = e^{-\sigma\sqrt{(h)}}$

$\Delta(dF-F) + Be^{rh} = C_d$

$\Delta(uF-F) + Be^{rh} = C_u$

$\Delta = (C_u - C_d)/[F(u-d)]$

$B = e^{-rh}[C_u(1-d)/(u-d) + C_d(u-1)/(u-d)]$; B is also the value of the option.

$p^* = (1-d)/(u-d)$

Definitions of variables:

r = annual continuously-compounded risk-free interest rate.

u = 1 + rate of capital gain on stock if futures price increases.

d = 1 + rate of capital loss on stock if futures price decreases.

σ = the annualized standard deviation of the continuously compounded return on the futures contract.

p* = the risk-neutral probability of an increase in the futures price.

h = one time period in the binomial model.

Δ (delta) = the number of units of the futures contract contained in the replicating portfolio for the option.

B = the number of dollars lent out in the replicating portfolio for the option - equivalent to the option value for options on futures contracts.

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 10, p. 333-334.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem BPOFC1.** Futures Contract Ω sells for $90 today. The annualized standard deviation of the continuously compounded return on the futures contract is 0.34. Using a three-period binomial option pricing model, find the price of Futures Contract Ω after 6 years if the contract always increases in price every time period.

**Solution BPOFC1.** We use the formula $u = e^{\sigma\sqrt{(h)}}$, where h = 2 and σ = 0.34. Thus, $u = e^{0.34\sqrt{(2)}} =$ u = 1.617420524.

So, after three periods, during each of which Futures Contract Ω goes up in price, the futures contract price will be $F_{uuu} = 1.617420524^3*90 =$ $\mathbf{F_{uuu} = 380.8126432}$

**Problem BPOFC2.** Futures Contract Ω sells for $90 today. The annualized standard deviation of the continuously compounded return on the futures contract is 0.34. Using a binomial option pricing model, find the risk-neutral probability that the price of Futures Contract Ω will increase during any given two-year period.

**Solution BPOFC2.** We are given that h = 2 and σ = 0.34.

We use the formula p* = (1-d)/(u-d), knowing from Solution BPOFC1 that

u = 1.617420524. To find d, we use the formula $d = e^{-\sigma\sqrt{(h)}} = e^{-0.34\sqrt{(2)}} =$ d = 0.6182684002

Thus, p* = (1-0.6182684002)/(1.617420524 - 0.6182684002) = **p* = 0.3820555355**

**Problem BPOFC3.** Futures Contract Ω sells for $90 today. The annualized standard deviation of the continuously compounded return on the futures contract is 0.34. The annual continuously compounded risk-free interest rate is 0.05. Using a three-period binomial option pricing model, find the price of one six-year European call option on Futures Contract Ω with a strike price of $80.

**Solution BPOFC3.** We are given that h = 2, r = 0.05, and σ = 0.34.

We assume a recombining binomial tree.

We know from Solutions BPOFC1-2 that u = 1.617420524 and d = 0.6182684002.

We already know from Solution BPOFC1 that

$F_{uuu} = 380.8126432$ and thus $C_{uuu} = 300.8126432$

$F_{duu} = 1.617420524^2*0.6182684002*90 = 145.5678472$ and thus $C_{duu} = 65.56784718$

$F_{ddu}$ = 1.617420524*0.6182684002$^2$*90 = 55.64415602 and thus $C_{ddu}$ = 0

Naturally, $F_{ddd} < F_{ddu}$, so $C_{ddd}$ = 0 as well.

We know from Solution BPOFC2 that p* = 0.3820555355

For a three-period binomial model, we apply the formula

$C = e^{-3rh}[(p^*)^3C_{uuu} + 3(p^*)^2(1-p^*)C_{duu} + 3(p^*)(1-p^*)^2C_{ddu} + (1-p^*)^3C_{ddd}]$.

Thus,

$C = e^{-3*0.05*2}[(0.3820555355)^3 300.8126432 + 3(0.3820555355)^2(1-0.3820555355)65.56784718]$ = **C = $25.57156033**

**Problem BPOFC4.** Futures Contract Ω sells for $90 today. The annualized standard deviation of the continuously compounded return on the futures contract is 0.34. The annual continuously compounded risk-free interest rate is 0.05. Using a one-period binomial option pricing model, find Δ for a replicating portfolio equivalent to one two-year European call option on Futures Contract Ω with a strike price of $30.

**Solution BPOFC4.** We are given that h = 2, r = 0.05, F = 90, and σ = 0.34.

We know from Solutions BPOFC1-2 that u = 1.617420524 and d = 0.6182684002.

$F_u$ = 1.617420524*90 = 145.5678472 and thus $C_u$ = 115.5678472

$F_d$ = 0.6182684002*90 = 55.64415602 and thus $C_d$ = 55.64415602

We use the formula $\Delta = (C_u - C_d)/[F(u-d)]$ =

(115.5678472 - 55.64415602)/[90(1.617420524 - 0.6182684002)] = **Δ = 0.6663838017**

**Problem BPOFC5.** Futures Contract Ξ sells for $49 today. The annual continuously compounded risk-free interest rate is 0.15. The price today of one particular three-month European call option on Futures Contract Ξ is $10. Δ for a replicating portfolio equivalent to one such option is 0.4. If in three months, Futures Contract Ξ will be worth 0.85 of its present amount, what will the price of the call option be? Use a one-period binomial option pricing model.

**Solution BPOFC5.** We are asked to find $\Delta(dF-F) + Be^{rh} = C_d$, given that F = 49, d = 0.85, Δ = 0.4, B = 10, r = 0.15, and h = 0.25.

Thus, $\Delta(dF-F) + Be^{rh} = 0.4(0.85*49 - 49) + 10e^{0.15*0.25}$ = **$C_d$ = $7.442119971**

# Section 23

# Exam-Style Questions on Binomial Option Pricing

The problems in this section were designed to be similar to problems from past versions of Exam 3F / Exam MFE. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

**Problem ESQBOP1.**

**Similar to Question 14 from the Casualty Actuarial Society's Spring 2007 Exam 3:**

A European call option on Opaque LLC stock has the following specifications:
Strike price = $578, current stock price = $581, time to expiration = 4 years, annual continuously compounded interest rate = 0.12, dividend yield = 0. You can use a two-period binomial model to calculate the option's price.

This is the binomial tree for the stock price movements:

----------------------981.89
----------755.3 -----------
581 -------------- 604.24
---------464.8 ------------
---------------------371.84

Find the price of one European call option on Opaque LLC stock.

**Solution ESQBOP1.**

First, it is useful to find u and d. Here, u = 755.3/581 = u = 1.3 and d = 464.8/581 = d = 0.8.

Since this is a two-period model, we know that h = 2. Furthermore, r = 0.12, and $\partial$ = 0.

We calculate $p^* = (e^{(r-\partial)h} - d)/(u - d) = (e^{(0.12)2} - 0.8)/(1.3 - 0.8) = p^* = 0.9424983006$

We note that $C_{uu} = 981.89 - 578 = C_{uu} = 403.89$ and $C_{du} = 604.24 - 578 = C_{du} = 26.24$

$C_{dd} = 0$, since the stock price in that case is lower than the strike price.

Now we use the formula $C = e^{-2rh}[(p^*)^2C_{uu} + 2(p^*)(1-p^*)C_{ud} + (1 - p^*)^2C_{dd}] =$

$e^{-2*0.12*2}[(0.9424983006)^2 403.89 + 2(0.9424983006)(1-0.9424983006)26.24 + 0] =$

**C = $223.7649974**

**Problem ESQBOP2.**

**Similar to Question 15 from the Casualty Actuarial Society's [Spring 2007 Exam 3](#):**

Within the framework of a binomial pricing model, you have the following data:

The price of one share of Transparent Co. is $200; the option strike price is $187. The annual continuously compounded risk-free interest rate is 0.07, and the stock's annual continuously-compounded dividend yield is 0.03. The volatility of the stock's price movements in 0.4. The options' time to expiration is 2 years, and the length of one time period within the binomial model is 6 months. Find the risk-neutral probability that the stock price will increase over one time period.

**Solution ESQBOP2.** First, we need to compute $u = e^{(r-\partial)h + \sigma\sqrt{(h)}}$ and $d = e^{(r-\partial)h - \sigma\sqrt{(h)}}$, given that r = 0.07, $\partial$ = 0.03, h = 0.5, so $(r - \partial)h = 0.02$. $\sigma = 0.4$, so $u = e^{(r-\partial)h + \sigma\sqrt{(h)}} = e^{0.02 + 0.4\sqrt{(0.5)}}$ = u = 1.353701527

$d = e^{0.02 - 0.4\sqrt{(0.5)}}$ = d = 0.7688628203.

Thus, $p^* = (e^{(r-\partial)h} - d)/(u - d) = (e^{0.02} - 0.7688628203)/(1.353701527 - 0.7688628203)$ = **p* = 0.4297569854**

**Solution ESQBOP3.**

**Similar to Question 16 from the Casualty Actuarial Society's [Spring 2007 Exam 3](#):**

The following binomial tree models the possible price movements of a stock, Q, that does not pay dividends:

----------544.5
363 -----------
-----------72.6
t = 0----- t =1

A European call option on Q has a strike price of $400 and expires at t =1. In the replicating portfolio for this option, what is the number of shares of stock Q present?

**Solution ESQBOP3.** We seek to find Δ. First, we note that $C_u$ = 544.5 - 400 = 144.5, whereas $C_d$ = 0, since $S_d$< 400. u = 544.5/363 = u = 1.5 and d = 72.6/363 = d = 0.2. S is given as 363.

We use the formula $\Delta = e^{-\partial h}(C_u - C_d)/[S(u-d)]$, which, since the stock pays no dividends, simplifies to $\Delta = (C_u - C_d)/[S(u-d)] = 144.5/[363(1.5-0.2)] = \Delta =$ **0.3062089426**

**Problem ESQBOP4.**

**Similar to Question 17 from the Casualty Actuarial Society's Spring 2007 Exam 3:**

A binomial tree with valuations made every 4 years is used to calculate the price of one 8-year European put option on the stock of Inscrutable Co. The stock price volatility is 0.5, and annual continuously compounded risk-free interest rate is 0.09. The stock pays no dividends. The put option's strike price is $56, and the stock price is $70. Find the price of such a put option.

**Solution ESQBOP4.** Here, $h = 4$, $\sigma = 0.5$, $\partial = 0$, and $r = 0.09$.

First, we need to compute $u = e^{(r-\partial)h + \sigma\sqrt{(h)}}$ and $d = e^{(r-\partial)h - \sigma\sqrt{(h)}}$.
$u = e^{(0.09)4 + 0.5\sqrt{(4)}} = u = 3.896193302$
$d = e^{(0.09)4 - 0.5\sqrt{(4)}} = d = 0.527292424$

We also note that $p^* = (e^{(0.09)4} - 0.527292424)/(3.896193302 - 0.527292424) =$
$p^* = 0.2689414214$
Now we find $S_{uu} = 3.896193302^2 * 70 = 1062.622557$, so $P_{uu} = 0$
$S_{du} = 3.896193302 * 0.527292424 * 70 = 143.8103247$, so $P_{du} = 0$
$S_{dd} = 0.527292424^2 * 70 = 19.46261103$, so $P_{dd} = 56 - 19.46261103 = P_{dd} = 36.53738897$
Now we use the formula $P = e^{-2rh}[(p^*)^2 P_{uu} + 2(p^*)(1-p^*)P_{ud} + (1 - p^*)^2 P_{dd}] =$
$e^{-2*0.09*4}[0 + 0 + (1 - 0.2689414214)^2 36.53738897] =$ **P = $9.50495001**

**Problem ESQBOP5.**

**Similar to Question 17 from the Casualty Actuarial Society's Fall 2007 Exam 3:**

In one year, the stock of Timorous, Inc., will be worth either $230 or $185. The stock is currently worth $200 and pays no dividends. The annual risk-free interest rate is 0.033, compounded continuously. Using the one-period binomial option pricing model, calculate the delta for a call option on Timorous, Inc., stock that expires in 1 year and has a strike price of $175.

**Solution ESQBOP5.** We use the formula $\Delta = e^{-\partial h}(C_u - C_d)/[S(u-d)]$, which, since the stock pays no dividends, simplifies to $\Delta = (C_u - C_d)/[S(u-d)]$.

$C_u = 230 - 175 = 55$, and $C_d = 185 - 175 = 10$. Furthermore,
$S = 200$, $u = 230/200 = 1.15$ and $d = 185/200 = 0.925$. Thus,
$\Delta = (55 - 10)/[200(1.15-0.925)] = \Delta =$ **1**.

(Yes, this is correct. It means that you would need to hold one share of stock in the replicating portfolio for the call option. Also as part of the portfolio, you would need to borrow some amount of money B, so the B term will be negative and the option price will be less than the stock price, as expected.)

# Section 24

# Exam-Style Questions on Binomial Option Pricing for Actuaries – Part 2

The problems in this section were designed to be similar to problems from past versions of Exam 3F / Exam MFE. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

**Problem MESQBOP1.**

**Similar to Question 18 from the Casualty Actuarial Society's Fall 2007 Exam 3:**

The stock of Devious Co. currently trades for $65 per share. The annual continuously compounded risk-free interest rate is 0.10. Every 2 years, the stock price either increases by 30% or decreases by 20%. The stock pays no dividends. Using a two-period binomial model, calculate the price of a 4-year American put option on Devious Co. stock with a strike price of $74.

**Solution MESQBOP1.**

We cannot bypass the intermediate values of put prices here, because for an American put, at each node in the binomial tree, we must choose the higher of two values, K - S or the otherwise equivalent European option price.

We are given that u = 1.3 and d = 0.8. Also, S = 65, r = 0.10, $\partial = 0$, h = 2, and K = 74.

We first find $p^* = (e^{(r-\partial)h} - d)/(u - d) = (e^{(0.1)2} - 0.8)/(1.3 - 0.8) = p^* = 0.8428055163$

$S_{uu} = 1.3^2*65 = 109.85$, implying that $P_{uu} = 0$
$S_{du} = 1.3*0.8*65 = 67.6$, implying that $P_{du} = 74 - 67.6 = P_{du} = 6.4$
$S_{dd} = 0.8^2*65 = 41.6$, implying that $P_{dd} = 74 - 41.6 = P_{dd} = 32.4$

We can calculate the equivalent European option prices two years from now:

$P_u = e^{-rh}[p^*P_{uu} + (1 - p^*)P_{du}] = e^{-0.1*2}[0 + (1 - 0.8428055163)6.4] = P_u = 0.8236797313$

If the stock price increases in two years, $S_u = 1.3*65 = 84.5$, so K - S < 0 and thus

$P_u = 0.8236797313$

$P_d = e^{-rh}[p*P_{du} + (1 - p*)P_{dd}] = e^{-0.1*2}[0.8428055163*6.4 + (1 - 0.8428055163)32.4] =$

$P_d = 8.586075728$. If the stock price declines in two years, $S_d = 0.8*65 = 52$, so K - S = 74 - 52 = 22. Thus, it will be optimal to exercise the option and so $P_d = 22$.

Thus, $P = e^{-rh}[p*P_u + (1 - p*)P_d] = e^{-0.1*2}[0.8428055163*0.8236797313 + (1 - 0.8428055163)22]$ = P = $3.399763456. But we compare this to the gain from exercising the option immediately, which is 74 - 65 = 9 > 3.399763456. Thus, the option today is worth **$9.**

**Problem MESQBOP2.**

**Similar to Question 19 from the Casualty Actuarial Society's Fall 2007 Exam 3:**

The price of a European call option on Impeccable LLC stock can be found using a one-period binomial model. The option expires in 5 months, and the stock pays no dividends. The option's strike price is $400, while the stock's current price is $385. The annual continuously compounded risk-free interest rate is 0.06. The annual stock price volatility is 0.5. Find the current value of the call option.

**Solution MESQBOP2.** Here, h = 5/12, σ = 0.5, ∂ = 0, and r = 0.06.

First, we need to compute $u = e^{(r-\partial)h + \sigma\sqrt{(h)}}$ and $d = e^{(r-\partial)h - \sigma\sqrt{(h)}}$.

$u = e^{(0.06)5/12 + 0.5\sqrt{(5/12)}} = u = 1.415876271$
$d = e^{(0.06)5/12 - 0.5\sqrt{(5/12)}} = d = 0.7424879687$
$p* = (e^{(r-\partial)h} - d)/(u - d) = (e^{(0.06)5/12} - 0.7424879687)/(1.415876271 - 0.7424879687) =$
$p* = 0.4200060364$.

It is evident that the call option will only have value in 5 months if the stock price increases to uS = 1.415876271*385 = 545.1123643. In that case, $C_u = 545.1123643 - 400 = C_u = 145.1123643$

We use the formula $C = e^{-rh}[p*C_u + (1 - p*)C_d] = e^{-0.06(5/12)}[0.4200060364*145.1123643 + 0] =$

**C = $59.44325579**

**Problem MESQBOP3.**

**Similar to Question 4 from the Society of Actuaries' Sample MFE Questions and Solutions:**

The stock of Hospitable Co. currently trades for $45 per share. The stock pays no dividends. The annual continuously compounded risk-free interest rate is 0.13. u = 1.4, where uis one plus the rate of capital gain on the stock per period if the stock price goes up. d = 0.7, where dis one plus the rate of capital lose on the stock per period if the stock price goes down. Using a two-period binomial model, calculate the price of a 3-year American call option on Hospitable Co. stock with a strike price of $49.

**Solution MESQBOP3.** Here, h = 1.5, $\partial = 0$, r = 0.13, u = 1.4, d = 0.7.

Thus, $p^* = (e^{(r-\partial)h} - d)/(u - d) = (e^{(0.13)1.5} - 0.7)/(1.4 - 0.7) = p^* = 0.7361585521$

$S_u = 1.4*45 = 63$

$S_{uu} = 1.4*63 = 88.2$, so $C_{uu} = 88.2-49 = C_{uu} = 39.2$

$S_{ud} = 0.7*63 = 44.1$, so $C_{du} = 0$

$S_d = 0.7*45 = 31.5$

$S_{dd} = 0.7*31.5 = 22.05$, so $C_{dd} = 0$

Thus, since $C_{du} = 0$ and $C_{dd} = 0$, $C_d = 0$ as well.

$C_u = e^{-rh}[p^*C_{uu} + (1 - p^*)C_{ud}] = e^{-0.13*1.5}[0.7361585521*39.2 + 0] = C_u = 23.7448814.$

This is greater than $S_u$ - K = 14.

Thus, $C = e^{-0.13*1.5}[0.7361585521*23.7448814 + 0] = C = 14.38314778.$

This is greater than S - K = 4. Thus, **C = $14.38314778**.

**Problem MESQBOP4.**

**Similar to Question 5 from the Society of Actuaries' Sample MFE Questions and Solutions:**

One Golden Hexagon (GH) currently trades for 998 Wooden Circles (WC). The volatility of this exchange rate is 0.2. The annual GH-denominated continuously compounded interest rate is 0.04, while the annual WC-denominated continuously compounded interest rate is 0.05. Using a three-period binomial model, calculate the price of a 3-year American put option on Golden Hexagons, denominated in WC. The put option has a strike price of 1023 WC.

**Solution MESQBOP4.**

We first find $u = e^{(r-f)h + \sigma\sqrt{(h)}}$. Here, r = 0.05, f = 0.04, h = 1, and σ = 0.2. S

Thus,

$u = e^{0.01 + 0.2\sqrt{(1)}} = u = 1.23367806$

$d = e^{(r-f)h - \sigma\sqrt{(h)}} = e^{0.01 - 0.2\sqrt{(1)}} = d = 0.8269591339$

Currently, the exchange rate x = 998.

$x_{duu}$ = $1.23367806^2$*0.8269591339*998 = 1256.08281, so $P_{duu}$ and $P_{uuu}$ are both 0, since the exchange rate will exceed the strike price if the stock goes up consistently or up twice and down once.

$x_{ddu}$ = 1.23367806*$0.8269591339^2$*998 = 841.977487, so $P_{ddu}$ = 1023 - 841.977487 =
$P_{ddu}$ = 181.022513
$x_{ddd}$ = $0.8269591339^3$*998 = 564.3943878, so $P_{ddd}$ = 1023 - 564.3943878 = $P_{ddd}$ = 458.6056122
p* = $(e^{(r-f)h}$ - d)/(u - d) =($e^{0.01}$ - 0.8269591339)/(1.23367806 - 0.8269591339) =
p* = 0.4501660026

Thus, it seems at first that $P_{dd}$ = $e^{-0.05}$[0.4501660026*181.022513 + (1 - 0.4501660026)458.6056122] = 317.3749751

But $x_{dd}$ = $0.8269591339^2$*998 = 682.4936864, so K - $x_{dd}$ = 1023 - 682.4936864 = 340.5063136 > 317.3749751, so $P_{dd}$ = 340.5063136

Since $P_{duu}$ and $P_{uuu}$ are both 0, $P_{uu}$ = 0.

Since $P_{du}$ = $e^{-0.05}$[0.4501660026* $P_{duu}$ + (1 - 0.4501660026)*$P_{ddu}$] =

$e^{-0.05}$[0 + (1 - 0.4501660026)*181.022513] = $P_{du}$ = 94.67808283 (European equivalent)

$x_{ud}$ = 1.23367806*0.8269591339*998 = 1018.160937, so (1023 - $x_{ud}$ ) < 94.67808283 and so

$P_{du}$ = 94.67808283.

Now we find

$P_u$ = $e^{-rh}$[p*$P_{uu}$ + (1 - p*)$P_{ud}$] = $e^{-0.05}$[0 + (1 - 0.4501660026)94.67808283] = $P_u$ = 49.51836774, which is greater than K - $x_u$ < 0.

$P_d$ = $e^{-rh}$[p*$P_{du}$ + (1 - p*)$P_{dd}$] =

$e^{-0.05}$[0.4501660026*94.67808283+ (1 - 0.4501660026)340.5063136] = $P_d$ = 218.6332359.

$x_d$ = 0.8269591339*998 = 825.3052157, so K - $x_d$ = 1023 - 825.3052157 = 197.6947843 < 218.6332359. Thus, $P_d$ = 218.6332359.

So P = $e^{-rh}$[p*$P_u$ + (1 - p*)$P_d$] =

$e^{-0.05}$[0.4501660026*49.51836774 + (1 - 0.4501660026)218.6332359] = P = 135.5534954, which is greater than K - x = 25. So **P = 135.5534954 WC**

This problem had enough work for two or three problems of a more typical (and reasonable) size, so I will conclude Section 24 with only 4 problems.

# Section 25

# Volatility and Early Exercise of American Options

On an American call option where the volatility is zero, it is optimal to defer exercise as long as the following condition holds:
$rK > \partial S$

It is optimal to exercise whenever

$S > rK/\partial$

American "call options are early-exercised in order to capture dividends on the underlying stock."

With put options, the reverse holds. It is optimal to exercise early when

$S < rK/\partial$

and it is optimal to defer exercise when $rK < \partial S$

Meaning of variables:

$r$ = annual continuously-compounded risk-free interest rate.

$\partial$ = annual continuously-compounded dividend yield.

$K$ = strike price of the option.

$S$ = underlying asset (stock) price.

When volatility is positive, the exercise bounds for lower volatility are lower on call options than the exercise bounds for higher volatility. The insurance effect of holding call options is greater the higher the volatility, and this effect is lost if the call is exercised. So if stock price volatility = $\sigma = 0.1$ and the lowest stock price at which it is optimal to exercise the call is \$200, then if the stock price volatility were $\sigma > 0.1$, we would expect the lowest stock price at which it is optimal to exercise the call to be > \$200.

With put options, lower volatility means that it is optimal to exercise at a *higher* stock price than would be optimal if volatility were higher. So with put options if stock price volatility = $\sigma = 0.1$ and the lowest stock price at which it is optimal to exercise the call is \$200, then if the stock price volatility were $\sigma > 0.1$, we would expect highest stock price at which it is optimal to exercise the call to be < \$200.

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 11, pp. 333-335, 365.

**Problem VEEAO1.** Impressive Co. stock currently trades for \$456 per share. The price movements of this stock are not volatile at all. The stock pays dividends with an annual continuously compounded yield of 0.04. The annual continuously compounded interest rate is 0.05. You own American call options on Impressive Co. stock and you know that exercising early is not optimal right now. The strike price of the call options must at least be greater than some number X. Find X.

**Solution VEEAO1.** We are given that the following condition holds: $rK > \partial S$ and $S = 456$, $\partial = 0.04$, $r = 0.05$. Thus, $0.05K > 0.04*456$, so $K > 0.04*456/0.05$ or $K > 364.8$. Thus, **X = \$364.8**

**Problem VEEAO2.** You know that if the price of the non-volatile Precocious LLC stock increased at all, it would be a good idea to exercise an American call option on Precocious LLC stock right now. The stock currently trades for \$430, and the annual continuously compounded interest rate is 0.09. The strike price of the option is \$344. Find the annual continuously compounded yield on the dividends of Precocious LLC stock.

**Solution VEEAO2.** If the price of Precocious LLC stock increased at all, then the condition

$S > rK/\partial$ would hold. Right now, $S = rK/\partial$ is the case, so $\partial = rK/S$, where $r = 0.09$, $K = 344$, and $S = 430$. Thus, $\partial = 0.09*344/430 =$ **$\partial$ = 0.072**

**Problem VEEAO3.** You own an American put option on the non-volatile Mysterious, Inc., stock. The strike price of the option is 23, and the stock's annual continuously compounded dividend yield is 0.4. The annual continuously compounded interest rate is 0.1. For which of these stock prices would it be optimal to exercise the option?

(a) S = 20
(b) S= 15.65
(c) S = 13.54
(d) S = 12.34
(e) S = 6.78
(f) S = 3.45

**Solution VEEAO3.** We know that for American put options, it is optimal to exercise early whenever $S < rK/\partial$. We know that $r = 0.1$, $K = 23$, and $\partial = 0.4$, so $rK/\partial = 5.75$. Of the prices listed, only $3.45 < 5.75$. Thus, only **(f) is the correct answer** and only (f) describes a price for which early exercise would be optimal.

**Problem VEEAO4.** Volatile Co. stock prices currently have a volatility of $\sigma = 0.3$. For American call options with a strike price of \$120 and time to expiration 1 year, you know that the lowest stock price where exercise is optimal is \$160. If the stock price volatility changes, for which of these volatilities will exercise still be optimal at a stock price of \$160? More than one correct answer is possible.

(a) $\sigma = 0.1$
(b) $\sigma = 0.2$
(c) $\sigma = 0.4$
(d) $\sigma = 0.5$
(e) $\sigma = 0.6$

**Solution VEEAO4.** Exercise will definitely *not* be optimal at \$160 if the volatility increases, because the lowest bound for optimal exercise price will be pushed up by such an increase. This rules out choices (c), (d), and (e). On the other hand, exercise will definitely be optimal at \$160 if the volatility decreases, because the lowest bound for optimal exercise price will be pushed down by such a decrease. Thus, both **(a) and (b) are correct answers.**

**Problem VEEAO5.** Volatile Co. stock prices currently have a volatility of $\sigma = 0.3$. For American put options with a strike price of \$90 and time to expiration 1 year, you know that the highest stock price where exercise is optimal is \$40. If the stock price volatility changes, for which of these volatilities will exercise still be optimal at a stock price of \$40? More than one correct answer is possible.

(a) $\sigma = 0.1$
(b) $\sigma = 0.2$
(c) $\sigma = 0.4$
(d) $\sigma = 0.5$
(e) $\sigma = 0.6$

**Solution VEEAO5.** Exercise will definitely not be optimal at \$40 if volatility increases, because the highest bound for optimal exercise price will be pushed up by such an increase. This rules out choices (c), (d), and (e). On the other hand, exercise will definitely be optimal at \$40 if the volatility decreases, because the highest bound for optimal exercise price will be pushed up by such a decrease. Thus, both **(a) and (b) are correct answers.**

# Section 26

# Comparing Risk-Neutral and Real Probabilities in the Binomial Model

In a risk-neutral situation, the risk-neutral probability of the stock price increasing during a period of the binomial model must satisfy this equation:

$(p^*)uSe^{\delta h} + (1-p^*)dSe^{\delta h} = Se^{rh}$

With real probabilities, for a stock that does not pay dividends,

$puS + (1-p)dS = Se^{\alpha h}$

Meaning of variables:

S = underlying asset (stock) price.
$p^* = (e^{(r-\partial)h} - d)/(u - d)$ = risk-neutral probability of stock price increase.
p = true probability of stock price increase.
u = 1 + rate of capital gain on stock if stock price increases.
d = 1 + rate of capital loss on stock if stock price decreases.
h = one time period in binomial model.
$\partial$ = annual continuously-compounded dividend yield.
$\alpha$ = the annual continuously compounded expected return on the stock.

Thus, the formula is as follows for p:
$p = (e^{\alpha h} - d)/(u - d)$

There is a constraint that enables the value of p to always be between 0 and 1:
$d < e^{\alpha h} < u$

**Problem CRNRPBM1.** The price of Deleterious Co. stock, which does not pay dividends, will change by a factor of either 1.5 or 0.4 in 2 years. Find the upper constraint on the annual continuously compounded expected return on the stock.

**Solution CRNRPBM1.** We use the inequality $d < e^{\alpha h} < u$, where we want to find the upper bound on $\alpha$, given that $e^{\alpha h} < u$. This upper bound X occurs when $e^{Xh} = u$ or $X = \ln(u)/h$. Here, u = 1.5 and h = 2, so X = ln(1.5)/2 = **X = 0.2027325541**

**Problem CRNRPBM2.** The price of Deleterious Co. stock, which does not pay dividends, will change by a factor of either 1.5 or 0.4 in 2 years. The annual continuously compounded expected return on the stock is 0.05. Find the real probability that the stock will increase in price in two years.

**Solution CRNRPBM2.** We use the formula p = $(e^{\alpha h} - d)/(u - d)$, where u = 1.5, d = 0.4, h = 2, and α = 0.05. Thus, p = $(e^{0.05*2} - 0.4)/(1.5 - 0.4)$ = **p = 0.641064471**.

**Problem CRNRPBM3.** Imperious LLC stock will change by a factor of either 1.9 or 0.2 in h years. The stock pays dividends on a continuously compounded basis with annual yield of 0.06. The annual continuously compounded rate of interest is 0.09. The risk-neutral probability of the stock price's increase in h years is 0.72. Find h.

**Solution CRNRPBM3.** To find h, we will need to use the formula p* = $(e^{(r-\partial)h} - d)/(u - d)$.

We can simplify p* using the values given for u = 1.9, d = 0.2, r = 0.09, and δ = 0.06, so (r - δ) = 0.03. Thus,

p* = 0.72 = $(e^{0.03h} - 0.2)/1.7$. Thus, $e^{0.03h}$ = 1.7*0.72 + 0.2 = 1.424 and h = ln(1.424)/0.03 = **h = 11.7823271 years**.

**Problem CRNRPBM4.** You live in a risk-neutral world, and just purchased a share of Lucrative Co. stock at a price of $1000. You do not know the annual continuously compounded risk-free interest rate (who does, in reality?), but you do know that the stock pays dividends with an annual continuously compounded yield of 0.12. You also plan to hold the stock for 13 years, at the end of which it will be either 3.4 or 0.1 of its present price. The risk-neutral probability of a stock price increase is 0.82. You *could have* invested the $1000 for 13 years in an account earning the annual continuously compounded risk-free interest rate. What would your account balance be at the end of 13 years if you did so?

**Solution CRNRPBM4.** We use the formula $(p^*)uSe^{\delta h} + (1-p^*)dSe^{\delta h} = Se^{rh}$, where we desire to find $Se^{rh}$. We are given that p* = 0.82, u = 3.4, d = 0.1, h = 13, ∂ = 0.12, and S = 100. Thus, $Se^{rh}$ = $0.82*3.4*1000e^{0.12*13} + 0.18*0.1*1000e^{0.12*13}$ = **$Se^{rh}$ = $13353.25241.**

**Problem CRNRPBM5.** At t =13 years, Lucrative Co. stopped paying dividends on its stock, and everybody suddenly became risk-averse. Lucrative Co. also changed its name to Not-So-Lucrative Co. In another 10 years, its stock price will change by a factor of 1.3 or by a factor of 0.3. You take $13353.25241and you invest it in Not-So-Lucrative Co. stock, knowing that the *real* probability of the stock price increasing over that time period is 0.76. You *could have* invested the $13353.25241 for 10 years in an account earning a return equal to the annual continuously compounded expected return on the stock, but you do not know what that return is. What would your account balance be at the end of 10 years if you did so?

**Solution CRNRPBM5.** We use the formula puS + (1-p)dS = $Se^{\alpha h}$, where we desire to find $Se^{\alpha h}$. We are given that p = 0.76, u = 1.3, d = 0.3, S = $13353.25241 (for all practical purposes, we can consider this the price of one share). Thus, $Se^{\alpha h}$ = 13353.25241(0.76*1.3 + 0.24*0.3) = **$Se^{\alpha h}$ = $14154.44756.**

# Section 27

# Option Valuation Using True Probabilities in the Binomial Model

In a non-risk-neutral world, where we use true probabilities instead of risk-neutral probabilities, if the expected return on a stock option is $\gamma$, then we can find $\gamma$ by taking a weighted average of the return of the assets in a replicating portfolio for the option. Recall from Section 15 that a replicating portfolio on an option consists of $\Delta$ in shares of the underlying asset (here, a stock) and B in lending. The following formula enables us to compute $\gamma$.

$e^{\gamma h} = [S\Delta/(S\Delta + B)]e^{\alpha h} + [B/(S\Delta + B)]e^{rh}$

In this case, the expected European call option payoff can be calculated in the binomial model as follows.

$C = e^{-\gamma h}[pC_u + (1-p)C_d]$, where $p = (e^{\alpha h} - d)/(u - d)$

This calculation gives the same ultimate result as the calculation which involves risk-neutral probabilities. So for all practical purposes, using risk-neutral probabilities in the binomial model is just as realistic as attempting to account for true probabilities (unless you are investing anything in the real world, in which case you *will* lose money if you rely solely on these formulas!).

**Meaning of Variables:**

S = underlying asset (stock) price.

$p^* = (e^{(r-\partial)h} - d)/(u - d)$ = risk-neutral probability of stock price increase.

p = true probability of stock price increase.

u = 1 + rate of capital gain on stock if stock price increases.

d = 1 + rate of capital loss on stock if stock price decreases.

h = one time period in binomial model.

r = annual continuously-compounded risk-free interest rate

$\alpha$ = the annual continuously compounded expected return on the stock.

C = price of the call option.

$C_u$ = price of the call option if the stock price increases.

$C_d$ = price of the call option if the stock price decreases.

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 11, pp. 347-348.

**Problem OVUTPBM1.** You know that the replicating portfolio for a call option on Dependable Co. stock consists of (2/3) shares and borrowing \$45. You also know that the annual continuously compounded expected return on the stock is 0.32 and the annual continuously-compounded risk-free interest rate is 0.03. The stock currently sells for \$450. Find the expected return on the call option over the course of one year using a one-period binomial model.

**Solution OVUTPBM1.** We use the formula $e^{\gamma h} = [S\Delta/(S\Delta + B)]e^{\alpha h} + [B/(S\Delta + B)]e^{rh}$, and we want to find $\gamma$. Here, h = 1, S = 450, r = 0.03, $\alpha$ = 0.32, B = -45 (money is borrowed), and $\Delta$ = 2/3.

Then $e^{\gamma h} = [S\Delta/(S\Delta + B)]e^{\alpha h} + [B/(S\Delta + B)]e^{rh} =$

$(450(2/3)/[450(2/3)-45])e^{0.32} - 45/[450(2/3)-45]e^{0.03} = 1.438305393 = e^{\gamma}$. Thus, $\gamma$ = ln(1.438305393) = **$\gamma$ = 0.3634656103** (nice return for one year!)

**Problem OVUTPBM2.** The stock of Dependable Co. currently sells for \$450. You also know that the annual continuously compounded expected return on the stock is 0.32 and the annual continuously-compounded risk-free interest rate is 0.03. You know that in three years, the stock price will change either by a factor of 0.34 or by a factor of 3.

The annual continuously-compounded expected return on a particular call option is 0.3634656103. This option has a strike price of \$465 and a time to expiration of three years. Find the price today of this option using a one-period binomial model.

**Solution OVUTPBM2.** First we find $p = (e^{\alpha h} - d)/(u - d)$, where $\alpha$ = 0.32, h = 3, d = 0.34, and u = 3. Thus, $p = (e^{0.32*3} - 0.34)/(3 - 3.4)$ = p = 0.85402124306.

If the stock price triples in 3 years, uS = 1350 and so $C_u$ = 1350 - 465 = 885.

If the stock price declines in 3 years, the call option will be worth $C_d$ = 0.

Also, we are given that $\gamma$ = 0.3634656103.

Thus, $C = e^{-\gamma h}[pC_u + (1-p)C_d] = e^{-0.3634656103*3}[0.85402124306*885 + 0] =$

**C = \$254.0845601**

**Problem OVUTPBM3.** A call option on Artificial LLC currently sells for \$32. In two years, it will sell for \$35 with a real probability of 0.93 and for \$13 with a real probability of 0.07. Using a one-period binomial model, find the annual continuously-compounded expected return on this option.

**Solution OVUTPBM3.** We use the formula $C = e^{-\gamma h}[pC_u + (1-p)C_d]$, where

$e^{-\gamma h} = C/[pC_u + (1-p)C_d]$ and $C = 32$, $C_u = 35$, $C_d = 13$, $h = 2$, and $p = 0.93$. Thus,

$e^{-2\gamma} = 32/[0.93*35 + 0.07*13] = 0.9563658099$. Thus, $\gamma = -\ln(0.9563658099)/2 =$

**$\gamma = 0.0223073964$**

**Problem OVUTPBM4.** A call option on Content-of-Character Co. currently sells for \$45. In three years, it will sell for \$67 or for \$32. The annual continuously-compounded expected return on this option is 0.12. Using a one-period binomial model, find the probability that the option will sell for \$67 in three years.

**Solution OVUTPBM4.** We use the formula $C = e^{-\gamma h}[pC_u + (1-p)C_d]$, rearranging it to solve for p: $Ce^{\gamma h} = pC_u - pC_d + C_d$

$Ce^{\gamma h} - C_d = p(C_u - C_d)$

$p = (Ce^{\gamma h} - C_d)/(C_u - C_d)$

Here, $\gamma = 0.12$, $C = 45$, $C_u = 67$, $C_d = 32$, $h = 3$, so $p = (45e^{0.12*3} - 32)/(67-32) =$

**$p = 0.9285663901$**

**Problem OVUTPBM5.** By performing some financial analysis of Elusive Co. stock, Maximus concludes that for a particular call option, $e^{\gamma h}*(S\Delta + B) = 0.05$ for $h = 3$. Maximus also knows that the annual continuously-compounded risk-free interest rate is 0.06 and the annual continuously compounded expected return on the stock is 0.12. The delta of the replicating portfolio for this option is (3/4), and the stock is currently worth \$990. How much in lending does a replicating portfolio for this call option contain?

**Solution OVUTPBM5.** We use the formula $e^{\gamma h} = [S\Delta/(S\Delta + B)]e^{\alpha h} + [B/(S\Delta + B)]e^{rh}$ , noting that $e^{\gamma h}*(S\Delta + B) = S\Delta e^{\alpha h} + Be^{rh} = 0.05$. We are also given $\Delta = 0.75$, $r = 0.06$, $h = 3$, $S = 990$, and $\alpha = 0.12$. We want to find $B = [0.05 - S\Delta e^{\alpha h}]e^{-rh} =$

$[0.05 - 990(3/4)e^{0.12*3}]e^{-0.06*3} =$ **$B = -888.8921286$**. (This means that the replicating portfolio requires one to *borrow* \$888.8921286.)

# Section 28

# The Random-Walk Model

Let a coin be flipped n times and let $Y_i$ denote the outcome of the ith flip. If the coin lands heads on the ith flip, then $Y_i = 1$. If the coin lands tails on the ith flip, then $Y_i = -1$. We can obtain the sum of the $Y_i$'s for the n flips ($Z_n$) as follows.

$Z_n = {}_{i=1}{}^{n}\sum Y_i$

Furthermore, to find $Y_n$, the outcome of the nth flip, we use the following formula:
$Z_n - Z_{n-1} = Y_n$
If $Y_n$ is heads: $Z_n - Z_{n-1} = +1$
If $Y_n$ is tails: $Z_n - Z_{n-1} = -1$

The random-walk model states that the more times we flip a coin, the likelier it is that we will be farther away from 0.

The random-walk model can be applied to stock price movements as well, though the above equations do not suffice to describe such movements. The binomial model is a special case of the random walk model that also incorporates the assumption that "*continuously compounded returns are a random walk.*" According to R. L. McDonald, there are the four properties of continuously compounded returns that the binomial model incorporates (where r = continuously compounded rate of return, S = stock price, and the subscripts denote time periods).

**Logarithmic function computes returns from prices:** $r_{t,t+h} = \ln(S_{t+h} / S_t)$
**Exponential function computes prices from returns:** $S_{t+h} = S_t e^{r\_(t,t+h)}$
**Continuously compounded returns are additive:** $r_{t,t+nh} = {}_{i=1}{}^{n}\sum r_{t+(i-1)h,t+ih}$
**Continuously compounded returns can be less than - 100%.** $e^r$ is always positive, even if r is a large negative number.

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 11, pp. 351-353.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem RWM1.** You flip a coin and get the following results: heads, heads, tails, tails, tails, heads, heads, heads, heads, tails. What is the sum $Y_i$'s for all of these flips?

**Solution RWM1.** We use the formula $Z_n = {}_{i=1}{}^{n}\sum Y_i$, which in this case can be applied by adding the number of heads (1's) and the number of tails ("-1"'s) resulting from the flips. 10 flips in all were made, 6 of which were heads and 4 of which were tails. Thus, $Z_n = 6(1) + 4(-1) = 6 - 4 = Z_{10} = 2$.

**Problem RWM2.** A coin was flipped 13 times, and you know that $Z_{13} = 6$, $Y_{12} = -1$, $Y_{11} = -1$, $Y_{10} = 1$, and $Y_9 = 1$. Find $Z_8$.

**Solution RWM2.** We apply the formula $Z_n - Z_{n-1} = Y_n$ multiple times to get.
$Z_{13} - Y_{12} - Y_{11} - Y_{10} - Y_9 = Z_8 = 6 - (-1) - (-1) - 1 - 1 = 6 - 0 =$ $\mathbf{Z_8 = 6}$.

**Problem RWM3.** The price of Lucrative Co. stock today is \$567. Five years ago, it was \$320. Find the continuously compounded return on Lucrative Co. stock over the past five years.

**Solution RWM3.** We use the formula $r_{t,t+h} = \ln(S_{t+h}/S_t) = \ln(567/320) =$
$\mathbf{r_{0,5} = 0.5720383079}$

**Problem RWM4.** The stock of Despicable Co. earns the following continuously compounded returns:
January 20034 to January 20035: -0.45
January 20035 to January 20036: -0.2
January 20036 to January 20037: 0.43
January 20037 to January 20038: 0.03
January 20038 to January 20039: 23
January 20039 to January 20040: -1.2
January 20040 to January 20041: -32

What is the continuously compounded return on the stock of Despicable Co. from January 20034 to January 20041?

**Solution RWM4.** We use the formula $r_{t,t+nh} = \sum_{i=1}^{n} r_{t+(i-1)h,t+ih}$ and simply add up the returns from each year: $r_{20034,20041} = -0.45 - 0.2 + 0.43 + 0.03 + 23 - 1.2 - 32 =$

$\mathbf{r_{20034,20041} = -10.39}$. (If you are alive in 20034, do not invest in this stock!)

**Problem RWM5.** The stock of Despicable Co. earns the following continuously compounded returns:
January 20034 to January 20035: -0.45
January 20035 to January 20036: -0.2
January 20036 to January 20037: 0.43
January 20037 to January 20038: 0.03
January 20038 to January 20039: 23
January 20039 to January 20040: -1.2
January 20040 to January 20041: -32

Hinjanmin purchases a share of Despicable Co. stock for \$56900 in January 20036 and wisely sells it on January 20039. How much money does he get after selling the stock?

**Solution RWM5.** We note that the three-year rate of return from January 20036 to January 20039 is 0.43 + 0.03 + 23 = 23.46. Thus, using the formula $S_{t+h} = S_t e^{r\_(t,t+h)}$, we get $S_{20039} = 56900e^{23.46} =$ **\$878,336,259,839,150** (making Hinjanmin a multi-trillionaire).

# Section 29

# Standard Deviation of Returns and Multi-Period Probabilities in the Binomial Model

The standard deviation of returns on an asset over time period of length h can be expressed as $\sigma_h = \sigma\sqrt{(h)}$, where σ is the standard deviation over a time period of length 1.

Using the binomial model, stock prices can be modeled as follows:

$S_{t+h} = S_t e^{(r-\delta)h \pm \sigma\sqrt{(h)}}$.

When we take the natural logs of both sides, we get $\ln(S_{t+h}/S_t) = (r-\delta)h \pm \sigma\sqrt{(h)}$.

The binomial model is an approximation of the *lognormal distribution.* According to R. L. McDonald, "The lognormal distribution is the probability distribution that arises from the assumption that *continuously compounded returns on the stock are normally distributed.*"

In a binomial tree, where the risk-neutral probability is p*, the probability of reaching the ith node can be expressed as

$p_{\text{ith node}} = [n!/((n-i)!i!)](p^*)^{n-1}(1-p^*)^i = C(n, i)(p^*)^{n-1}(1-p^*)^i$

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 11, pp. 354-358.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem SDRMPPBM1.** The standard deviation of returns on Frivolous Co. stock over 1 year is 0.67. Find the standard deviation of returns on Frivolous Co. stock over 12 years.

**Solution SDRMPPBM1.** We use the formula $\sigma_h = \sigma\sqrt{(h)}$, where h = 12, and σ = 0.67. Thus, $\sigma_{12} = 0.67\sqrt{(12)} = \boldsymbol{\sigma_{12} = 2.320948082}$

**Problem SDRMPPBM2.** The standard deviation of returns on Meticulous Co. stock over 10 years is 0.02. The standard deviation of returns on Frivolous Co. stock over Z years is 0.15. Find Z.

**Solution SDRMPPBM2.**

We use the formula $\sigma_h = \sigma\sqrt{(h)}$, but here h is expressed in 10-year periods. So Z = 10h.

$\sqrt{(h)} = \sigma_h/\sigma = 0.15/0.02 = \sqrt{(h)} = 7.5$, so h = 56.25 and **Z = 562.5 years**.

**Problem SDRMPPBM3.** The standard deviation of returns on Meticulous Co. stock over 10 years is 0.02; the annual continuously-compounded interest rate is 0.03, and the stock pays dividends at an annual continuously-compounded of 0.01. The stock price is currently $120/share. If the stock price increases in 10 years, what will it be?

**Solution SDRMPPBM3.** We use the formula

$S_{t+h} = S_t e^{(r- \delta)h + \sigma\sqrt{(h)}}$, where $S_t = 120$, $r = 0.03$, $\delta = 0.01$, $\sigma = 0.02$, $h = 10$, so

$S_{t+10} = 120e^{(0.03- 0.02)10 + 0.02\sqrt{(10)}} = S_{t+10} =$ **$141.27909**

**Problem SDRMPPBM4.** You can use a 15-period binomial tree to model the price movements of Stock Q. For each time period, the risk-neutral probability of an upward movement in the stock price is 0.54. Find the probability that stock price will be at the 8th node of the binomial tree at the end of 15 periods.

**Solution SDRMPPBM4.** We use the formula $p_{\text{ith node}} = C(n, i)(p^*)^{n-i}(1-p^*)^i$, where n = 15, i = 8, p = 0.54. Thus, $p_{\text{8th node}} = C(15, 8)\ (0.54)^7(0.46)^8 =$ $\mathbf{p_{8th\ node} = 0.172729907}$.

**Problem SDRMPPBM5.** You can use a 32-period binomial tree to model the price movements of Stock R. For each time period, the risk-neutral probability of an upward movement in the stock price is 0.78. Find the probability that stock price will be at the 11th node of the binomial tree at the end of 32 periods.

**Solution SDRMPPBM5.** We use the formula $p_{\text{ith node}} = C(n, i)(p^*)^{n-i}(1-p^*)^i$, where n = 32, i = 11, p = 0.78. Thus, $p_{\text{11th node}} = C(32, 11)(0.78)^{21}(0.22)^{11} =$ $\mathbf{p_{11th\ node} = 0.0408609315}$

# Section 30

# Alternative Binomial Trees

There are several alternative ways to construct a binomial tree.

**The Cox-Rubinstein binomial tree** can be constructed using these formulas:

$u = e^{\sigma\sqrt{(h)}}$

$d = e^{-\sigma\sqrt{(h)}}$

This model breaks down if h is too large or σ is too small, such that $e^{rh} > e^{\sigma\sqrt{(h)}}$.

**The lognormal tree** can be constructed using these formulas:

$u = e^{(r- \delta-0.5\sigma^2)h + \sigma\sqrt{(h)}}$

$d = e^{(r- \delta-0.5\sigma^2)h - \sigma\sqrt{(h)}}$

All methods of binomial tree construction give the same ratio of u to d:

$u/d = e^{2\sigma\sqrt{(h)}}$

$\ln(u/d) = 2\sigma\sqrt{(h)}$

**Meaning of variables:**

u = 1 + rate of capital gain on stock if stock price increases.

d = 1 + rate of capital loss on stock if stock price decreases.

h = one time period in binomial model.

r = annual continuously-compounded risk-free interest rate.

$\partial$ = annual continuously-compounded dividend yield.

σ = annual stock price volatility.

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 11, pp. 359.

**Problem ABT1.** The stock price of Particular Co. has volatility of 0.3. The stock currently trades for $1230/share. Using 2 months as one time period and a three-period binomial model, calculate the price of the stock if it goes up twice and down once.

**Solution ABT1.** We use the formulas $u = e^{\sigma\sqrt{(h)}}$ and $d = e^{-\sigma\sqrt{(h)}}$, where h = 1/6 and c = 0.3. Thus, $u = e^{0.3\sqrt{(1/6)}} = 1.130290283$, and $d = e^{-0.3\sqrt{(1/6)}} = 0.8847284766$. Thus, $S_{uud}$ = 1.130290283*1.130290283*0.8847284766*1230 = **$1390.257048**

**Problem ABT2.** During 54 periods in a binomial model, the stock price of Imperious LLC has gone up 33 times and gone down 21 times. The current price of Imperious LLC stock is $32/share. The stock price volatility is 0.2, and one time period in the binomial model is 6 months. Using a Cox-Rubinstein binomial tree, calculate the original price of Imperious LLC stock.

**Solution ABT2.** We note that with a Cox-Rubinstein binomial tree, $ud = e^{\sigma\sqrt{(h)}}e^{-\sigma\sqrt{(h)}} = 1$, so going up 33 times and down 21 times is equivalent to going up 12 times. $u^{12} = e^{12\sigma\sqrt{(h)}} = e^{12*0.2\sqrt{(1/2)}} =$ 5.457857289. The current price is 32, so the price 54 periods ago was 32/5.457857289 = **$5.863106766.**

**Problem ABT3.** During the course of 34 time periods of 1 year each in the binomial model, two stocks that were identically priced at the beginning diverged in their prices. Stock A went up consistently, while Stock B went down consistently. The volatility of both stocks' prices is 0.09. Find the ratio of the price of Stock A to the price of Stock B.

**Solution ABT3.** We use the formula $u/d = e^{2\sigma\sqrt{(h)}}$, noting that we seek $u^{34}/d^{34} = (u/d)^{34} = e^{68\sigma\sqrt{(h)}} = e^{68*0.09\sqrt{(1)}} =$ **$u^{34}/d^{34}$ = 454.8646945**

**Problem ABT4.** The stock prices of Furious LLC can be modeled via a lognormal tree with 6 years as one time period. The annual continuously compounded risk-free interest rate is 0.08, and the stock pays dividends with an annual continuously compounded yield of 0.01. The stock price volatility is 0.43. The stock price is currently $454 per share. Find the stock price in 72 years if it goes up 7 times and down 5 times.

**Solution ABT4.** We use the formulas

$u = e^{(r-\delta-0.5\sigma^2)h + \sigma\sqrt{(h)}}$ and $d = e^{(r-\delta-0.5\sigma^2)h - \sigma\sqrt{(h)}}$

Here, r = 0.08, $\partial$ = 0.01, h = 6, S = 454, σ = 0.43. Thus,

$u = e^{(0.08-0.01-0.5*0.43^2)6 + 0.43\sqrt{(6)}} = u = 2.505731203$

$d = e^{(0.08-0.01-0.5*0.43^2)6 - 0.43\sqrt{(6)}} = d = 0.3048362324$

Thus, our desired price is $u^7d^5S = 2.505731203^7 0.3048362324^5 *454 =$ **$741.1910185**

**Problem ABT5.** The stock prices of Obsequious Co. can be modeled via a lognormal tree with 1 day as one time period. The annual continuously compounded risk-free interest rate is 0.4, and the stock pays dividends with an annual continuously compounded yield of 0.2. The stock price volatility is 0.90. Today, Obsequious Co.'s stock price is $9000/share. During the past year, the price went up 43 times and down 322 times. Find the price of Obsequious Co.'s stock one non-leap year ago.

**Solution ABT5.** We use the formulas

$u = e^{(r- \delta-0.5\sigma^2)h + \sigma\sqrt{(h)}}$ and $d = e^{(r- \delta-0.5\sigma^2)h - \sigma\sqrt{(h)}}$

Here, $r = 0.4$, $\partial = 0.2$, $h = 1/365$, $\sigma = 0.9$. Thus,

$u = e^{(0.4- 0.2- 0.5*0.9^2)/365 + 0.9\sqrt{(1/365)}} = 1.047646803$

$d = e^{(0.4- 0.2- 0.5*0.9^2)/365 - 0.9\sqrt{(1/365)}} = 0.9534485668$

Using these values for u and d, $d^{322}u^{43} = 0.00000159573726$

Thus, the stock price one year ago was 9000/0.00000159573726 = **$5,640,026,227 per share.** (Obsequious Co. stock *really* took a hit over the past year!)

# Section 31

# Constructing Binomial Trees with Discrete Dividends

When dividends are paid on a stock on a discrete rather than on a continuous basis, we can use Schroder's method for constructing binomial trees.

The up and down movements (u and d) of the stock price S can be modeled as follows:
$u = e^{rh + \sigma\sqrt{(h)}}$

$d = e^{rh - \sigma\sqrt{(h)}}$

The discrete dividend is, however, taken into account in calculating the stock price at each of the nodes of the binomial tree. The formula for calculating the stock price is

$S_t = F^P{}_{t,T} + De^{-r(V-t)}$

The actual stock price volatility in this case is not the same as the prepaid forward price volatility. The relationship between them can be expressed as

$\sigma_F = (S/F^P)\sigma_S$

It is the *prepaid forward price volatility* that is used in computing u and d in constructing the binomial tree.

The advantage of Schroder's approach is that it produces a *recombining* binomial tree where, once the nodes have been determined, the option price calculation follows the usual approach in the binomial model.

**Meaning of variables:**

S = stock price.

u = 1 + rate of capital gain on stock if stock price increases.

d = 1 + rate of capital loss on stock if stock price decreases.

h = one time period in binomial model.

r = annual continuously-compounded risk-free interest rate.

T = time to expiration of the option.

$V$ = time at which the dividend is paid, where $V < T$.

$\sigma_S$ = annual stock price volatility.

$\sigma_F$ = annual prepaid forward price volatility.

$F^P_{t,T}$ = time t prepaid forward price for a forward contract expiring at time T.

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 11, pp. 363-365.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem CBTWDD1.** The price of Hideous Co. stock has volatility of 0.45. Hideous Co. stock currently trades for $556 per share, while a prepaid forward contract on Hideous Co. stock currently trades for $523. Find the volatility of the price of a prepaid forward contract on Hideous Co. stock.

**Solution CBTWDD1.** We use the formula $\sigma_F = (S/F^P)\sigma_S$, where $S = 556$, $F^P = 523$, $\sigma_S = 0.45$. Thus, $\sigma_F = (556/523)0.45$ = $\boldsymbol{\sigma_F = 0.4783938815}$

**Problem CBTWDD2.** The price of Hideous Co. stock has volatility of 0.45. Hideous Co. stock currently trades for $556 per share, while a prepaid forward contract on Hideous Co. stock currently trades for $523. If a dividend of $37 per share is to be paid on Hideous Co. stock in 2 years, find the annual continuously compounded interest rate.

**Solution CBTWDD2.** We use the formula $S_t = F^P_{t,T} + De^{-r(V-t)}$ and rearrange it as

$De^{-r(V-t)} = S_t - F^P_{t,T}$. Here, $S_0 = 556$, $F^P_{0,T} = 523$, $V = 2$, $D = 37$. So

$37e^{-2r} = 556 - 523 = 33$. Thus, $e^{-2r} = 33/37$ and $r = -\ln(33/37)/2$ = $\mathbf{r = 0.0572051756}$

**Problem CBTWDD3.** The price of Hideous Co. stock has volatility of 0.45. Hideous Co. stock currently trades for $556 per share, while a prepaid forward contract on Hideous Co. stock currently trades for $523. If a dividend of $37 per share is to be paid on Hideous Co. stock in 2 years, find the price of Hideous Co. stock in one year if the stock price declines during that time period. Use a binomial model where one period is equal to one year.

**Solution CBTWDD3.** We note that the desired σ here is $\sigma_F = 0.4783938815$, found in Solution CBTWDD1. Also the desired r here is $r = 0.0572051756$, found in Solution CBTWDD2. Also, $h = 1$.

We calculate $d = e^{rh - \sigma\sqrt{(h)}} = e^{0.0572051756 - 0.4783938815} = d = 0.6562662484$.

We use the prepaid forward price $F^P_{0,T} = 523$ to calculate $F^P_{1,T} = dF^P_{0,T} = 0.6562662484*523 = 343.2272479$

Now we find $S_1 = F^P{}_{1,T} + De^{-r(V-1)} = 343.2272479 + 37e^{-0.0572051756(2-1)}$ = $\mathbf{S_1 = \$378.1700583}$

**Problem CBTWDD4.** Various Industries will pay a dividend of $4 per share in two years. Currently, the stock of Various Industries trades for $69 per share. The annual continuously-compounded interest rate is 0.08, and prepaid forward price volatility on Various Industries stock is 0.05. Find the price of Various Industries stock in one year if the stock goes up. Use a binomial model where one period is equal to one year.

**Solution CBTWDD4.** First we find the current prepaid forward price: $S_t = F^P{}_{t,T} + De^{-r(V-t)}$ implies that $F^P{}_{t,T} = S_t - De^{-r(V-t)}$, where V = 2, t = 0, r = 0.08, $S_t = 69$, and D = 4.

Thus, $F^P{}_{0,T} = 69-4e^{-0.08(2)} = F^P{}_{0,T} = 65.59142484$

Now we find u using σ = 0.05 and h = 1:

$u = e^{rh + \sigma\sqrt{(h)}} = e^{0.08 + 0.05} = u = 1.13882838$

So the *prepaid forward* price in one year if the stock goes up will be
$F^P{}_{1,T} = uF^P{}_{0,T} = 1.13882838*65.59142484 = F^P{}_{1,T} = 74.69737631$.

Now we apply the formula $S_1 = F^P{}_{1,T} + De^{-r(V-1)} = 74.69737631 + 4e^{-0.08(2-1)}$ = $\mathbf{S_1 = \$78.3898417}$

**Problem CBTWDD5.** Torpid LLC stock currently trades for $2 per share, and *stockprice volatility* is 0.08. The annual continuously-compounded interest rate is 0.03. Torpid LLC will pay a dividend of $0.5 per share in 1 year. If the stock price of Torpid LLC stock goes down in two months, what will the stock price be? Use a binomial model where one period is equal to two months.

**Solution CBTWDD5.** First we find the current prepaid forward price: $S_t = F^P{}_{t,T} + De^{-r(V-t)}$ implies that $F^P{}_{t,T} = S_t - De^{-r(V-t)}$, where V = 1, t = 0, r = 0.03, $S_t = 2$, and D = 0.5.

Thus, $F^P{}_{0,T} = 2-0.5e^{-0.03} = F^P{}_{0,T} = 1.514777233$.

We note that the desired σ here is $\sigma_F = (S/F^P)\sigma_S$, where S = 2, $F^P{}_{0,T} = 1.514777233$, and $\sigma_S$ = 0.08. Thus, $\sigma_F = (2/1.514777233)0.08 = \sigma_F = 0.1056260924$

Now we find d using σ = 0.1056260924 and h = 1/6:

$d = e^{0.03/6 - 0.1056260924\sqrt{(1/6)}} = e^{-0.0381216717} = d = 0.9625958131$

So the *prepaid forward* price in 2 months if the stock goes down will be
$F^P{}_{1/6,T} = 0.9625958131F^P{}_{0,T} = 0.9625958131*1.514777233 = F^P{}_{1/6,T} = 1.458118223$

Now we apply the formula $S_{1/6} = F^P{}_{1/6,T} + De^{-r(V-1/6)} = 1.458118223 + 0.5e^{-0.03(5/6)}$ =

$\mathbf{S_{1/6} = \$1.945773179}$

# Section 32

# Review of Put-Call Parity and Binomial Option Pricing

This section will review all the concepts we have worked with thus far. It applies the ideas of put-call parity and binomial option pricing discussed in Sections 1 through 31.

The problems in this section were designed to be similar to problems from past versions of Exam 3F / Exam MFE. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

**Problem RPCPBOP1.**

**Similar to Question 1 from the Society of Actuaries' May 2007 Exam MFE:**

On October 1, 3451, the stock of Respectable Co. has a price of $300/share. Two equal dividends will be paid on May 1, 3452 and July 1, 3452. A European put option on Respectable Co. stock with strike price of $290 expiring in one year sells for $30, while a European call option on Respectable Co. stock with strike price of $290 expiring in one year sells for $53. The annual continuously-compounded risk-free interest rate is 12%. Find the amount of each dividend.

**Solution RPCPBOP1.** We use the formula for put-call parity on stock option:
$C(K, T) - P(K, T) = [S_0 - PV_{0,T}(Div)] - e^{-rT}K$, where T = 1 year and the dividends will be paid in 7/12 years and in 9/12 = ¾ years. K = 290, C(K, T) = 53, P(K, T) = 30, $S_0 = 300$, r = 0.12.
Let D be the amount of each dividend.

Thus, $PV_{0,T}(Div) = S_0 - e^{-rT}K - C(K, T) + P(K, T) = 300 - e^{-0.12}290 - 53 + 30 = PV_{0,T}(Div) =$ $19.79307335 = D(e^{-0.12*7/12} + e^{-0.12*9/12})$ and $D = 19.79307335/(e^{-0.12*7/12} + e^{-0.12*9/12})$ = **D = $10.72025418.**

**Problem RPCPBOP2.**

**Similar to Question 2 from the Society of Actuaries' May 2007 Exam MFE:**

You are aware of the following information in the one-period binomial model for the price of the stock of The Firm Firm. The time period in the binomial model is two years, and the stock does not pay any dividends. In two years, the stock price will change by a factor of 0.333 or by a factor of 4.36. The annual continuously compounded expected return on the stock is 0.23. Find the true probability of the stock price going up in two years.

**Solution RPCPBOP2.** We use the formula $p = (e^{\alpha h} - d)/(u - d)$ for the true probability of the stock price's increase. We are given u = 4.36, d = 0.333, h = 2, α = 0.23. Thus,

$p = (e^{2*0.23} - 0.333)/(4.36 - 0.333)$ = **p = 0.3106714639**

**Problem RPCPBOP3.**

**Similar to Question 4 from the Society of Actuaries' May 2007 Exam MFE:**

For the stock of Cautious Co., the annual continuously compounded dividend yield is 0.11, and the annual continuously compounded risk-free interest rate is 0.04. The stock currently trades for $367 per share. One-year European call options on Cautious Co. stock have the following prices (C) corresponding to various strike prices (K):

K = $350, C = $14

K = $370, C = $5

K = $450, C = $0.23

K = $1000, C = $0

K = $1100, C = $0

You own five put options with each of the above strike prices:
Put A: K = $350

Put B: K = $370

Put C: K = $450

Put D: K = $1000

Put E: K = $1100

Each of the options can only be exercised either immediately or in two years. Which of these options is it optimal to exercise immediately? More than one answer may be correct.

**Solution RPCPBOP3.** First, we can use put-call parity to determine optimality of exercise by comparing K-S for each put with the value of P(K, T). If K - S > P(K, T), immediate exercise is optimal.

We can eliminate Put A right away, because K - S = -17 < 0.

For Put B, K - S = 370 - 367 = 3.

$P(K, T) = C(K, T) + Ke^{-rT} - S_0e^{-\partial T}$, where T = 3, K = 370, C(K, T) = 14, r = 0.04, T = 2.

Thus, $P(370, 2) = 5 + 370e^{-0.04*2} - 367e^{-0.11*2} = P(370, 2) = 52.02864931 > 3$, so it is not optimal to exercise Put B.

For Put C, K - S = 450 - 367 = 83

$P(450, 2) = 0.23 + 450e^{-0.04*2} - 367e^{-0.11*2} = P(450, 2) = 121.107957 > 83$, so it is not optimal to exercise Put C.

For Put D, K - S = 1000 - 367 = 633

$P(1000, 2) = 1000e^{-0.04*2} - 367e^{-0.11*2} = P(1000, 2) = 628.5919475 < 633$, so it *is* optimal to exercise Put D.

For Put E, K - S = 1100 - 367 = 733

$P(1100, 2) = 1100e^{-0.04*2} - 367e^{-0.11*2} = P(1100, 2) = 720.9035822 < 733$, so it *is* optimal to exercise Put E.

Thus, **it is optimal to immediately exercise Puts D and E.**

**Problem RPCPBOP4.**

**Similar to Question 11 from the Society of Actuaries' May 2007 Exam MFE:**

The price movements of Jubilant Co. stock follow a two-period binomial model, where every period the stock price can change by a factor of 0.78 or 2. The stock pays no dividends, and the current stock price is $100. The annual continuously-compounded risk-free interest rate is 0.03. Each period in the binomial model is two years. Find the price of a four-year American put option on Jubilant Co. stock with a strike price of $300.

**Solution RPCPBOP4.**

We first find the stock prices in the binomial tree:
S = 100, u = 2, d = 0.78.

So $S_u = 200$ and $S_{uu} = 400$

$S_d = 78$, $S_{dd} = 60.84$, and $S_{du} = 156$.

Thus, at the final nodes of the binomial tree, we have

$P_{uu} = 0$, $P_{du} = 300 - 156 = 144$, and $P_{dd} = 300 - 60.84 = 239.16$

The risk-neutral probability of a stock price increase is (for r = 0.03, h = 2, and $\partial = 0$)

$p^* = (e^{(r-\partial)h} - d)/(u - d) = (e^{0.03*2} - 0.78)/(2 - 0.78) = p^* = 0.2310135627$

$P_u = e^{-rh}[p^*P_{uu} + (1 - p^*)P_{du}] = e^{-0.03*2}[0 + (1 - 0.2310135627)144] = P_u = 104.2853981 > 300 - S_u = 300 - 200 = 100$. Thus, $P_u = 104.2853981$.

$P_d = e^{-rh}[p^*P_{du} + (1 - p^*)P_{dd}] = e^{-0.03*2}[0.2310135627*144 + (1 - 0.2310135627)239.16] = P_d = 204.5293601$ (or so it seems at first). However, $300 - S_d = 300 - 78 = 222 > 204.5293601$. Thus, because the put is American, the true value of $P_d$ is 222.

$P = e^{-rh}[p^*P_u + (1 - p^*)P_d] = e^{-0.03*2}[0.2310135627*104.2853981 + (1 - 0.2310135627)222] = P = 183.4616929$. However, the value of exercising the put immediately is $K - S = 300 - 100 = 200 > 183.4616929$. Thus, the price of this put is **$200.**

**Problem RPCPBOP5.**

**Similar to Question 14 from the Society of Actuaries' May 2007 Exam MFE:**

Payoff on a straddle option is the absolute value of (K - S) on the expiration date, where K is the strike price of the option and S is the stock price at expiration. Stock Ψ, which pays no dividends, currently sells for $444 per share. In 65 years, the stock will sell for $567 or $230. The annual continuously-compounded risk-free interest rate is 0.0004. A straddle option on Stock Ψ with strike price of $500 can only be exercised in 65 years. Find the straddle's current price.

**Solution RPCPBOP5.** This problem is a one-period binomial option pricing problem in disguise. In 65 years, the payoff to the straddle will be either 567 - 500 = 67 (in the event of an up move) or 500 - 230 = 270 (in the event of a down move).

We find $p^* = (e^{(r-\partial)h} - d)/(u - d) = (e^{0.0004*65} - 230/444)/(567/444 - 230/444) =$

$p^* = 0.6697192318$

The straddle price $A = e^{-rh}[p^*A_u + (1 - p^*)A_d] =$

$e^{-0.0004*65}[0.6697192318*67 + (1 - 0.6697192318)270] =$ **A = $130.6066918**

# Section 33

# The Black-Scholes Formula

The Black-Scholes formula for option pricing is valid under the following six assumptions, as stated by R. L. McDonald:

"1. Continuously compounded returns on the stock are normally distributed and independent over time.
"2. The volatility of continuously compounded returns is known and constant.
"3. Future dividends are known, either as a dollar amount or as a fixed dividend yield.
"4. The risk-free interest rate is known and constant.
"5. There are no transaction costs or taxes.
"6. It is possible short-sell costlessly and to borrow at the risk-free rate."

The Black-Scholes formula for the call price is

$C(S, K, \sigma, r, T, \partial) = Se^{-\partial T}N(d_1) - Ke^{-rT}N(d_2)$
where $d_1 = [\ln(S/K) + (r - \partial + 0.5\sigma^2)T]/[\sigma\sqrt{(T)}]$ and $d_2 = d_1 - \sigma\sqrt{(T)}$

The Black-Scholes formula for the put price is

$P(S, K, \sigma, r, T, \partial) = Ke^{-rT}N(-d_2) - Se^{-\partial T}N(-d_1)$

We can also get the put formula via put-call parity:
$P(S, K, \sigma, r, T, \partial) = C(S, K, \sigma, r, T, \partial) + Ke^{-rT} - Se^{-\partial T}$

**Meaning of variables:**

S = current stock price.

K = strike price of the option.

C = call option price.

P = put option price.

$\sigma$ = annual stock price volatility.

r = annual continuously compounded risk-free interest rate.

T = time to expiration.

$\partial$ = annual continuously compounded dividend yield.

**A note on the function N(x):** N(x) is called the *cumulative normal distribution function.* It is the probability that a randomly chosen number in the standard normal distribution (where mean = 0 and variance = 1) is less than x. It is impossible to directly integrate the normal probability distribution function to find N(x). On the exam, you will be given a table of values for N(x), so such integration will not be necessary. Here, however, we will use the Microsoft Excel function NormSDist. Try entering "=NormSDist(1)" into a cell in MS Excel. The result should be 0.84134474. Using this method will give us greater accuracy than using a table would.

We also note that N(-x) = 1 - N(x), since the probability of being below -x within the standard normal distribution is the same as the probability of being above x within the standard normal distribution.

We will proceed gently with applying the formula, its components, and its variants for different underlying assets. There will be plenty of uses for this formula throughout future sections of this study guide.

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 12, pp. 375-379.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem BSF1.** The stock of Blackscholesian Co. currently sells for $1500 per share. The annual stock price volatility is 0.2, and the annual continuously compounded risk-free interest rate is 0.05. The stock's annual continuously compounded dividend yield is 0.03. Find the value of $d_1$ in the Black-Scholes formula for the price of a call option on Blackscholesian Co. stock with strike price $1600 and time to expiration of 3 years.

**Solution BSF1.** We use the formula $d_1 = [\ln(S/K) + (r - \partial + 0.5\sigma^2)T]/[\sigma\sqrt{(T)}]$, where we are given that S = 1500, K = 1600, r = 0.05, $\partial = 0.03$, $\sigma = 0.2$, and T = 3.

Thus, $d_1 = [\ln(1500/1600) + (0.05 - 0.03 + 0.5*0.2^2)3]/[0.2\sqrt{(3)}]$ = $\mathbf{d_1 = 0.1601034988}$

**Problem BSF2.** The stock of Blackscholesian Co. currently sells for $1500 per share. The annual stock price volatility is 0.2, and the annual continuously compounded risk-free interest rate is 0.05. The stock's annual continuously compounded dividend yield is 0.03. Find the value of $d_2$ in the Black-Scholes formula for the price of a call option on Blackscholesian Co. stock with strike price $1600 and time to expiration of 3 years.

**Solution BSF2.** We use the formula $d_2 = d_1 - \sigma\sqrt{(T)}$, where, from Solution BSF1, $d_1$ = 0.1601034988 and we are given that T = 3 and $\sigma = 0.2$. Thus, $d_2 = 0.1601034988 - 0.2\sqrt{(3)}$ =

$\mathbf{d_2 = -0.1863066628}$

**Problem BSF3.** The stock of Blackscholesian Co. currently sells for $1500 per share. The annual stock price volatility is 0.2, and the annual continuously compounded risk-free interest rate is 0.05. The stock's annual continuously compounded dividend yield is 0.03. Use the Black-

Scholes formula to find the price of a call option on Blackscholesian Co. stock with strike price \$1600 and time to expiration of 3 years.

**Solution BSF3.** We use the formula $C(S, K, \sigma, r, T, \partial) = Se^{-\partial T}N(d_1) - Ke^{-rT}N(d_2)$, where we know that S = 1500, K = 1600, r = 0.05, $\partial$ = 0.03, σ = 0.2, and T = 3. From Solution BSF1, $d_1$ = 0.1601034988. From Solution BSF2, $d_2$ = -0.1863066628.

We find $N(d_1)$ via MS Excel using the input "=NormSDist(0.1601034988)" = $N(d_1)$ = 0.563600238.

We find $N(d_2)$ via MS Excel using the input "=NormSDist(-0.1863066628)" = $N(d_2)$ = 0.426102153

Thus, $C(S, K, \sigma, r, T, \partial) = 1500e^{-0.03*3}0.563600238 - 1600e^{-0.05*3}0.426102153 =$

**$C(S, K, \sigma, r, T, \partial)$ = \$185.8385153**

**Problem BSF4.** The stock of Blackscholesian Co. currently sells for \$1500 per share. The annual stock price volatility is 0.2, and the annual continuously compounded risk-free interest rate is 0.05. The stock's annual continuously compounded dividend yield is 0.03. Find the price of a *put* option on Blackscholesian Co. stock with strike price \$1600 and time to expiration of 3 years.

**Solution BSF4.** We recall that, since the call price is known from Solution BSF3, we can use put-call parity to get the put price: $P(S, K, \sigma, r, T, \partial) = C(S, K, \sigma, r, T, \partial) + Ke^{-rT} - Se^{-\partial T}$

where $C(S, K, \sigma, r, T, \partial)$ = 185.8385153, S = 1500, K = 1600, r = 0.05, $\partial$ = 0.03, and T = 3.

Thus, $P(S, K, \sigma, r, T, \partial) = 185.8385153 + 1600e^{-0.05*3} - 1500e^{-0.03*3} =$

**$P(S, K, \sigma, r, T, \partial)$ = \$192.0744997**

**Problem BSF5.** The stock of Blackscholesian Co. currently sells for \$1500 per share. The annual stock price volatility is 0.2, and the annual continuously compounded risk-free interest rate is 0.05. The stock's annual continuously compounded dividend yield is 0.03. Within the Black-Scholes formula for the price of a *put* option on Blackscholesian Co. stock with strike price \$1600 and time to expiration of 3 years, find the value of $N(-d_2)$.

**Solution BSF5.** We use the formula N(-x) = 1 - N(x). We know from Solution BSF3 that $N(d_2)$ = 0.426102153. Thus, $N(-d_2) = 1 - N(d_2)$ = 1 - 0.426102153 = **$N(-d_2)$ = 0.573897847**

# Section 34

# The Black-Scholes Formula Using Prepaid Forward Prices

It is possible to express the Black-Scholes formula using prepaid forward prices for the stock and for the strike asset.

We note that $F^P_{0,T}(S) = Se^{-\partial T}$ and $F^P_{0,T}(K) = Ke^{-rT}$.

Thus, the Black-Scholes formula for the call price is

$C(F^P_{0,T}(S), F^P_{0,T}(K), \sigma, T) = F^P_{0,T}(S)N(d_1) - F^P_{0,T}(K)N(d_2)$
where $d_1 = [\ln(F^P_{0,T}(S)/F^P_{0,T}(K)) + 0.5\sigma^2T]/[\sigma\sqrt{(T)}]$ and $d_2 = d_1 - \sigma\sqrt{(T)}$

The Black-Scholes formula for the put price is

$P(F^P_{0,T}(S), F^P_{0,T}(K), \sigma, T) = F^P_{0,T}(K)N(-d_2)- F^P_{0,T}(S)N(-d_1)$

We can also get the put formula via put-call parity:
$P(F^P_{0,T}(S), F^P_{0,T}(K), \sigma, T) = C(F^P_{0,T}(S), F^P_{0,T}(K), \sigma, T) + F^P_{0,T}(K) - F^P_{0,T}(S)$

This formula has two advantages over the standard Black-Scholes formula.

1. The interest rate and dividend yield do not appear explicitly in the formula; the formula requires only four parameters to be known rather than six.

2. This formula allows for calculating option prices for options where the strike asset is something other than cash.

**Meaning of variables:**

S = current stock price.

K = strike price of the option.

C = call option price.

P = put option price.

σ = annual stock price volatility.

r = annual continuously compounded risk-free interest rate.

T = time to expiration.

$\partial$ = annual continuously compounded dividend yield.

$F^P_{0,T}(S)$ = price of prepaid forward on the stock (underlying asset).

$F^P_{0,T}(K)$ = price of prepaid forward on the strike asset.

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 12, pp. 379-380.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem BSFUPFP1.** Put options are written, using the stock of Ferocious LLC as the underlying asset and the stock of Timid Co. as the strike asset. The current prepaid forward price on Ferocious LLC stock is $330. The current prepaid forward price on Timid LLC stock is $310.

The annual underlying asset price volatility relevant for the Black-Scholes formula is 0.1, and the options' time to expiration is 4 years. Calculate $d_1$ in the Black-Scholes formula for the price of such a put option.

**Solution BSFUPFP1.** We use the equation $d_1 = [\ln(F^P_{0,T}(S)/F^P_{0,T}(K)) + 0.5\sigma^2T]/[\sigma\sqrt{(T)}]$, where $F^P_{0,T}(S) = 330$, $F^P_{0,T}(K) = 310$, T = 4, and σ = 0.1.

Thus, $d_1 = [\ln(330/310) + 0.5*0.1^2*4]/[0.1\sqrt{(4)}]$ = $\mathbf{d_1 = 0.4126017849}$

**Problem BSFUPFP2.** Put options are written, using the stock of Ferocious LLC as the underlying asset and the stock of Timid Co. as the strike asset. The current prepaid forward price on Ferocious LLC stock is $330. The current prepaid forward price on Timid LLC stock is $310.

The annual underlying asset price volatility relevant for the Black-Scholes formula is 0.1, and the options' time to expiration is 4 years. Calculate $d_2$ in the Black-Scholes formula for the price of such a put option.

**Solution BSFUPFP2.** We use the equation $d_2 = d_1 - \sigma\sqrt{(T)}$, where $d_1 = 0.4126017849$ from Solution BSFUPFP1, T = 4, and σ = 0.1. Thus, $d_2 = 0.4126017849 - 0.1\sqrt{(4)}$ =

$\mathbf{d_2 = 0.2126017849}$

**Problem BSFUPFP3.** Put options are written, using the stock of Ferocious LLC as the underlying asset and the stock of Timid Co. as the strike asset. The current prepaid forward price on Ferocious LLC stock is $330. The current prepaid forward price on Timid LLC stock is $310.

The annual underlying asset price volatility relevant for the Black-Scholes formula is 0.1, and the options' time to expiration is 4 years. Use the Black-Scholes formula to find the price of such a put option.

**Solution BSFUPFP3.** The Black-Scholes formula for the put price is
$P(F^P{}_{0,T}(S), F^P0,T(K), \sigma, T) = F^P{}_{0,T}(K)N(-d_2)- F^P{}_{0,T}(S)N(-d_1)$

We are given that $F^P{}_{0,T}(S) = 330$, $F^P{}_{0,T}(K) = 310$, $T = 4$, and $\sigma = 0.1$.

Furthermore, from Solution BSFUPFP1, $d_1 = 0.4126017849$. In MS Excel, using the input "=NormSDist(-0.4126017849)", we find that $N(-d_1) = 0.339949236$

Furthermore, from Solution BSFUPFP2, $d_2 = 0.2126017849$. In MS Excel, using the input "=NormSDist(-0.2126017849)", we find that $N(-d_2) = 0.415818821$

Thus, $P(F^P{}_{0,T}(S), F^P0,T(K), \sigma, T) = 310*0.415818821 - 330*0.339949236 =$
$\mathbf{P(F^P{}_{0,T}(S), F^P0,T(K), \sigma, T) = \$16.72058663}$

**Problem BSFUPFP4.** Call options are written, using the stock of Ferocious LLC as the underlying asset and the stock of Timid Co. as the strike asset. The current prepaid forward price on Ferocious LLC stock is $330. The current prepaid forward price on Timid LLC stock is $310.

The annual underlying asset price volatility relevant for the Black-Scholes formula is 0.1, and the options' time to expiration is 4 years. Use the Black-Scholes formula to find the price of such a call option.

**Solution BSFUPFP4.** Since the put price is known from Solution BSFUPFP3 to be 16.72058663, we can calculate the call price using put-call parity:

$P(F^P{}_{0,T}(S), F^P{}_{0,T}(K), \sigma, T) = C(F^P{}_{0,T}(S), F^P{}_{0,T}(K), \sigma, T) + F^P{}_{0,T}(K) - F^P{}_{0,T}(S)$, which implies that $C(F^P{}_{0,T}(S), F^P{}_{0,T}(K), \sigma, T) = P(F^P{}_{0,T}(S), F^P{}_{0,T}(K), \sigma, T) - F^P{}_{0,T}(K) + F^P{}_{0,T}(S)$, where $F^P{}_{0,T}(K) = 310$ and $F^P{}_{0,T}(S) = 330$. Thus, $C(F^P{}_{0,T}(S), F^P{}_{0,T}(K), \sigma, T) = 16.72058663 - 310 + 330 =$
$\mathbf{C(F^P{}_{0,T}(S), F^P{}_{0,T}(K), \sigma, T) = \$36.72058663}$

**Problem BSFUPFP5.** Use the Black-Scholes formula with prepaid forward prices to find the price of a call option whose underlying asset price is not volatile at all. (That is, $\sigma = 0$.)

**Solution BSFUPFP5.** The Black-Scholes formula for the call price is

$C(F^P{}_{0,T}(S), F^P{}_{0,T}(K), \sigma, T) = F^P{}_{0,T}(S)N(d_1) - F^P{}_{0,T}(K)N(d_2)$
where $d_1 = [\ln(F^P{}_{0,T}(S)/F^P{}_{0,T}(K)) + 0.5\sigma^2T]/[\sigma\sqrt{(T)}]$ and $d_2 = d_1 - \sigma\sqrt{(T)}$,
where $\sigma = 0$, $d_1 = [\ln(F^P{}_{0,T}(S)/F^P{}_{0,T}(K)) + 0]/[0\sqrt{(T)}]$. This entails division of some nonzero quantity by zero, which means that $d_1 = +\infty$. Likewise, $d_2 = d_1 - \sigma\sqrt{(T)} = +\infty - 0 = d_2 = +\infty$

Thus, $N(d_1) = N(d_2) = N(+\infty) = 1$. (The probability of x being less than positive infinity within the standard normal distribution is 1.)

Thus, $C(F^P{}_{0,T}(S), F^P{}_{0,T}(K), 0, T) = F^P{}_{0,T}(S)*1 - F^P{}_{0,T}(K)*1$ and so

$\mathbf{C(F^P{}_{0,T}(S), F^P{}_{0,T}(K), 0, T) = F^P{}_{0,T}(S) - F^P{}_{0,T}(K)}$

# Section 35

# The Black-Scholes Formula for Options on Stocks with Discrete Dividends

When stocks pay discrete dividends, we recall that the following relationship holds between the prepaid forward price and the stock price:
$F^P_{0,T}(S) = S_0 - PV_{0,T}(Div)$

For time to expiration T and annual continuously compounded risk-free interest rate, the following equality still holds with respect to the prepaid forward on the strike asset:

$F^P_{0,T}(K) = Ke^{-rT}$.

Making the substitutions above, we can now use the Black-Scholes formula from Section 34 - the formula that uses prepaid forward prices.

Thus, the Black-Scholes formula for the call price is

$C(F^P_{0,T}(S), F^P_{0,T}(K), \sigma, T) = F^P_{0,T}(S)N(d_1) - F^P_{0,T}(K)N(d_2)$
where $d_1 = [\ln(F^P_{0,T}(S)/F^P_{0,T}(K)) + 0.5\sigma^2T]/[\sigma\sqrt{(T)}]$ and $d_2 = d_1 - \sigma\sqrt{(T)}$

The Black-Scholes formula for the put price is

$P(F^P_{0,T}(S), F^P_{0,T}(K), \sigma, T) = F^P_{0,T}(K)N(-d_2)- F^P_{0,T}(S)N(-d_1)$

We can also get the put formula via put-call parity:
$P(F^P_{0,T}(S), F^P_{0,T}(K), \sigma, T) = C(F^P_{0,T}(S), F^P_{0,T}(K), \sigma, T) + F^P_{0,T}(K) - F^P_{0,T}(S)$

**Meaning of variables:**

S = current stock price.

K = strike price of the option.

C = call option price.

P = put option price.

σ = annual stock price volatility.

r = annual continuously compounded risk-free interest rate.

T = time to expiration.

$\partial$ = annual continuously compounded dividend yield.

$PV_{0,T}(Div)$ = present value of future discrete dividends.

$F^P_{0,T}(S)$ = price of prepaid forward on the stock (underlying asset).

$F^P_{0,T}(K)$ = price of prepaid forward on the strike asset.

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 12, p. 380.

**Problem BSFOSDD1.** The stock of Auspicious Co. currently trades for $221 per share. The stock will pay a dividend of $40 in 2 years and another dividend of $32 in 6 years. The annual continuously compounded risk-free interest rate is 0.05, and the annual price volatility relevant for the Black-Scholes equation is 0.3. Call options are written on Auspicious Co. stock with a strike price of $250 and time to expiration of 8 years. Calculate $d_1$ in the Black-Scholes formula for the price of one such call option.

**Solution BSFOSDD1.** First, we convert the asset and strike price terms into prepaid forward prices. $F^P_{0,T}(S) = S_0 - PV_{0,T}(Div) = 221 - 40e^{-2*0.05} - 32e^{-6*0.05} = F^P_{0,T}(S) = 161.1003202$

$F^P_{0,T}(K) = Ke^{-rT} = 250e^{-8*0.05} = 167.5800115$. Here, T = 8 and σ = 0.3.

Now we use the formula $d_1 = [\ln(F^P_{0,T}(S)/F^P_{0,T}(K)) + 0.5\sigma^2T]/[\sigma\sqrt{(T)}] =$

$[\ln(161.1003202/167.5800115) + 0.5*0.3^2*8]/[0.3\sqrt{(8)}] =$ $\mathbf{d_1 = 0.3777910782}$

**Problem BSFOSDD2.** The stock of Auspicious Co. currently trades for $221 per share. The stock will pay a dividend of $40 in 2 years and another dividend of $32 in 6 years. The annual continuously compounded risk-free interest rate is 0.05, and the annual price volatility relevant for the Black-Scholes equation is 0.3. Call options are written on Auspicious Co. stock with a strike price of $250 and time to expiration of 8 years. Calculate $d_2$ in the Black-Scholes formula for the price of one such call option.

**Solution BSFOSDD2.** We use the formula $d_2 = d_1 - \sigma\sqrt{(T)}$, where T = 8, σ = 0.3, and

$d_1 = 0.3777910782$ from Solution BSFOSDD1. So $d_2 = d_1 - \sigma\sqrt{(T)} = 0.3777910782 - 0.3\sqrt{(8)} =$

$\mathbf{d_2 = -0.4707370592}$

**Problem BSFOSDD3.** The stock of Auspicious Co. currently trades for $221 per share. The stock will pay a dividend of $40 in 2 years and another dividend of $32 in 6 years. The annual continuously compounded risk-free interest rate is 0.05, and the annual price volatility relevant for the Black-Scholes equation is 0.3. Call options are written on Auspicious Co. stock with a strike price of $250 and time to expiration of 8 years. Use the Black-Scholes formula to find the price of one such call option.

**Solution BSFOSDD3.** We use the formula
$C(F^P_{0,T}(S), F^P_{0,T}(K), \sigma, T) = F^P_{0,T}(S)N(d_1) - F^P_{0,T}(K)N(d_2)$. We know from Solutions BSFOSDD1-2 that $d_1 = 0.3777910782$, $d_2 = -0.4707370592$, $F^P_{0,T}(S) = 161.1003202$, $F^P_{0,T}(K) = 167.5800115$.
In MS Excel, using the input "=NormSDist(0.3777910782)", we find that $N(d_1) = 0.647207046$
In MS Excel, using the input "=NormSDist(-0.4707370592)", we find that $N(d_2) = 0.318914268$
Thus, $C(F^P_{0,T}(S), F^P_{0,T}(K), \sigma, T) = 161.1003202*0.647207046 - 167.5800115*0.318914268 =$
$\mathbf{C(F^P_{0,T}(S), F^P_{0,T}(K), \sigma, T) = \$50.82160565}$

**Problem BSFOSDD4.** The stock of Auspicious Co. currently trades for $221 per share. The stock will pay a dividend of $40 in 2 years and another dividend of $32 in 6 years. The annual continuously compounded risk-free interest rate is 0.05, and the annual price volatility relevant for the Black-Scholes equation is 0.3. Put options are written on Auspicious Co. stock with a strike price of $250 and time to expiration of 8 years. Use the Black-Scholes formula to find the price of one such put option.

**Solution BSFOSDD4.** Since the corresponding call option price is known (from Solution BSFOSDD3), we can use put-call parity to find the put option price.

$P(F^P_{0,T}(S), F^P_{0,T}(K), \sigma, T) = C(F^P_{0,T}(S), F^P_{0,T}(K), \sigma, T) + F^P_{0,T}(K) - F^P_{0,T}(S) =$

$50.82160565 + 167.5800115 - 161.1003202 = \mathbf{P(F^P_{0,T}(S), F^P_{0,T}(K), \sigma, T) = \$57.30129695}$

**Problem BSFOSDD5.** You are given that $d_1$ in the Black-Scholes formula for the price of a particular call option on Unstoppable Co. is 0.4. You also know that the current stock price of Unstoppable Co. is $56, the relevant annual price volatility is 0.03, the price of the prepaid forward on the strike asset is $42, and the option's time to expiration is 2 years. The annual continuously-compounded interest rate is 0.03. The stock of Unstoppable Co. will pay one discrete dividend in 1 year. Find the size of the dividend.

**Solution BSFOSDD5.** We use the formula $d_1 = [\ln(F^P_{0,T}(S)/F^P_{0,T}(K)) + 0.5\sigma^2T]/[\sigma\sqrt{(T)}]$, which we rearrange as follows: $d_1\sigma\sqrt{(T)} = [\ln(F^P_{0,T}(S)/F^P_{0,T}(K)) + 0.5\sigma^2T]$
$d_1\sigma\sqrt{(T)} - 0.5\sigma^2T = \ln(F^P_{0,T}(S)/F^P_{0,T}(K))$
$\exp(d_1\sigma\sqrt{(T)} - 0.5\sigma^2T) = F^P_{0,T}(S)/F^P_{0,T}(K)$
$F^P_{0,T}(K)\exp(d_1\sigma\sqrt{(T)} - 0.5\sigma^2T) = F^P_{0,T}(S)$
We are given that $F^P_{0,T}(K) = 42$, $d_1 = 0.4$, $\sigma = 0.03$, $T = 2$. Thus,
$F^P_{0,T}(K)\exp(d_1\sigma\sqrt{(T)} - 0.5\sigma^2T) = 42\exp(0.4*0.03\sqrt{(2)} - 0.5*0.03^2*2) = F^P_{0,T}(S) = 42.68041633$
Now we use the formula $F^P_{0,T}(S) = S_0 - PV_{0,T}(Div)$ and rearrange it thus:
$PV_{0,T}(Div) = S_0 - F^P_{0,T}(S) = 56 - 42.68041633 = 13.31958367$
$PV_{0,T}(Div) = Div*e^{-1*0.03} = 13.31958367$
So $Div = 13.31958367e^{0.03} =$ **Div = 13.72522538**

# Section 36

# The Garman-Kohlhagen Formula for Pricing Currency Options

The Garman-Kohlhagen Formula is a variant on the Black-Scholes option pricing formula, applied to finding the prices of currency options.

We note that the prepaid forward price for a given currency (our underlying asset in this case) can be expressed as $F^P_{0,T}(x) = x_0e^{-fT}$, where $x_t$ is the exchange rate (in "domestic currency" per unit of "foreign" currency at time t), T is the forward's time to expiration, and f ($r_f$ in McDonald's book) is the "foreign" currency risk-free interest rate. Here the "foreign" currency risk-free interest rate f is analogous to the continuously compounded dividend yield $\partial$ in the Black-Scholes equation.

The Garman-Kohlhagen Formula for the price of a call option is
$C(x, K, \sigma, r, T, f) = xe^{-fT}N(d_1) - Ke^{-rT}N(d_2)$
where $d_1 = [\ln(x/K) + (r - f + 0.5\sigma^2)T]/[\sigma\sqrt{(T)}]$ and $d_2 = d_1 - \sigma\sqrt{(T)}$

The Garman-Kohlhagen formula for the put price is
$P(x, K, \sigma, r, T, f) = Ke^{-rT}N(-d_2) - xe^{-fT}N(-d_1)$

We can also get the put formula via put-call parity:
$P(x, K, \sigma, r, T, f) = C(x, K, \sigma, r, T, f) + Ke^{-rT} - xe^{-fT}$

**Meaning of variables:**

x = currency exchange rate (in "domestic currency" per unit of "foreign" currency at time t).

K = strike price (strike exchange rate) of the option.

C = call option price.

σ = annual exchange rate volatility.

r = annual continuously compounded "domestic" currency risk-free interest rate.

T = time to expiration.

f = annual continuously compounded "foreign" currency risk-free interest rate.

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 12, p. 381.

**Problem GKFPCO1.** One piece of stone from Yap (YPS) currently trades for 800 cigarettes (CG). You own a cigarette-denominated YPS call option with time to expiration of 5 years and strike price of 850 CG. The annual continuously compounded cigarette-denominated risk-free interest rate is 0.23. The annual continuously compounded YPS-denominated risk-free interest rate is 0.05. The exchange rate volatility relevant for the Garman-Kohlhagen Formula is 0.1. Find the value of $d_1$ in the Garman-Kohlhagen Formula for such a call option.

**Solution GKFPCO1.** We use the formula $d_1 = [\ln(x/K) + (r - f + 0.5\sigma^2)T]/[\sigma\sqrt{(T)}]$, where x = 800, K = 850, r = 0.23, f = 0.05, σ = 0.1, T = 5. Thus,

$d_1 = [\ln(x/K) + (r - f + 0.5\sigma^2)T]/[\sigma\sqrt{(T)}] = [\ln(800/850) + (0.23 - 0.05 + 0.5*0.1^2)5]/[0.1\sqrt{(5)}] =$

$\mathbf{d_1 = 3.865604207}$

**Problem GKFPCO2.** One piece of stone from Yap (YPS) currently trades for 800 cigarettes (CG). You own a cigarette-denominated YPS call option with time to expiration of 5 years and strike price of 850 CG. The annual continuously compounded cigarette-denominated risk-free interest rate is 0.23. The annual continuously compounded YPS-denominated risk-free interest rate is 0.05. The exchange rate volatility relevant for the Garman-Kohlhagen Formula is 0.1. Find the value of $d_2$ in the Garman-Kohlhagen Formula for such a call option.

**Solution GKFPCO2.** We use the formula $d_2 = d_1 - \sigma\sqrt{(T)}$, where σ = 0.1, T = 5, and $d_1$ = 3.865604207 from Solution GKFPCO1. Thus, $d_2 = 3.865604207 - 0.1\sqrt{(5)} =$ $\mathbf{d_2 = 3.641997409}$

**Problem GKFPCO3.** One piece of stone from Yap (YPS) currently trades for 800 cigarettes (CG). You own a cigarette-denominated YPS call option with time to expiration of 5 years and strike price of 850 CG. The annual continuously compounded cigarette-denominated risk-free interest rate is 0.23. The annual continuously compounded YPS-denominated risk-free interest rate is 0.05. The exchange rate volatility relevant for the Garman-Kohlhagen Formula is 0.1. Use the Garman-Kohlhagen Formula to find the price of such a call option.

**Solution GKFPCO3.** We use the formula $C(x, K, \sigma, r, T, f) = xe^{-fT}N(d_1) - Ke^{-rT}N(d_2)$

We are given that x = 800, K = 850, r = 0.23, f = 0.05, σ = 0.1, T = 5.

Furthermore, from Solutions GKFPCO1-2, $d_1$ = 3.865604207 and $d_2$ = 3.641997409.

In MS Excel, using the input "=NormSDist(3.865604207)", we find that $N(d_1)$ = 0.999944572

In MS Excel, using the input "=NormSDist(3.641997409)", we find that $N(d_2)$ = 0.9998647

Thus, $C(x, K, \sigma, r, T, f) = 800e^{-0.05*5}0.999944572 - 850e^{-0.23*5}0.9998647 =$

**C(x, K, σ, r, T, f) = $353.9012534**

**Problem GKFPCO4.** One piece of stone from Yap (YPS) currently trades for 800 cigarettes (CG). Amon-Ra owns a cigarette-denominated YPS put option with time to expiration of 5 years and strike price of 850 CG. The annual continuously compounded cigarette-denominated risk-free interest rate is 0.23. The annual continuously compounded YPS-denominated risk-free interest rate is 0.05. The exchange rate volatility relevant for the Garman-Kohlhagen Formula is 0.1. Use the Garman-Kohlhagen Formula to find the price of such a put option.

**Solution GKFPCO4.** Since the corresponding call option price is known (from Solution BSFOSDD3), we can use put-call parity to find the put option price.

$P(x, K, \sigma, r, T, f) = C(x, K, \sigma, r, T, f) + Ke^{-rT} - xe^{-fT} = 353.9012534 + 850e^{-0.23*5} - 800e^{-0.05*5} =$

**$P(x, K, \sigma, r, T, f) = \$0.0018809151$**

**Problem GKFPCO5.** For a currency option priced using the Garman-Kohlhagen Formula, volatility σ suddenly increased by a factor of 2, leaving all other factors the same. It is the case that $d_1 = (0.2 + 8\sigma^2)/4\sigma$. By how much has $d_2$ changed as a result of the volatility increase?

**Solution GKFPCO5.** We note that the denominator of the equation for $d_1$ is $4\sigma = \sigma\sqrt{(T)}$.

Thus, $d_2 = d_1 - \sigma\sqrt{(T)} = d_1 - 4\sigma = (0.2 + 8\sigma^2)/4\sigma - 4\sigma = 0.2/4\sigma + 2\sigma - 4\sigma = d_2 = 1/20\sigma - 2\sigma$

If σ increases to 2σ, the increase in $d_2$ is $[1/(20*2\sigma) - 2(2\sigma)] - [1/20\sigma - 2\sigma] =$

$1/40\sigma - 4\sigma - 1/20\sigma + 2\sigma =$ **$-1/40\sigma - 4\sigma$**. Thus, **$d_2$ decreased by $1/40\sigma + 4\sigma$.**

# Section 37

# The Black Formula for Pricing Options on Futures Contracts

The Black formula for pricing European options on a futures contract is as follows:

For call options:
$C(F, K, \sigma, r, T, r) = Fe^{-rT}N(d_1) - Ke^{-rT}N(d_2)$, where
$d_1 = [\ln(F/K) + (0.5\sigma^2)T]/[\sigma\sqrt{(T)}]$ and $d_2 = d_1 - \sigma\sqrt{(T)}$

For put options:
$P(F, K, \sigma, r, T, r) = Ke^{-rT}N(-d_2) - Fe^{-rT}N(-d_1)$

Also, put-call parity holds:
$P(F, K, \sigma, r, T, r) = C(F, K, \sigma, r, T, r) + (K - F)e^{-rT}$

**Meanings of variables:**

F = futures contract price.
K = strike price of the option.
C = call option price.
P = put option price.
σ = annual futures contract price volatility.
r = annual continuously compounded currency risk-free interest rate.
T = time to expiration.

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 12, pp. 381-382.

**Problem BFPOFC1.** Futures contracts on superwidgets currently trade for $444 per superwidget. The annual futures contract price volatility is 0.15, and the annual continuously compounded currency risk-free interest rate is 0.03. European put options are written on superwidget futures contracts, with a strike price of $454 and time to expiration of 2 years. Find the value of $d_1$ in the Black formula for the price of such a put option.

**Solution BFPOFC1.** We use the formula $d_1 = [\ln(F/K) + (0.5\sigma^2)T]/[\sigma\sqrt{(T)}]$, where F = 444, K = 454, T = 2, σ = 0.15. Thus, $d_1 = [\ln(444/454) + (0.5*0.15^2)2]/[0.15\sqrt{(2)}]$ = $\mathbf{d_1 = 0.001071806}$

**Problem BFPOFC2.** Futures contracts on superwidgets currently trade for $444 per superwidget. The annual futures contract price volatility is 0.15, and the annual continuously compounded currency risk-free interest rate is 0.03. European put options are written on superwidget futures contracts, with a strike price of $454 and time to expiration of 2 years. Find the value of $d_2$ in the Black formula for the price of such a put option.

**Solution BFPOFC2.** We use the formula $d_2 = d_1 - \sigma\sqrt{(T)}$, where $d_1 = 0.001071806$ from Solution BFPOFC1, T = 2, and σ = 0.15. Thus, $d_2 = 0.001071806 - 0.15\sqrt{(2)}$ = $\mathbf{d_2 = -0.21110602284}$

**Problem BFPOFC3.** Futures contracts on superwidgets currently trade for $444 per superwidget. The annual futures contract price volatility is 0.15, and the annual continuously compounded currency risk-free interest rate is 0.03. European put options are written on superwidget futures contracts, with a strike price of $454 and time to expiration of 2 years. Use the Black formula to find the price of one such put option.

**Solution BFPOFC3.** We use the formula $P(F, K, \sigma, r, T, r) = Ke^{-rT}N(-d_2) - Fe^{-rT}N(-d_1)$, where it is given that F = 444, K = 454, r = 0.03, T = 2, σ = 0.15. Furthermore, from Solutions BFPOFC1-2 that $d_1 = 0.001071806$ and $d_2 = -0.21110602284$.

In MS Excel, using the input "=NormSDist(-0.001071806)", we find that $N(-d_1) = 0.499572409$
In MS Excel, using the input "=NormSDist(0.21110602284)", we find that $N(-d_2) = 0.5835977$
Thus, $P(F, K, \sigma, r, T, r) = 454e^{-0.03*2}0.5835977 - 444e^{-0.03*2}0.499572409$ = **P(F, K, σ, r, T, r) = $40.63074147**

**Problem BFPOFC4.** Futures contracts on superwidgets currently trade for $444 per superwidget. The annual futures contract price volatility is 0.15, and the annual continuously compounded currency risk-free interest rate is 0.03. European call options are written on superwidget futures contracts, with a strike price of $454 and time to expiration of 2 years. Find the price of one such call option.

**Solution BFPOFC4.** We use put-call parity: $P(F, K, \sigma, r, T, r) = C(F, K, \sigma, r, T, r) + (K - F)e^{-rT}$, rearranging the formula thus: $C(F, K, \sigma, r, T, r) = P(F, K, \sigma, r, T, r) + (F - K)e^{-rT}$. Since F = 444 and K = 454, (F - K) = -10. We are also given that T = 2, r = 0.03, and, from Solution BFPOFC3, P(F, K, σ, r, T, r) = 40.63074147.

Thus, $C(F, K, \sigma, r, T, r) = 40.63074147 - 10e^{-0.03*2}$ = **C(F, K, σ, r, T, r) = 31.21309613**

**Problem BFPOFC5.** The dividend yield in the Black-Scholes formula for stock option pricing is analogous to which of these variables in other related formulas? More than one answer may be correct.

(a) The risk-free interest rate in the Black-Scholes formula for stock option pricing.
(b) The risk-free interest rate in the Black formula for futures contract option pricing.
(c) The domestic risk-free interest rate in the Garman-Kohlhagen formula for currency option pricing.
(d) The foreign risk-free interest rate in the Garman-Kohlhagen formula for currency option pricing.
(e) The volatility in the Black formula for futures contract option pricing.

**Solution BFPOFC5.** We are looking for equivalents to $\partial$ in the Black-Scholes formula C(S, K, σ, r, T, $\partial$) = $Se^{-\partial T}N(d_1) - Ke^{-rT}N(d_2)$. Clearly, r is not necessarily the same as $\partial$ in this formula, so (a) is incorrect. In all variants of the Black-Scholes, formula, σ is distinct from $\partial$, so (e) is incorrect. In the Garman-Kohlhagen formula, C(x, K, σ, r, T, f) = $xe^{-fT}N(d_1) - Ke^{-rT}N(d_2)$, the *foreign* risk-free interest rate f is analogous to $\partial$, while the *domestic* interest rate r is used in the manner of r in all other variants of the Black-Scholes formula. Thus, (c) is incorrect and (d) is correct. In the Black formula, C(F, K, σ, r, T, r) = $Fe^{-rT}N(d_1) - Ke^{-rT}N(d_2)$, r is used in place of both r and $\partial$, so (b) is correct. Thus, **(b) and (d) are correct answers.**

# Section 38

# Exam-Style Questions on the Black-Scholes Formula

The problems in this section were designed to be similar to problems from past versions of Exam 3F / Exam MFE. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

**Problem ESQBSF1.**

**Similar to Question 20 from the Casualty Actuarial Society's Spring 2007 Exam 3:**

You are aware of the following information for the dividend-paying stock of Astonishing Co. and a European call option on that stock. The current stock price is $600, and the call option strike price is $654. The stock's volatility is 0.4, and the expected annual return on the stock is 15%. The annual continuously-compounded risk-free interest rate is 0.12, and the stock's annual continuously-compounded dividend yield is 0.04. The call option expires in 2 years. Find the call price using the Black-Scholes formula.

**Solution ESQBSF1.**

We note that the given expected annual return on the stock is entirely superfluous to solving this problem.

First we find $d_1 = [\ln(S/K) + (r - \partial + 0.5\sigma^2)T]/[\sigma\sqrt{(T)}] =$

$[\ln(600/654) + (0.12 - 0.04 + 0.5*0.4^2)2]/[0.4\sqrt{(2)}] = d_1 = 0.4133433415$

Now we find $d_2 = d_1 - \sigma\sqrt{(T)} = 0.4133433415 - 0.4\sqrt{(2)} = d_2 = -0.1523420835$

In MS Excel, using the input "=NormSDist(0.4133433415)", we find that $N(d_1) = 0.660322421$

In MS Excel, using the input "=NormSDist(-0.1523420835)", we find that $N(d_2) = 0.43945855$

Now we use the Black-Scholes formula:

$C(S, K, \sigma, r, T, \partial) = Se^{-\partial T}N(d_1) - Ke^{-rT}N(d_2) = 600e^{-0.04*2}0.660322421 - 654e^{-0.12*2}0.43945855 =$

**$C(S, K, \sigma, r, T, \partial) = \$139.6511706$**

**Problem ESQBSF2.**

**Similar to Question 21 from the Casualty Actuarial Society's Spring 2007 Exam 3:**

The stock of Intrepid LLC does not pay dividends. The stock currently trades for $5500 per share, with price volatility of 0.21. The annual continuously-compounded risk-free interest rate is 0.02. Use the Black-Scholes formula to find the price of a put option on Intrepid LLC stock with a strike price of $5430 and time to expiration of 1 year.

**Solution ESQBSF2.**

First, we find $d_1 = [\ln(S/K) + (r - \partial + 0.5\sigma^2)T]/[\sigma\sqrt{(T)}] =$

$[\ln(5500/5430) + (0.02 - 0 + 0.5*0.21^2)1]/[0.21\sqrt{(1)}] = d_1 = 0.2612331347$

Now we find $d_2 = d_1 - \sigma\sqrt{(T)} = 0.2612331347 - 0.21 = d_2 = 0.0512331347$

In MS Excel, using the input "=NormSDist(-0.2612331347)", we find that $N(-d_1) = 0.39695642$

In MS Excel, using the input "=NormSDist(-0.0512331347)", we find that $N(-d_2) = 0.479569806$

Now we use the Black-Scholes formula: $P(S, K, \sigma, r, T, \partial) = Ke^{-rT}N(-d_2) - Se^{-\partial T}N(-d_1)$

$= 5430e^{-0.02}0.479569806 - 5500*0.39695642 =$ $\mathbf{P(S, K, \sigma, r, T, \partial) = \$369.2398137}$

**Problem ESQBSF3.**

**Similar to Question 20 from the Casualty Actuarial Society's Fall 2007 Exam 3:**

Which of these statements are assumptions of the Black-Scholes option pricing model? More than one answer may be possible.

(a) A risk premium must be paid for borrowing money.
(b) The volatility of continuously compounded returns is a random variable modeled by the normal probability distribution.
(c) All taxes and transaction costs are the same positive quantity for every transaction.
(d) The risk-free interest rate is known and constant.
(e) Continuously compounded returns on the stock are normally distributed and independent over time.
(f) The stock can only pay dividends on a continuously compounded basis; discrete dividends are not allowed.

**Solution ESQBSF3.**

According to R. L. McDonald, the six assumptions of the Black-Scholes model are

"1. Continuously compounded returns on the stock are normally distributed and independent over time.

"2. The volatility of continuously compounded returns is known and constant.

"3. Future dividends are known, either as a dollar amount or as a fixed dividend yield.

"4. The risk-free interest rate is known and constant.

"5. There are no transaction costs or taxes.

"6. It is possible short-sell costlessly and to borrow at the risk-free rate."

We note that assumption 4 corresponds to statement (d) and assumption 1 corresponds to statement (e). So (d) and (e) are true. Statement (a) contradicts assumption 6, so (a) is false. (b) contradicts assumption 2, so (b) is false. (c) contradicts assumption 5, so (c) is false. (f) is not true, as Section 35 describes how to apply the Black-Scholes model to stocks that pay discrete dividends. So **(d) and (e) are correct answers.**

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 12, pp. 375-379.

**Problem ESQBSF4.**

**Similar to Question 21 from the Casualty Actuarial Society's Fall 2007 Exam 3:**

You know the following information about exchanging Platinum Icosagons (PI) for Titanium Dodecahedra (TD). The spot exchange rate is 1.23 PI/TD. The annual continuously-compounded PI-denominated interest rate is 0.01.The annual continuously-compounded TD-denominated interest rate is 0.005. The exchange rate volatility is 0.02. (These are commodity moneys, so exchange rate volatility should be low.) Polyneices wishes to buy 756 PI-demoniated TD call options. The options have a strike price of 1.10 PI/TD and expire in 3 years. Use the Garman-Kohlhagen formula to find the price of the block of 756 options.

**Solution ESQBSF4.** First we find $d_1 = [\ln(x/K) + (r - f + 0.5\sigma^2)T]/[\sigma\sqrt{(T)}] =$

$[\ln(1.23/1.10) + (0.01 - 0.005 + 0.5*0.02^2)3]/[0.02\sqrt{(3)}] = d_1 = 3.674949633$

Now we find $d_2 = d_1 - \sigma\sqrt{(T)} = 3.674949633 - 0.02\sqrt{(3)} = d_2 = 3.640308616$

In MS Excel, using the input "=NormSDist(3.674949633)", we find that $N(d_1) = 0.99988102$

In MS Excel, using the input "=NormSDist(3.640308616)", we find that $N(d_2) = 0.99986381$

Now we use the Garman-Kohlhagen formula: $C(x, K, \sigma, r, T, f) = xe^{-fT}N(d_1) - Ke^{-rT}N(d_2) =$

$1.23e^{-0.005*3}0.99988102 - 1.10e^{-0.01*3}0.99986381 = C(x, K, \sigma, r, T, f) = 0.1441988137$

We want to find 756C = 756*0.1441988137 = **$109.0143031**

**Problem ESQBSF5.**

**Similar to Question 6 from the Society of Actuaries' Sample MFE Questions and Solutions:**

Athena wishes to purchase 204 European call options on the stock of Mythology Co. The stock currently trades for $40 per share, and pays dividends at an annual continuously-compounded yield of 0.11. The annual continuously-compounded risk-free interest rate is 0.09, and the relevant measure of volatility is 0.1.

The strike price of the options is $43, and the options will expire in 5 years. Use the Black-Scholes formula to find the price of the block of 204 call options.

**Solution ESQBSF5.**

First, we find $d_1 = [\ln(40/43) + (0.09 - 0.11 + 0.5*0.1^2)5]/[0.1\sqrt{(5)}] = d_1 = -0.6588380276$

Now we find $d_2 = d_1 - \sigma\sqrt{(T)} = -0.0119823657 - 0.1\sqrt{(5)} = d_2 = -0.8824448253$

In MS Excel, using the input "=NormSDist(-0.6588380276)", we find that $N(d_1) = 0.254999892$

In MS Excel, using the input "=NormSDist(-0.8824448253)", we find that $N(d_2) = 0.188768152$

Now we use the Black-Scholes formula:

$C(S, K, \sigma, r, T, \partial) = Se^{-\partial T}N(d_1) - Ke^{-rT}N(d_2) = 40e^{-0.11*5}0.254999892 - 43e^{-0.09*5}0.188768152 =$

$C(S, K, \sigma, r, T, \partial) = 0.7092383961$

We want to find 204C = 204*0.7092383961 = **$144.6846328**

# Section 39

# Option Greeks: Delta

The option Greek delta ($\Delta$) "measures the option price change when the stock price increases by \$1" (McDonald 2006, p. 382).

Delta is also the number of shares in the replicating portfolio for an option - otherwise known as the *share-equivalent* of the option.

An option that is in-the-money will be more sensitive to price changes than an option that is out-of-the-money. The more deeply an option is in-the-money, the more likely it is to be exercised, and delta approaches 1 in that case.

For an out-of-the-money option that is unlikely to be exercised, delta approaches 0.

As time to expiration increases, delta is smaller at high stock prices and greater at low stock prices. (McDonald 2006, p. 383).

The formula for a call option's Delta is

$\Delta_{call} = e^{-\partial T}N(d_1)$

where $d_1 = [\ln(S/K) + (r - \partial + 0.5\sigma^2)T]/[\sigma\sqrt{(T)}]$ and $d_2 = d_1 - \sigma\sqrt{(T)}$

A replicating portfolio for a call option involves holding $\Delta$ shares and borrowing B dollars.

Here, $B = Ke^{-rT}N(d_2)$, so the cost of the replicating portfolio is the Black-Scholes price of the call option:

$C = Se^{-\partial T}N(d_1) - Ke^{-rT}N(d_2)$

The formula for a put option's delta can be derived via put-call parity:

$\Delta_{put} = \Delta_{call} - e^{-\partial T} = e^{-\partial T}N(d_1) - e^{-\partial T} = e^{-\partial T}(N(d_1) -1) = \boldsymbol{\Delta_{put} = -e^{-\partial T}N(-d_1)}$

We note that delta changes with the stock price and so the replicating portfolio for an option must be adjusted every time the stock price changes.

**Meaning of variables:**

S = current stock price.

K = strike price of the option.

C = call option price.

P = put option price.

σ = annual stock price volatility.

r = annual continuously compounded risk-free interest rate.

T = time to expiration.

$\partial$ = annual continuously compounded dividend yield.

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 12, pp. 382-384.

**Problem OGD1.** The stock of Delta Corporation has a price of $506. Certain call options on Delta Corporation stock have a delta of 0.4 and trade for $33 per option. The stock price suddenly increases to $508. Assuming that this move did not substantially alter delta, what is the new price of the call option?

**Solution OGD1.** $\Delta_{call}$ = 0.4 means that the option price increases by $0.4 for every increase of $1 in the stock price. Since the stock price has increased by $2, the option price increased by $0.8, and the new option price is **$33.8.**

**Problem OGD2.** The Black-Scholes price for a certain call option on Vacuous LLC stock is $50. The stock currently trades for $1000 per share, and it is known that $452 must be borrowed in the replicating portfolio for this option. Find the delta of the option.

**Solution OGD2.** The Black-Scholes price for the option is $C = 50 = Se^{-\partial T}N(d_1) - Ke^{-rT}N(d_2)$. We are given that S = 1000 and $Ke^{-rT}N(d_2) = 452$. We are left to find $\Delta_{call} = e^{-\partial T}N(d_1)$.

$50 = 1000\Delta_{call} - 452$, so $502 = 1000\Delta_{call}$ and $\boldsymbol{\Delta_{call} = 0.502}$.

**Problem OGD3.** The stock of Voracious Co. currently trades for $95 per share. The annual continuously compounded risk-free interest rate is 0.06, and the stock pays dividends with an annual continuously compounded yield of 0.03. The price volatility relevant for the Black-Scholes formula is 0.32. Find the delta of a call option on Voracious Co. stock with strike price of $101 and time to expiration of 3 years.

**Solution OGD3.** First, we find

$d_1 = [\ln(S/K) + (r - \partial + 0.5\sigma^2)T]/[\sigma\sqrt{(T)}] = [\ln(95/101) + (0.06 - 0.03 + 0.5*0.32^2)3]/[0.32\sqrt{(3)}] =$
$d_1 = 0.3290109439$

In MS Excel, using the input "=NormSDist(0.3290109439)", we find that $N(d_1) = 0.628926227$

Now we use the formula

$\Delta_{call} = e^{-\partial T}N(d_1) = e^{-0.03*3}0.628926227 = \mathbf{\Delta_{call} = 0.5747952921}$

**Problem OGD4.** The stock of Voracious Co. currently trades for \$95 per share. The annual continuously compounded risk-free interest rate is 0.06, and the stock pays dividends with an annual continuously compounded yield of 0.03. The price volatility relevant for the Black-Scholes formula is 0.32. Find the delta of a put option on Voracious Co. stock with strike price of \$101 and time to expiration of 3 years.

**Solution OGD4.** Since the call delta is known from Solution OGD3, we use the put-call parity formula to find the put delta: $\Delta_{put} = \Delta_{call} - e^{-\partial T} = 0.5747952921 - e^{-0.03*3} = \mathbf{\Delta_{put} = -0.3391358932}$

**Problem OGD5.** For otherwise equivalent call options on a particular stock, for which of these values of strike price (K) and time to expiration (T) would you expect delta to be the highest? The stock price at both T = 0.3 and T = 0.2 is \$50.

(a) K = \$43, T = 0.3
(b) K = \$43, T = 0.2
(c) K = \$55, T = 0.3
(d) K = \$55, T = 0.2
(e) K = \$50, T = 0.3
(f) K = \$50, T = 0.2

**Solution OGD5.** The more an option is in-the-money, the higher the value of delta. So, other things equal, a \$43-strike option should have a higher delta than a \$50-strike or a \$55-strike option. As expiration time gets closer, delta becomes higher at low strike prices because there is less of a likelihood that an in-the-money option will become out-of-the-money prior to expiration. Thus, the highest delta should occur for the stock with the lowest strike price and lowest time to expiration, i.e., **answer (b).**

# Section 40

# Option Greeks: Gamma and Vega

The option Greek gamma ($\Gamma$) "measures the change in delta when the stock price increases by \$1" (McDonald 2006, p. 382).

Whether a call or a put is purchased, gamma is positive. Thus, delta increases as the stock price increases.

"For a call, delta approaches 1 as the stock price increases. For a put, delta approaches 0 as the stock price increases" (McDonald 2006, p. 384).

For a call and a put with the same strike price and time to expiration, gamma is the same.

Deep in-the-money and out-of-the money options both have values of gamma close to 0.

The option Greek vega "measures the change in the option price when there is an increase in volatility of one percentage point" (where one percentage point = 0.01).

R. L. McDonald suggests the following mnemonic device to help memorize the definition of vega: "vega" and "volatility" begin with the same letter.

The higher the volatility of the underlying asset price, the higher the price of *both* the call and put options on that asset. Thus, vega is positive for purchased calls and puts.

According to R. L. McDonald, "vega tends to be greater for at-the-money options, and greater for options with moderate than with short times to expiration" - but this is not the case with very long-lived options (386).

Due to put-call parity, the vega for calls and puts with the same strike prices and times to expiration is the same.

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 12, pp. 382-386.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem OGGV1.** The stock of Delta-Gamma Corporation has a price of \$506. Certain call options on Delta-Gamma Corporation stock have a delta of 0.4 and trade for \$33 per option. These options also have a gamma of 0.03. The stock price suddenly increases to \$508. What is the new call option delta?

**SolutionOGGV1.** A gamma of 0.03 means that delta increases by 0.03 whenever the stock price increases by $1. Since the stock price has increased by $2, delta has increased by 0.06, and the new delta is **0.46.**

**Problem OGGV2.** The stock of Vega Corporation has a price of $567 and a volatility of 0.45. A certain put option on Vega Corporation has a price of $78 and a vega of 0.23. Suddenly, volatility increases to 0.51. Find the new put option price.

**Solution OGGV2.** A vega of 0.23 means that for every increase in volatility by 0.01, the option price will increase by 0.23. Volatility here has increased by 6 percentage points, to the put option price increased by 6*0.23 = 1.38 and the new put option price is 78 + 1.38 = **$79.38**

**Problem OGGV3.** Certain call options on the stock of Simultaneous Co. have a delta of 0.3, a gamma of 0, and a vega of 0.11. The stock currently trades at $34 per share with volatility of 0.4, and a call option trades at $3 per contract. Suddenly, the stock price increases to $36, and volatility falls to 0.2. Find the new call option price.

**Solution OGGV3.** Since the gamma of this option is 0, a change in the stock price does not affect delta. Thus, the effect of the stock price change is to increase the option price by (36-34)Δ = 2*0.3 = 0.6

But the effect of the volatility decrease is to decrease the option price by (0.4 - 0.2)*vega/0.01 =

20*vega = 20*0.11 = 2.2

Thus, the new option price is 3 + 0.6 - 2.2 = **$1.40**

**Problem OGGV4.** The stock of Gamma LLC currently trades for $60 per share. For which of these otherwise equivalent options and strike prices (K) is the gamma the highest?

(a) Call, K = 2
(b) Put, K = 20
(c) Call, K = 45
(d) Put, K = 61
(e) Call, K = 98
(f) Put, K = 102

**Solution OGGV4.** The farther an option is in-the-money *or* out-of-the-money, the closer the option's gamma is to 0. Options (a), (c), and (f) are significantly in-the-money, whereas options (b) and (e) and significantly out-of-the-money. (d), however, is extremely close to being right at-the-money. So the gamma for (d) should be the highest, and **(d) is the correct answer.**

**Problem OGGV5.** The stock of Vega Corporation has a price of $567. For which of these strike prices (K) and times to expiration (T, in years) is the vega for one of these otherwise equivalent call options most likely to be the highest?

(a) K = 564, T = 0.2
(b) K = 564, T = 1
(c) K = 564, T = 30
(d) K = 598, T = 0.2
(e) K = 598, T = 1
(f) K = 598, T = 30

**Solution OGGV5.** Vega tends to be greatest for at-the money options with moderately long times to expiration. A call with K = 564 is closer to being at-the-money than a call with K = 598. Furthermore, T = 0.2 is a short time period, whereas T = 1 is a longer time period without being extremely long. For very long-lived options, vega tends to be *smaller* than the vega for moderately lived options that are otherwise equivalent. Thus, the greatest value of vega most likely pertains to **answer (b).**

# Section 41

# Option Greeks: Theta, Rho, Psi, and Greek Measures for Portfolios

### Theta

The option Greek theta (θ) "measures the change in the option price when there is a decrease in the time to maturity of 1 day" (McDonald 2006, p. 383).

R. L. McDonald suggests the following mnemonic device to help memorize the definition of theta: "theta" and "time" begin with the same letter.

In most cases, theta is negative - that is, an option loses value as expiration time approaches. The fastest time decay at expiration occurs for at-the-money options.

There are a few exceptions to this rule. Theta can be positive for deep in-the-money European puts and deep in-the-money European calls when the underlying asset has a high dividend yield.

### Rho

The option Greek rho (ρ) "measures the change in the option price when there is an increase in the interest rate of 1 percentage point (100 basis points)" (McDonald 2006, p. 383).

R. L. McDonald suggests the following mnemonic device to help memorize the definition of rho: "rho" begins with the letter r, which is commonly used to denote the annual continuously-compounded risk-free interest rate.

For an ordinary call option, rho is positive; for a put, rho is negative.

### Psi

The option Greek psi (Ψ) "measures the change in the option price when there is an increase in the continuous dividend yield of 1 percentage point (100 basis points)" (McDonald 2006, p. 383).

For call options, psi is negative. For put options, psi is positive. "The absolute value of psi increases with time to expiration" (McDonald 2006, p. 388).

### Greek Measures for Portfolios

"The Greek measure of a portfolio is the sum of the Greeks of the individual portfolio components" (McDonald 2006, p. 388). For delta, this can be expressed by the formula

$\Delta_{portfolio} = {}_{i=1}{}^{n}\Sigma\omega_i\Delta_i$

where $\omega_i$ is the quantity of each option i and $\Delta_i$ is the delta of each option i and the portfolio contains n options.

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 12, pp. 383-389.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem OGTRPGMP1.** The stock of Theta-Rho-Psi Co. pays dividends at an annual continuously compounded yield of 0.12. The annual continuously compounded risk-free interest rate is 0.34. Certain call options on the stock of Theta-Rho-Psi Co. have time to expiration of 99 days, $\theta = -0.03$, $\rho = 0.11$, $\Psi = -0.04$. One call option currently trades for \$56. Find the price of the call option 65 days from expiration, all other things equal.

**Solution OGTRPGMP1.** We want to calculate the option price once 99-65 = 34 days have passed. The price change is $34\theta = -0.03*34 = -1.02$. Thus, the new option price is 56 - 1.02 = **\$54.98**

**Problem OGTRPGMP2.** The stock of Theta-Rho-Psi Co. pays dividends at an annual continuously compounded yield of 0.12. The annual continuously compounded risk-free interest rate is 0.34. Certain call options on the stock of Theta-Rho-Psi Co. have time to expiration of 99 days, $\theta = -0.03$, $\rho = 0.11$, $\Psi = -0.04$. One call option currently trades for \$56. Find the price of the call option if the interest rate suddenly increases to 0.66, all other things equal.

**Solution OGTRPGMP2.** The change in the interest rate is 0.66 - 0.34 = 0.32, so the change in the option price is $32\rho = 32*0.11 = 3.52$ and the new option price is 56 + 3.52 = **\$59.52**

**Problem OGTRPGMP3.** The stock of Theta-Rho-Psi Co. pays dividends at an annual continuously compounded yield of 0.12. The annual continuously compounded risk-free interest rate is 0.34. Certain call options on the stock of Theta-Rho-Psi Co. have time to expiration of 99 days, $\theta = -0.03$, $\rho = 0.11$, $\Psi = -0.04$. One call option currently trades for \$56. Find the price of the call option if the stock's dividend yield suddenly decreases to 0.02, all other things equal.

**Solution OGTRPGMP3.** The change in the dividend yield is 0.02 - 0.12 = -0.1. So the change in the option price is $-10\Psi = -10*-0.04 = 0.4$ and the new option price is 56 + 0.4 = **\$56.40**

**Problem OGTRPGMP4.** The stock of Theta-Rho-Psi Co. pays dividends at an annual continuously compounded yield of 0.12. The annual continuously compounded risk-free interest rate is 0.34. Certain call options on the stock of Theta-Rho-Psi Co. have time to expiration of 99 days, $\theta = -0.03$, $\rho = 0.11$, $\Psi = -0.04$. One call option currently trades for \$56. Find the price of the call option if the stock's dividend yield increases to 0.45, the interest rate drops to 0.01, and time to expiration becomes 12 days, all other things equal.

**Solution OGTRPGMP4.** We consider the effect of the dividend yield increase on the option price: The change in the dividend yield is 0.45 - 0.12 = 0.33. So the change in the option price from this alone is $33\Psi = 33*-0.04 = -1.32$

We consider the effect of the interest rate decrease on the option price: The change in the interest rate is 0.01 - 0.34 = -0.33. So the change in the option price from this alone is $-33\rho = -33*0.11 = -3.63$

Also, 99-12 = 87 days have passed, so from this alone the option price changed by $87\theta =$

$87*-0.03 = -2.61$.

Thus, the new option price is 56 - 1.32 - 3.63 - 2.61 = **$48.44**

**Problem OGTRPGMP5.** You own 45 call options on Asset A, 14 put options on Asset B, 44 put options on Asset C, and 784 call options on Asset D. Asset A call options have $\Delta = 0.22$. Asset B put options have $\Delta = -0.82$. Asset C put options have $\Delta = -0.33$. Asset D call options have $\Delta = 0.01$. Find the delta of your entire portfolio.

**Solution OGTRPGMP5.** We use the formula

$\Delta_{portfolio} = \sum_{i=1}^{n}\omega_i\Delta_i = 45*0.22 + 14*-0.82 + 44*-0.33 + 784*0.01 = \boldsymbol{\Delta_{portfolio} = -8.26}$

# Section 42

# Option Elasticity and Option Volatility

The formula for option elasticity $\Omega$ is

$\Omega$ = [% change in option price]/[% change in stock price] = $S\Delta/C$, where C is the option price, S is the stock price, and $\Delta$ is the option delta.

For a call option, $\Omega \geq 1$. $\Omega$ decreases as the strike price K increases.

For a put option, $\Omega \leq 0$.

The volatility of an option can be expressed as

$\sigma_{option} = \sigma_{stock} * |\Omega|$. That is, option price volatility is stock price volatility multiplied by the absolute value of option elasticity.

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 12, pp. 391-394.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem OEOV1.** The stock of Assiduous Co. trades for $123 per share with a price volatility of 0.3. Certain call options on Assiduous Co. stock have a delta of 0.44 and a price of $20. Find the elasticity of such a call option.

**Solution OEOV1.** We use the formula $\Omega = S\Delta/C = 123*0.44/20 =$ $\mathbf{\Omega = 2.706}$

**Problem OEOV2.** The stock of Assiduous Co. trades for $123 per share with a price volatility of 0.3. Certain call options on Assiduous Co. stock have a delta of 0.44 and a price of $20. Find the volatility of such a call option.

**Solution OEOV2.** From Solution OEOV1, we know that $\Omega = 2.706 = |\Omega|$. Also, we are given that $\sigma_{stock} = 0.3$. Thus, $\sigma_{option} = \sigma_{stock} * |\Omega| = 0.3*2.706 =$ $\boldsymbol{\sigma_{option} = 0.8118}$

**Problem OEOV3.** The stock of Ingenious Co. trades for $550 per share with a prepaid forward price volatility (the volatility relevant for the Black-Scholes formula) of 0.2. Certain call options on Ingenious Co. stock have a strike price of $523 and a time to expiration of 3 years. The stock pays dividends at an annual continuously compounded yield of 0.03, and the annual continuously compounded interest rate is 0.07. Find the elasticity of such a call option.

**Solution OEOV3.** We can use the Black-Scholes formula to find the call option price.

First we find $d_1 = [\ln(S/K) + (r - \partial + 0.5\sigma^2)T]/[\sigma\sqrt{(T)}] =$

$[\ln(550/523) + (0.07 - 0.03 + 0.5*0.2^2)3]/[0.2\sqrt{(3)}] = d_1 = 0.6649251083$

Now we find $d_2 = d_1 - \sigma\sqrt{(T)} = 0.6649251083 - 0.2\sqrt{(3)} = d_2 = 0.3185149468$

In MS Excel, using the input "=NormSDist(0.6649251083)", we find that $N(d_1) = 0.746950872$

In MS Excel, using the input "=NormSDist(0.3185149468)", we find that $N(d_2) = 0.624952755$

Now we use the Black-Scholes formula:

$C(S, K, \sigma, r, T, \partial) = Se^{-\partial T}N(d_1) - Ke^{-rT}N(d_2) = 550e^{-0.03*3}0.746950872 - 523e^{-0.07*3}0.624952755 =$
$C = 110.5242361$

We can also find $\Delta_{call} = e^{-\partial T}N(d_1) = e^{-0.03*3}0.746950872 = \Delta_{call} = 0.6826616958$

Now we use the formula $\Omega = S\Delta/C = 550*0.6826616958/110.5242361 =$ **$\Omega = 3.39711855$**

**Problem OEOV4.** The stock of Ingenious Co. trades for \$550 per share with a price volatility of 0.2. Certain call options on Ingenious Co. stock have a strike price of \$523 and a time to expiration of 3 years. The stock pays dividends at an annual continuously compounded yield of 0.03, and the annual continuously compounded interest rate is 0.07. Find the volatility of such a call option.

**Solution OEOV4.** From Solution OEOV3, we know that $\Omega = 3.39711855 = |\Omega|$. Also, we are given that $\sigma_{prepaid\ forward} = 0.2$. But we need to find

$\sigma_{stock} = (F^P/S)\sigma_{prepaid\ forward} = (Se^{-\partial T}/S)\sigma_{prepaid\ forward} = e^{-\partial T}\sigma_{prepaid\ forward} = e^{-0.03*3}0.2 =$
$\sigma_{stock} = 0.1827862371$

Thus, $\sigma_{option} = \sigma_{stock}*|\Omega| = 0.1827862371*3.39711855 =$ **$\sigma_{option} = 0.6209465166$**

**Problem OEOV5.** The stock of Ingenious Co. trades for \$550 per share with a prepaid forward price volatility (the volatility relevant for the Black-Scholes formula) of 0.2. Certain put options on Ingenious Co. stock have a strike price of \$523 and a time to expiration of 3 years. The stock pays dividends at an annual continuously compounded yield of 0.03, and the annual continuously compounded interest rate is 0.07. Find the elasticity of such a put option.

**Solution OEOV5.** Since the call price is known from Solution OEOV3, we can use put-call parity to calculate the put price: $P(S, K, \sigma, r, T, \partial) = C(S, K, \sigma, r, T, \partial) + Ke^{-rT} - Se^{-\partial T} =$
$110.5242361 + 523e^{-0.07*3} - 550e^{-0.03*3} = P = \$31.79764484$

Now we find $\Delta_{put} = \Delta_{call} - e^{-\partial T} = 0.6826616958 - e^{-0.03*3} = \Delta_{put} = -0.2312694895$

Now we use the formula $\Omega = S\Delta/P = 550*-0.2312694895/31.79764484 =$ **$\Omega = -4.000240265$**

# Section 43

# The Risk Premium and Sharpe Ratio of an Option

The risk premium of an option can be expressed as

$\gamma - r = (\alpha - r)\Omega$

The Sharpe ratio of any asset is $(\alpha - r)/\sigma$. The Sharpe ratio for a call is the same as the Sharpe ratio for the underlying asset. If the option is a put, the sign of the Sharpe ratio is reversed, so the Sharpe ratio for a put becomes $(r - \alpha)/\sigma$.

**Meaning of variables:**

$\gamma$ = expected annual continuously compounded return on the option.

$\alpha$ = expected annual continuously compounded return on the underlying asset (most often a stock).

$\Omega$ = option elasticity.

r = annual continuously compounded risk-free interest rate.

$\sigma$ = annual asset price volatility.

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 12, pp. 394-395.

**Problem RPSRO1.** You expect to get an annual continuously compounded return of 0.3 on the stock of Curious Co. The stock has annual price volatility of 0.22. The annual continuously compounded risk-free interest rate is 0.02. A certain call option on Curious Co. stock has elasticity of 2.3. Find the expected annual continuously compounded return on the call option.

**Solution RPSRO1.** We use the formula $\gamma - r = (\alpha - r)\Omega$, which we rearrange as

$\gamma = (\alpha - r)\Omega + r$ to find $\gamma = (0.3 - 0.02)2.3 + 0.02 =$ $\boldsymbol{\gamma = 0.664}$

**Problem RPSRO2.** You expect to get an annual continuously compounded return of 0.3 on the stock of Curious Co. The stock has annual price volatility of 0.22. The annual continuously compounded risk-free interest rate is 0.02. A certain call option on Curious Co. stock has elasticity of 2.3. Find the Sharpe ratio of the call option.

**Solution RPSRO2.** We use the formula $(\alpha - r)/\sigma$, which applies to the call option as well as the underlying stock. Here, $(\alpha - r)/\sigma = (0.3 - 0.02)/0.22 =$ **Sharpe ratio = 1.272727272727**

**Problem RPSRO3.** The stock of Delirious LLC has a Sharpe ratio of 0.77 and annual price volatility of 0.11. The annual continuously compounded risk-free interest rate is 0.033. For a certain call option on Delirious LLC stock, the elasticity is 3.4. Find the expected annual continuously compounded return on the option.

**Solution RPSRO3.** First, we find $\alpha$. We know that $(\alpha - r)/\sigma = 0.77$, so $\alpha = 0.77\sigma + r =$

$0.77*0.11 + 0.033 = \alpha = 0.1177$. Now we use the formula $\gamma = (\alpha - r)\Omega + r$ to find

$\gamma = (0.1177 - 0.033)3.4 + 0.033 =$ **$\gamma$ = 0.32098**

**Problem RPSRO4.** You expect an annual continuously compounded return of 0.145 on the stock of Deleterious, Inc., and an annual continuously compounded return of 0.33 on a certain call option on that stock. The option elasticity is 4.44. Find the annual continuously compounded risk-free interest rate.

**Solution RPSRO4.** The formula $\gamma = (\alpha - r)\Omega + r$ can be expressed as $\gamma = \Omega\alpha + (1 - \Omega)r$, which we can rearrange as follows: $r = (\gamma - \Omega\alpha)/(1 - \Omega) = (0.33 - 4.44*0.145)/(1 - 4.44) =$

**r = 0.0912209302**

**Problem RPSRO5.** You know that the Sharpe Ratio for a certain call option on Insidious Co. stock is 0.55 and the annual continuously compounded expected return on the option is 0.4. The option elasticity is 2.23. The annual continuously compounded risk-free interest rate is 0.05. Find the annual stock price volatility.

**Solution RPSRO5.** First, we find $\alpha$. The formula $\gamma = (\alpha - r)\Omega + r$ can be expressed as $\gamma = \Omega\alpha + (1 - \Omega)r$, which we can rearrange as follows: $\alpha = [\gamma - (1 - \Omega)r]/\Omega = [0.4 - (1- 2.23)0.05]/2.23 =$

$\alpha = 0.2069506726$. We know that $0.55 = (\alpha - r)/\sigma$, so $\sigma = (\alpha - r)/0.55 =$

$(0.2069506726 - 0.05)/0.55 =$ **$\sigma$ = 0.2853648594**

# Section 44

# The Elasticity and Risk Premium of an Option Portfolio

The elasticity of a portfolio of call options can be expressed as

$\Omega_{portfolio} = {}_{i=1}{}^{n}\Sigma\omega_i\Omega_i$ where $\Omega_i$ is the elasticity of the ith call option and $\omega_i$ is the percentage of the portfolio comprised of the ith call option.

The risk premium on the portfolio - where all call options are based on the same underlying asset - is

$\gamma - r = \Omega_{portfolio}(\alpha - r)$

**Meaning of variables:**

$\gamma$ = expected annual continuously compounded return on the option.

$\alpha$ = expected annual continuously compounded return on the underlying asset (most often a stock).

$\Omega$ = option elasticity.

r = annual continuously compounded risk-free interest rate.

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 12, p. 395 and this erratum.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem ERPOP1.** You own a portfolio of calls options on Tenacious Co. stock. The portfolio consists of 444 Options A, 334 Options B, and 3434 Options C. Options A have elasticity of 4.4. Options B have elasticity of 5.5. Options C have elasticity of 1.22. Find the elasticity of this portfolio.

**Solution ERPOP1.** There are a total of 444 + 334 + 3434 = 4212 options in the portfolio. Options A comprise (444/4212) = 0.1054131054 of the portfolio; Options B comprise (334/4212) = 0.079297246 of the portfolio; Options C comprise (3434/4212) = 0.8152896486 of the portfolio. We use the formula $\Omega_{portfolio} = {}_{i=1}{}^{n}\Sigma\omega_i\Omega_i$ = 0.1054131054*4.4 + 0.079297246*5.5 + 0.8152896486*1.22 = $\mathbf{\Omega_{portfolio}}$ **= 1.894605888**

**Problem ERPOP2.** You own a portfolio of calls options on Tenacious Co. stock. The portfolio consists of 444 Options A, 334 Options B, and 3434 Options C. Options A have elasticity of 4.4. Options B have elasticity of 5.5. Options C have elasticity of 1.22. The expected annual continuously compounded return on the stock is 0.24, and the annual continuously compounded risk-free interest rate is 0.05. Find the risk premium on this option portfolio.

**Solution ERPOP2.** We use the formula $\gamma - r = \Omega_{portfolio}(\alpha - r)$. We know from Solution ERPOP1 that $\Omega_{portfolio} = 1.894605888$. Thus, $\gamma - r = 1.894605888(0.24 - 0.05) =$ $\boldsymbol{\gamma - r = 0.3599751187}$

**Problem ERPOP3.** Your option portfolio consists of two distinct types of call options on Imperious LLC stock - Option D and Option E. Option D has elasticity 3.45, and Option E has elasticity 5.55. You have 100 options in the portfolio, and you know that the portfolio has elasticity of 4.374. How many Options D are in the portfolio?

**Solution ERPOP3.** Let D be the number of Options D in the portfolio.
By the formula $\Omega_{portfolio} = {}_{i=1}{}^{n}\Sigma\omega_i\Omega_i$, we know that $4.374 = (D/100)3.45 + [(100 - D)/100]*5.55$
Thus, $4.374 = 0.0345D + 5.55 - 0.0555D$
$-1.176 = -0.021D$ and $D = -1.176/-0.021 =$ $\mathbf{D = 56.}$

**Problem ERPOP4.** Your option portfolio consists of three distinct types of call options on Imperious LLC stock - Option F, Option G, and Option H. You own 34 Options F with elasticity 2.4 and 45 Options G with elasticity 4.7. Option H has elasticity 5.9. How many options in total must be in the portfolio in order for portfolio elasticity to be 4.17?

**Solution ERPOP4.** Let H be the number of Options H in the portfolio.
By the formula $\Omega_{portfolio} = {}_{i=1}{}^{n}\Sigma\omega_i\Omega_i$, we know that
$4.17 = [34/(79 + H)]2.4 + [45/(79 + H)]4.7 + [H/(79 + H)]5.9$
$4.17(79 + H) = 293.1 + 5.9H$
$329.43 + 4.17H = 293.1 + 5.9H$
$36.33 = 1.73 H$, so $H = 21$ and the total number of options in the portfolio is $34 + 45 + 21 =$ **100 options.**

**Problem ERPOP5.** You know that the risk premium on your call option portfolio is 0.56 and the expected annual continuously compounded return on the underlying asset - superwidgets - is 0.43. The annual continuously compounded risk-free interest rate is 0.09. The option portfolio consists of 44 Options I with elasticity 1.64 and 56 Options J with elasticity X. Find X.

**Solution ERPOP5.** First, we find portfolio elasticity using the formula $\gamma - r = \Omega_{portfolio}(\alpha - r)$, which we rearrange to $\Omega_{portfolio} = (\gamma - r)/(\alpha - r) = (0.56 - 0.09)/(0.43 - 0.09) =$
$\Omega_{portfolio} = 1.382352941$

Now we use the formula $\Omega_{portfolio} = {}_{i=1}{}^{n}\Sigma\omega_i\Omega_i$. Since we conveniently have 100 options in the portfolio, the fraction consisting of I is 0.44 and the fraction consisting of J is 0.56. Thus,
$1.382352941 = 0.44*1.64 + 0.56X$
$0.6607529412 = 0.56X$.
Thus, $\mathbf{X = 1.179915966}$

# Section 45

# Calendar Spreads and Implied Volatility

A **calendar spread** involves selling a call option and buying a call option with the same strike price but a greater time to expiration. A purchased calendar spread takes advantage of time decay and the fact that the sold option will lose its value faster than the purchased option. Once the shorter-lived option expires, the buyer of the spread can make the most profit if the stock price remains unchanged. (Assuming, initially, that the spread consisted of options whose strike price was equal to the stock price when the spread was entered into.)

**Implied volatility** is "the volatility that would explain the observed option price." That is,

Market Option Price = $C(S, K, \sigma_{implied}, r, T, \partial)$. So we assume that the option price can be modeled by the Black-Scholes formula, and the other variables - stock price (S), strike price (K), annual continuously compounded risk-free interest rate (r), time to expiration (T), and annual continuously compounded dividend yield ($\partial$) are known. Using these assumptions, we try to find $\sigma_{implied}$. There is no way to solve directly for implied volatility in the Black-Scholes formula.

When using the Black-Scholes or the binomial model, it is possible to confine implied volatility within particular boundaries by calculating option prices for two different implied volatilities. If one of these prices is greater than the observed option price and the other is less than the observed option price, then we know that the implied volatility is somewhere between the two values for which calculations were made.

On the actuarial exam, you will be given several ranges within which implied volatility might fall. Test the extreme values of those ranges and see if the observed option price falls somewhere in between the prices calculated by considering each of these extreme values. If it does, then you have obtained the correct value of implied volatility.

**Volatility skew** is a "systematic change in implied volatility across strike prices."

For European options, calls and puts with the same time to expiration and strike price must have the same implied volatility.

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 12, p. 398-402.

Some of the problems in this section were designed to be similar to problems from past versions of Exam 3F / Exam MFE. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem CSIV1.** The current price of Stable Co. stock is \$60. You purchase a calendar spread on this stock by selling one \$60-strike call option with time to expiration of 2 months and $\theta$ = -0.05. The premium on this option is \$3.45. You also purchase a \$60-strike call option with time to expiration of 1 year at $\theta$ = -0.04. The premium on this option is \$18.88. The annual continuously compounded risk-free interest rate is 0.06. Find your profit 2 months from now, once the sold option has expired, if the stock price remains at \$60 and nothing else has changed. Assume that there are 30 days in every month.

**Solution CSIV1.** Two months from now, at a stock price of \$60, the written call option will be at-the-money and so will not be exercised. You keep the entire premium on the option - i.e., \$3.45, and this value could have accumulated interest over two months. Thus, your profit from selling the 2-month option is $3.45e^{0.06/6}$ = 3.484673076.

You also lose some money on your purchased 1-year option because of time decay. Due to time decay, the option price changes by $60\theta = 60(-0.04) = -2.4$

You further lose some money in the form of the interest that could have accumulated on the \$18.88 premium you paid for the purchased call option, if instead you invested the money at the risk-free interest rate. That interest is $\$18.88(e^{0.06/6} - 1) = 0.1897471546$,

So your net gain from entering into this spread is 3.484673076 -2.4 - 0.1897471546 = **\$0.8949259215.**

**Problem CSIV2.** You own a calendar spread on a the stock of Ludicrous Co, which you bought when the stock was priced at \$22. The spread consists of a written call option with a strike price of \$22 and a longer-lived purchased call option with a strike price of \$22. Upon the expiration of the shorter-lived option, at which stock price will you make the most money on the calendar spread?

(a) \$3
(b) \$12
(c) \$22
(d) \$23
(e) \$33

**Solution CSIV2.** The calendar spread will make you the most money if the stock price is identical to the strike price of the options, so the correct answer is **(c): \$22.**

**Problem CSIV3.** The stock of Gregarious LLC currently costs \$555 per share. In one year, it might increase to \$569. The annual continuously compounded risk-free interest rate is 0.02, and the stock pays dividends at an annual continuously compounded yield of 0.01. Find the implied volatility of this stock using a one-period binomial model.

**Solution CSIV3.** In the one-period binomial model, $u = e^{(r-\partial)h + \sigma\sqrt{(h)}} = e^{(0.02 - 0.01)1 + \sigma\sqrt{(1)}}$. We know that $u = 569/555 = 1.025225225 = e^{0.01 + \sigma}$. Thus, $\sigma_{implied} = \ln(1.025225225) - 0.01 =$

$\boldsymbol{\sigma_{implied} = 0.0149123204}$

**Problem CSIV4.** The stock of Imperious LLC currently costs $228 per share. The annual continuously compounded risk-free interest rate is 0.11, and the stock pays dividends at an annual continuously compounded yield of 0.03. The stock price will be $281 next year if it increase. What will the stock price be next year if it decreases? Use a one-period binomial model.

**Solution CSIV4.** In the one-period binomial model, $u = e^{(r-\partial)h + \sigma\sqrt{(h)}} = e^{(0.11 - 0.03)1 + \sigma\sqrt{(1)}} =$

$e^{0.08+ \sigma} = 281/228$, so $\sigma_{implied} = \ln(281/228) - 0.08 = \sigma_{implied} = 0.1290090404$

Thus, $d = e^{(r-\partial)h - \sigma\sqrt{(h)}} = e^{0.08 - 0.1290090404} = d = 0.9521725217$.

If the stock price decreases, it will be 228*0.9521725217 = **$217.0953349**

**Problem CSIV5.**

**Similar to Question 23 from the Casualty Actuarial Society's Fall 2007 Exam 3:**

Lethargic LLC stock currently trades for $300 per share. The annual continuously compounded risk-free interest rate is 0.09, and the stock pays dividends at an annual continuously compounded yield of 0.03. You know that a $300-strike call option on Lethargic LLC stock expiring in one year has a price of $85.61554928. Given that the implied volatility of the stock is greater than 0.06, calculate this implied volatility using a one-period binomial option pricing model.

(a) Between 0.35 and 0.45
(b) Between 0.45 and 0.55
(c) Between 0.55 and 0.65
(d) Between 0.65 and 0.75
(e) Between 0.75 and 0.85

**Solution CSIV5.** We test the option prices that would be generated by volatilities of 0.35, 0.45, 0.55, 0.65, 0.75, and 0.85.

We know that $u = e^{(r-\partial)h + \sigma\sqrt{(h)}}$ and $d = e^{(r-\partial)h - \sigma\sqrt{(h)}}$

Here, $h = 1$, $\sigma\sqrt{(h)} = \sigma$ and $(r - \partial)h = 0.09 - 0.03 = 0.06$. Because we know that $\sigma > 0.06$, we know that

$d < e^{0.06 - 0.06}$, so $d < 1$ and the $300-strike call will be worthless if the stock declines in price. Thus, in calculating the option price we need to only apply the formula

$C = e^{-rh}[p^*C_u + (1 - p^*)C_d] = e^{-0.09}[p^*C_u + (1 - p^*)0] = e^{-0.09}[(p^*)(300u - 300)] =$

$C = e^{-0.09}[300(e^{0.06}- d)e^{0.06 + \sigma} /(u - d) - 300(e^{0.06}- d)e^{0.06 + \sigma} /(u^*(u - d))] =$

$C(\sigma) = e^{-0.09}[300(e^{0.06}- e^{0.06- \sigma})e^{0.06 + \sigma} /(e^{0.06 + \sigma} - e^{0.06- \sigma}) - 300(e^{0.06}- d)e^{0.06 + \sigma} /(e^{0.06 + \sigma}(e^{0.06 + \sigma} - e^{0.06- \sigma}))]$

Now we can insert various values of σ to find corresponding values of C(σ).

C(0.35) = 57.44319699

C(0.45) = 71.02295123

C(0.55) = 84.3057477

C(0.65) = 97.23721038

We need not go further, because our desired call price, $85.61554928, is between C(0.55) and C(0.65). Thus, implied volatility is between 0.55 and 0.65, and the correct answer is **(c).**

# Section 46

# Revised Exam-Style Questions on Option Elasticity, Option Volatility, and the Black-Scholes Formula

The problems in this section were designed to be similar to problems from past versions of Exam 3F / Exam MFE. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

**Problem ESQOEOVBSF1.**

**Similar to Question 22 from the Casualty Actuarial Society's [Fall 2007 Exam 3](#):**

Evasive Co. stock currently trades for $77 per share. The stock pays no dividends, and the annual continuously-compounded risk-free interest rate is 0.05. The stock price volatility is 0.44. A call option on Evasive Co. stock has a strike price of $74 and time to expiration of 2 years. Calculate Ω, the elasticity of this call option within the Black-Scholes framework.

**Solution ESQOEOVBSF1.**

In order to use the formula $\Omega = S\Delta/C$, we first need to find C and Δ.

To find C, we use the Black-Scholes formula, where

$d_1 = [\ln(S/K) + (r - \partial + 0.5\sigma^2)T]/[\sigma\sqrt{(T)}] = [\ln(77/74) + (0.05 - 0 + 0.5*0.44^2)2]/[0.44\sqrt{(2)}] =$

$d_1 = 0.5356981973$

$d_2 = d_1 - \sigma\sqrt{(T)} = 0.5356981973 - 0.44\sqrt{(2)} = d_2 = -0.0865557701$

In MS Excel, using the input "=NormSDist(0.5356981973)", we find that $N(d_1) = 0.70391645$

In MS Excel, using the input "=NormSDist(-0.0865557701)", we find that $N(d_2) = 0.465512247$

Now we use the Black-Scholes formula:

$C(S, K, \sigma, r, T, \partial) = Se^{-\partial T}N(d_1) - Ke^{-rT}N(d_2) = 77*0.70391645 - 74*e^{-2*0.05}0.465512247 =$

C = 23.03181208

Also, $\Delta_{call} = e^{-\partial T}N(d_1)$, which in this case is just $N(d_1) = 0.70391645$. Thus,

$\Omega = S\Delta/C = 77*0.70391645/23.03181208 =$ **$\Omega$ = 2.353334877**

**Problem ESQOEOVBSF2.**

**Similar to Question 29 from the Casualty Actuarial Society's [Fall 2007 Exam 3](Fall 2007 Exam 3):**

You know the following about the stock of Selective LLC and a certain call option on it. The stock price is $55, and the option's strike price is $65. The option expires in 1 year. The stock pays no dividends, and the annual continuously-compounded risk-free interest rate is 0.09. You expect an annual continuously-compounded return of 0.18 on the stock, and you estimate that stock volatility is 0.33. Find the Sharpe ratio of the call option.

**Solution ESQOEOVBSF2.**

The Sharpe ratio for a call option is the same as the Sharpe ratio for the underlying stock, i.e.,

$(\alpha - r)/\sigma = (0.18 - 0.09)/0.33 =$ **0.272727272727**

**Problem ESQOEOVBSF3.**

**Similar to Question 5 from the Society of Actuaries' May 2007 Exam MFE:**

The stock of Methodical, Inc., currently trades for $500 per share. The stock pays no dividends, and the annual continuously-compounded risk-free interest rate is 0.13. The stock price volatility is 0.39. A certain call option on Methodical, Inc., stock has a strike price of $433 and time to expiration of 1 year. Find the volatility of this call option within the Black-Scholes framework.

**Solution ESQOEOVBSF3.**

The volatility of an option can be expressed as

$\sigma_{option} = \sigma_{stock} * |\Omega|$. We need to find $\Omega$ before using this formula.

In order to use the formula $\Omega = S\Delta/C$, we first need to find C and $\Delta$.

To find C, we use the Black-Scholes formula, where

$d_1 = [\ln(S/K) + (r - \partial + 0.5\sigma^2)T]/[\sigma\sqrt{(T)}] = [\ln(500/433) + 1(0.13 - 0 + 0.5*0.39^2)]/ [0.39\sqrt{(1)}] =$

$d_1 = 0.897231719$ and $d_2 = d_1 - \sigma\sqrt{(T)} = 0.897231719 - 0.39 = d_2 = 0.507231719$

In MS Excel, using the input "=NormSDist(0.897231719)", we find that $N(d_1) = 0.815202393$. Since the stock pays no dividends, this is also Δ.

In MS Excel, using the input "=NormSDist(0.507231719)", we find that $N(d_2) = 0.694003889$.

Now we use the Black-Scholes formula:

$C(S, K, \sigma, r, T, \partial) = Se^{-\partial T}N(d_1) - Ke^{-rT}N(d_2) = 500*0.815202393 - 433e^{-0.13}0.694003889 = C = 143.7302847$ and $\Omega = S\Delta/C = 500*0.815202393/143.7302847 = \Omega = 2.835875524$

$\sigma_{option} = \sigma_{stock} * |\Omega| = 0.39*2.835875524 = \boldsymbol{\sigma_{option} = 1.105991455}$

**Problem ESQOEOVBSF4.**

**Similar to Question 15 from the Society of Actuaries' May 2007 Exam MFE:**

The stock of Insidious LLC currently trades for $10 per share. In 4 years, the stock will pay a dividend of $0.50. It pays no other dividends. The volatility relevant for the Black-Scholes formula is 0.37. The annual continuously-compounded risk-free interest rate is 0.07. A certain put option on Insidious LLC stock has a strike price of $10 and time to expiration of 8 years. Find the price of such an option using the Black-Scholes formula.

**Solution ESQOEOVBSF4.**

We can use the Black-Scholes formula with prepaid forwards:
$F^P_{0,T}(S) = 10 - 0.50e^{-0.07*4} = 9.622108129$

$F^P_{0,T}(K) = 10e^{-0.07*8} = 5.712090638$

$d_1 = [\ln(F^P_{0,T}(S)/F^P_{0,T}(K)) + 0.5\sigma^2T]/[\sigma\sqrt{(T)}] =$

$[\ln(9.622108129/5.712090638) + 0.5*0.37^2*8]/[0.37\sqrt{(8)}] = d_1 = 1.021557442$

$d_2 = d_1 - \sigma\sqrt{(T)} = 1.021557442 - 0.37\sqrt{(8)} = d_2 = -0.0249605942$

In MS Excel, using the input "=NormSDist(-1.021557442)", we find that $N(-d_1) = 0.153495219$.

In MS Excel, using the input "=NormSDist(0.0249605942)", we find that $N(-d_2) = 0.50995685$.

The Black-Scholes formula for the put price is

$P(F^P_{0,T}(S), F^P_{0,T}(K), \sigma, T) = F^P_{0,T}(K)N(-d_2) - F^P_{0,T}(S)N(-d_1) =$

$5.712090638*0.50995685 - 9.622108129*0.153495219 =$ **P = $1.435972154**

**Problem ESQOEOVBSF5.**

**Similar to Question 8 from the Society of Actuaries' Sample MFE Questions and Solutions:**

Treacherous Co. stock currently sells for $500 per share. The stock pays no dividends, and its price volatility is 0.34. A certain call option on Treacherous Co. stock has a strike price of $700 and time to expiration of 4 years. The call option has a delta of 0.5. Which of these expressions represents the price of this option?

(a) $251.3558251_{-\infty}{}^{0.68}\int(e^{-(x^\wedge 2)/2})dx$ - 380.0556182
(b) 250 - $251.3558251_{-\infty}{}^{0.68}\int(e^{-(x^\wedge 2)/2})dx$
(c) 250 - $630.0556182_{-\infty}{}^{0.68}\int(e^{-(x^\wedge 2)/2})dx$
(d) 250 - $380.0556182_{-\infty}{}^{0.68}\int(e^{-(x^\wedge 2)/2})dx$
(e) $630.0556182_{-\infty}{}^{0.68}\int(e^{-(x^\wedge 2)/2})dx$ - 380.0556182

**Solution ESQOEOVBSF5.**

This is indeed a treacherous problem. The annual continuously compounded risk-free interest rate (r) is unknown and needs to be found. Fortunately, it is possible to do so. The stock pays no dividends, so $\Delta_{call} = N(d_1)$, which is given to be 0.5. Since $N(d_1) = 0.5$, $d_1 = 0$.

$d_1 = [\ln(S/K) + (r - \partial + 0.5\sigma^2)T]/[\sigma\sqrt{(T)}] = 0$ in this case.

Thus, $[\ln(500/700) + 4r + 4*0.5*0.34^2]/[0.34\sqrt{(4)}] = 0$

and $\ln(500/700) + 4r + 4*0.5*0.34^2 = 0$

and 4r = 0.1052722366. Thus, r = 0.0263180592

$d_2 = d_1 - \sigma\sqrt{(T)} = 0 - 0.34\sqrt{(4)} = d_2 = -0.68$

Now the function N(t) can be represented as the following integral:

$[1/\sqrt{(2\pi)}]_{-\infty}{}^{t}\int(e^{-(x^\wedge 2)/2})dx$.

However, we note that the expression in the integral in all of our possible answers is *not* $N(d_2)$; rather, it is $N(-d_2) = 1 - N(d_2)$. However, we can express $N(d_2)$ as $1 - N(-d_2)$.

Now we are ready to use the Black-Scholes formula:
$C(S, K, \sigma, r, T, \partial) = Se^{-\partial T}N(d_1) - Ke^{-rT}N(d_2) = S\Delta_{call} - Ke^{-rT}[1 - N(-d_2)] =$

$500*0.5 - 700e^{-4*0.0263180592}[1 - [1/\sqrt{(2\pi)}]_{-\infty}{}^{0.68}\int(e^{-(x^\wedge 2)/2})dx] =$

$250 - 630.0556182 + 251.3558251_{-\infty}{}^{0.68}\int(e^{-(x^\wedge 2)/2})dx =$

$\mathbf{251.3558251_{-\infty}{}^{0.68}\int(e^{-(x^\wedge 2)/2})dx - 380.0556182}$. So **(a)** is the correct answer.

# Section 47

# The Delta-Gamma Approximation

A **market-maker** sells assets or contracts to buyers and buys them from sellers. He is an intermediary between the buyers and sellers. A market-maker's function is in contrast to **proprietary trading,** which is "trading to express an investment strategy" (McDonald, p. 414).

A **delta-hedged** position is a position designed to earn the risk-free rate of interest and is used to offset the risk of an option position.

The **delta-gamma approximation** is used to estimate option price movements if the underlying stock price changes.

The delta-gamma approximation for call options can be expressed via the following formula:

$C(S_{t+h}) = C(S_t) + \epsilon\Delta(S_t) + (1/2)\epsilon^2\Gamma(S_t)$

For a put option, the same formula holds, but delta is now negative - so the put price will decrease if the stock price increases.

**Meanings of variables:**
$S_t$ = stock price at time t.
$S_{t+h}$ = stock price at time t+h.
C = call option price.
$\epsilon$ = stock price change from time t to time t + h.
$\Delta$ = option delta.
$\Gamma$ = option gamma.

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 13, pp. 413-425.

**Problem DGA1.** The stock of Delta-Gamma Co. currently trades for $657 per share. A certain call option on the stock of Delta-Gamma Co. has a price of $120, a delta of 0.47, and a gamma of 0.01. Use a delta-gamma approximation to find the price of the call option if, after 1 second, the stock of Delta-Gamma Co. suddenly begins trading at $699 per share.

**Solution DGA1.** 1 second is an infinitesimally small unit of time, so the delta-gamma approximation will be reasonably accurate here. Here, $\epsilon$ = 699 - 657 = 42. Thus,
$C(S_{t+h}) = C(S_t) + \epsilon\Delta(S_t) + (1/2)\epsilon^2\Gamma(S_t) = 120 + 42*0.47 + (1/2)(42^2)*0.01 =$
**$C(S_{t+h})$ = $148.56**

**Problem DGA2.** The stock of Frivolous LLC currently trades for $1200 per share. A certain call option on the stock of Frivolous LLC has a price of $35, a delta of 0.72, and a certain value of gamma. When the stock price suddenly falls to \$1178, the call price falls to \$23. Using the delta-gamma approximation, what is the gamma of this call option?

**Solution DGA2.** We use the formula $C(S_{t+h}) = C(S_t) + \epsilon\Delta(S_t) + (1/2)\epsilon^2\Gamma(S_t)$, where $\Delta(S_t) = 0.72$, $\epsilon = -22$, $C(S_t) = \$35$, and $C(S_{t+h}) = \$23$. We rearrange the formula thus:
$C(S_{t+h}) - C(S_t) - \epsilon\Delta(S_t) = (1/2)\epsilon^2\Gamma(S_t)$ and $\Gamma(S_t) = [C(S_{t+h}) - C(S_t) - \epsilon\Delta(S_t)]/[(1/2)\epsilon^2]$. Thus,
$\Gamma(S_t) = [23 - 35 - (-22)(0.72)]/(1/2)(-22)^2] =$ $\mathbf{\Gamma(S_t) = 0.0158677686}$

**Problem DGA3.** The stock of Precarious LLC currently trades for \$13 per share. A certain call option on the stock of Frivolous LLC has a price of \$1.34, a gamma of 0.025, and a certain value of delta. When the stock price suddenly rises to \$19 per share, the call option price increases to \$5.67. Using the delta-gamma approximation, what is the original delta of this call option?

**Solution DGA3.** We use the formula $C(S_{t+h}) = C(S_t) + \epsilon\Delta(S_t) + (1/2)\epsilon^2\Gamma(S_t)$, where $\Gamma(S_t) = 0.025$, $\epsilon = 6$, $C(S_t) = \$1.34$, and $C(S_{t+h}) = \$5.67$.

We rearrange the formula thus:
$C(S_{t+h}) - C(S_t) - (1/2)\epsilon^2\Gamma(S_t) = \epsilon\Delta(S_t)$ and $\Delta(S_t) = [C(S_{t+h}) - C(S_t) - (1/2)\epsilon^2\Gamma(S_t)]/\epsilon$
Thus, $\Delta(S_t) = [5.67 - 1.34 - (1/2)6^2*0.025]/6 =$ $\mathbf{\Delta(S_t) = 0.646666666667}$

**Problem DGA4.** The stock of Imperious LLC suddenly began to trade for \$1440 per share. Previously, it traded for \$X per share. When the stock traded for \$X per share, a certain call option on the stock had a price of \$200. Now the call option has a price of \$250. The call option has a delta of 0.55 and a gamma of 0.003. Use the delta-gamma approximation to find X.

**Solution DGA4.** We use the formula $C(S_{t+h}) = C(S_t) + \epsilon\Delta(S_t) + (1/2)\epsilon^2\Gamma(S_t)$, where we desire to find $\epsilon$. We are given that $C(S_{t+h}) = 250$, $C(S_t) = 200$, $\Delta(S_t) = 0.55$, and $\Gamma(S_t) = 0.003$. Thus, $250 = 200 + 0.55\epsilon + 0.0015\epsilon^2$ and $0.0015\epsilon^2 + 0.55\epsilon - 50 = 0$. By the quadratic formula, the relevant (positive) $\epsilon = 75.4029116$, and so X = 1440 - 75.4029116 = **X = \$1364.597088**

**Problem DGA5.** When the stock of Odious Co. suddenly decreased in price by \$6 per share, a certain put option on the stock of Odious Co. increased in price to \$5.99. The put option had an original $\Delta$ of -0.49 and a gamma of 0.002. Find the original put option price using the delta-gamma approximation.

**Solution DGA5.** Here, $\epsilon = -6$, $\Delta = -0.49$, $\Gamma = 0.002$, and $P(S_{t+h}) = 5.99$
We use the formula $P(S_{t+h}) = P(S_t) + \epsilon\Delta(S_t) + (1/2)\epsilon^2\Gamma(S_t)$, which we can rearrange thus:
$P(S_t) = P(S_{t+h}) - \epsilon\Delta(S_t) - (1/2)\epsilon^2\Gamma(S_t) = 5.99 - (-6)*(-0.49) + (1/2)(-6^2)(0.002) =$
$\mathbf{P(S_t) = \$3.086}$

# Section 48

# The Delta-Gamma-Theta Approximation

When considerable periods of time pass between the moments at which the original price of an option occurs and the new option price occurs, time decay - whose effects are represented by the option Greek theta - must be taken into account. Hence, the delta-gamma approximation from Section 47 can be amplified into the delta-gamma-theta approximation for the new option price and the market-maker's profit on the option when the underlying stock price changes by $\epsilon$ and a time interval of duration h has passed.

$C(S_{t+h}, T - t - h) = C(S_t, T - t) + \epsilon\Delta(S_t, T - t) + (1/2)\epsilon^2\Gamma(S_t, T - t) + h\theta(S_t, T - t)$

When a market-maker has purchased $\Delta$ shares and short-sold the call, his profit is

$\text{Profit} = -(0.5\epsilon^2\Gamma_t + \theta_t h + rh[\Delta_t S_t - C(S_t)])$

We can make a substitution for $\epsilon^2$:

$\epsilon^2 = \sigma^2 S_t^{\,2} h$

Thus, $\text{Profit} = -(0.5\sigma^2 S_t^{\,2}\Gamma_t + \theta_t + r[\Delta_t S_t - C(S_t)])h$

**Meaning of variables:**

$S_t$ = stock price at time t.

C = call option price.

$\epsilon$ = stock price change from time t to time t + h.

$\Delta$ = option delta.

$\Gamma$ = option gamma.

$\theta$ = option theta

h = time interval under consideration.

r = annual continuously compounded risk-free interest rate

$\sigma$ = annual standard deviation of the stock price movement.

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 13, pp. 425-427.

**Problem DGTA1.** 22 days ago, the stock of Vindictive Co. traded for $511 per share. A certain call option on the stock had a delta of 0.66, a gamma of 0.001, and a daily theta of -0.03. The option used to trade for $59. Now the stock trades for $556. The annual continuously compounded risk-free interest rate is 0.08. Find the new option price using the delta-gamma-theta approximation.

**Solution DGTA1.** We use the equation

$C(S_{t+h}, T - t - h) = C(S_t, T - t) + \epsilon\Delta(S_t, T - t) + (1/2)\epsilon^2\Gamma(S_t, T - t) + h\theta(S_t, T - t)$.

Here, $\epsilon = 45$, h = 22 days, $\theta = -0.03$, $\Delta = 0.66$, $\Gamma = 0.001$, and $C(S_t, T - t) = 59$.

Thus, $C(S_{t+h}, T - t - h) = 59 + 45*0.66 + (1/2)(45)^2*0.001 + 22(-0.03) =$

$\mathbf{C(S_{t+h}, T - t - h) = \$89.0525}$

**Problem DGTA2.** 22 days ago, the stock of Vindictive Co. traded for $511 per share. A certain call option on the stock had a delta of 0.66, a gamma of 0.001, and a daily theta of -0.03. The option used to trade for $59. Now the stock trades for $556. The annual continuously compounded risk-free interest rate is 0.08. A hypothetical market maker is has purchased delta shares and short-sold the call. Find what a market-maker's profit on one such option would be using the delta-gamma-theta approximation.

**Solution DGTA2.** We use the equation Profit = $-(0.5\epsilon^2\Gamma_t + \theta_t h + rh[\Delta_t S_t - C(S_t)])$. We note that we can use h = 22 days when we multiply h by a *daily* theta, but we must use h = 22/365 years when multiplying h by an *annual* interest rate. Thus,

Profit = $-(0.5(45)^2 0.001 + (-0.03)22 + 0.08(22/365)[0.66*511 - 59]) =$ **-$1.694246849**

**Problem DGTA3.** 22 days ago, the stock of Vindictive Co. traded for $511 per share. A certain call option on the stock had a delta of 0.66, a gamma of 0.001, and a daily theta of -0.03. The option used to trade for $59. Now the stock trades for $556. The annual continuously compounded risk-free interest rate is 0.08. A hypothetical market maker is has purchased delta shares and short-sold the call. What is the annual standard deviation of the stock price movement?

**Solution DGTA3.** We use the equation $\epsilon^2 = \sigma^2 S_t^2 h$. Here, h = 22/365, $\epsilon = 45$, $S_t = 511$. We rearrange the equation thus: $\sigma^2 = (\epsilon^2)/(S_t^2 h)$ and $\sigma = \sqrt{[(\epsilon^2)/(S_t^2 h)]} = \sqrt{[(45^2)/(511^2 22/365)]}$ = $\boldsymbol{\sigma} =$ **0.3586961419**

**Problem DGTA4.** Active Co. stock has a price volatility of 0.55. A certain call option on Active Co. stock today costs $71.80. The option has a delta of 0.32, a gamma of 0.001, and a daily theta of -0.06. The stock price today is $3000 per share, and the annual continuously compounded risk-free interest rate is 0.1. Find what a market-maker's profit on one such option would be after 1 year using the delta-gamma-theta approximation.

**Solution DGTA4.** Here, h = 1, $\sigma = 0.55$, $S_t = 3000$, $C(S_t) = 71.80$, $\Delta_t = 0.32$, $\Gamma_t = 0.01$, $\theta_t = -0.06$. Thus, Profit = $-(0.5\sigma^2 S_t^2 \Gamma_t + \theta_t + r[\Delta_t S_t - C(S_t)])h$ =

$-(0.5*1*0.55^2*3000^2*0.001 + -0.06*365 + 0.1*1[0.32*3000 - 71.80])$ =

**Profit =-$1428.17**

**Problem DGTA5.** The stock of Reactive Co. currently trades for $678 per share. 98 days ago, it traded for $450 per share. Imhotep owns a call option on Reactive Co. stock that cost him $56 when he bought it 98 days ago. Now the call option trades for $100. The option has a delta of 0.33 and a gamma of 0.006. What is the daily option theta? Use the delta-gamma-theta approximation.

**Solution DGTA5.** Here, $\epsilon = 678 - 450 = 228$ and h = 98 days

We use the equation

$C(S_{t+h}, T - t - h) = C(S_t, T - t) + \epsilon\Delta(S_t, T - t) + (1/2)\epsilon^2\Gamma(S_t, T - t) + h\theta(S_t, T - t)$, which we rearrange thus:
$[C(S_{t+h}, T - t - h) - C(S_t, T - t) - \epsilon\Delta(S_t, T - t) - (1/2)\epsilon^2\Gamma(S_t, T - t)]/h = \theta(S_t, T - t)$ =

$(100 - 56 - 228*0.33 - (1/2)(228)^2 0.006)/98$ = $\boldsymbol{\theta(S_t, T - t) = -1.910122449}$

# Section 49

# The Black-Scholes Partial Differential Equation

The Black-Scholes partial differential equation or Black-Scholes equation (as opposed to the Black-Scholes *formula*) is as follows:
$rC(S_t) = (1/2)\sigma^2 S_t^2 \Gamma_t + rS_t\Delta_t + \theta$

This equation holds for American and European call s and puts, but not at times when it is optimal to exercise the options early.

The Black-Scholes equation has the following assumptions:

1. The underlying asset does not pay any dividends.
2. The option does not pay any dividends.
3. The interest rate and volatility are constant.
4. The stock moves one standard deviation over a small time interval.

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 13, pp. 429-430.

**Meaning of variables:**

$S_t$ = stock price at time t.
C = call option price.
$\Delta$ = option delta.
$\Gamma$ = option gamma.
$\theta$ = option theta
r = annual continuously compounded risk-free interest rate
$\sigma$ = annual standard deviation of the stock price movement.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem BSPDF1.** The stock of Company Ξ has a current price of $56 per share, and the standard deviation of its price is 0.34. A certain call option on this stock has a delta of 0.42, a gamma of 0.001, and a theta of -0.05. The annual continuously compounded risk-free interest rate is 0.06. What is the price of this call option, as found using the Black-Scholes equation?

**Solution BSPDF1.** We use the formula $rC(S_t) = (1/2)\sigma^2 S_t^2 \Gamma_t + rS_t\Delta_t + \theta$, which we rearrange thus: $C(S_t) = [(1/2)\sigma^2 S_t^2 \Gamma_t + rS_t\Delta_t + \theta]/r = [(1/2)0.34^2 56^2 0.001 + 0.06*56*0.42 - 0.05]/0.06 =$ C(56) = **$25.70768**

**Problem BSPDF2.** Assume that the Black-Scholes framework holds. The stock of Vicious Co. has a price of $93 per share, and the price has a standard deviation of 0.53. A certain call option on Vicious Co. stock has a price of $4, a delta of 0.53, and a gamma of 0.01. The annual continuously compounded risk-free interest rate is 0.02. What is the theta for this call option?

**Solution BSPDF2.** We use the formula $rC(S_t) = (1/2)\sigma^2 S_t^2 \Gamma_t + rS_t\Delta_t + \theta$, which we rearrange thus: $\theta = rC(S_t) - (1/2)\sigma^2 S_t^2 \Gamma_t - rS_t\Delta_t$. Thus, $\theta = 0.02*4 - (1/2)0.53^2 93^2 0.01 - 0.02*93*0.53 =$ $\mathbf{\theta = -13.0533205}$

**Problem BSPDF3.** Assume that the Black-Scholes framework holds. The price of the stock of Devious Co. has a standard deviation of 0.74. A certain call option on this stock has a price of $25, a delta of 0.29, a gamma of 0.02, and a theta of -0.04. The annual continuously compounded risk-free interest rate is 0.08. What is the current price of one share of Devious Co. stock?

**Solution BSPDF3.** We use the formula $rC(S_t) = (1/2)\sigma^2 S_t^2 \Gamma_t + rS_t\Delta_t + \theta$, which we rearrange thus: $0 = (1/2)\sigma^2 S_t^2 \Gamma_t + rS_t\Delta_t - (rC(S_t) - \theta)$. We find that

$rC(S_t) - \theta = 0.08*25 + 0.04 = 2.04$, $(1/2)\sigma^2\Gamma_t = (1/2)0.74^2 0.02 = 0.005476$, and $r\Delta_t = 0.08*0.29 = 0.0232$. We thus have the following equation:

$0.005476S_t^2 + 0.0232S_t - 2.04 = 0$. By the quadratic formula, the relevant positive value of

$S_t$ is $\mathbf{S_t = \$17.29872709}$

**Problem BSPDF4.** Assume that the Black-Scholes framework holds. The price of the stock of Imperious LLC is $516 per share and has a standard deviation of 0.22. A certain call option on this stock has a price of $67, a delta of 0.03, and a theta of -0.05. The annual continuously compounded risk-free interest rate is 0.1. What is the gamma of this option?

**Solution BSPDF4.** We use the formula $rC(S_t) = (1/2)\sigma^2 S_t^2 \Gamma_t + rS_t\Delta_t + \theta$, which we rearrange thus: $rC(S_t) - \theta - rS_t\Delta_t = (1/2)\sigma^2 S_t^2 \Gamma_t$, so $\Gamma_t = [(rC(S_t) - \theta - rS_t\Delta_t)/((1/2)\sigma^2 S_t^2)] =$

$[(0.1*67 + 0.05 - 0.1*516*0.03)/((1/2)0.22^2 516^2)] = 5.202/6443.3952 =$ $\mathbf{\Gamma_t = 0.0008073383424}$

**Problem BSPDF5.** The stock of Obsequious Co. has a price of $40, and this price has a standard deviation of 0.43. A certain call option in this stock has a price of $5, a gamma of 0.0004, and a theta of -0.02. The annual continuously compounded risk-free interest rate is 0.056. Find the delta of this call option.

**Solution BSPDF5.** We use the formula $rC(S_t) = (1/2)\sigma^2 S_t^2 \Gamma_t + rS_t\Delta_t + \theta$, which we rearrange thus: $(rC(S_t) - \theta - (1/2)\sigma^2 S_t^2 \Gamma_t)/rS_t = \Delta_t$

Thus, $(rC(S_t) - \theta - (1/2)\sigma^2 S_t^2 \Gamma_t)/(rS_t) = (0.056*5 + 0.02 - (1/2)0.43^2 40^2 0.0004)/(0.056*40) =$ $\mathbf{\Delta_t = 0.1075142857}$.

# Section 50

# The Return and Variance of the Return to a Delta-Hedged Market-Maker

The period i return to a delta-hedged market-maker who has *purchased* a call - $R_{h,i}$ - can be written as $R_{h,i} = (1/2)S^2\sigma^2\Gamma(x_i^2-1)h$

For a delta-hedged market-maker who has *written* a call, the period i return is the negative of the previous expression: $R_{h,i} = -(1/2)S^2\sigma^2\Gamma(x_i^2-1)h$

The variance of this return is

$Var(R_{h,i}) = (1/2)(S^2\sigma^2\Gamma h)^2$

It is assumed that $x_i$ is uncorrelated across time.

**Meanings of variables:**
$R_{h,i}$ = The period i return to a delta-hedged market-maker who has written a call.
S = stock price.
σ = standard deviation of the stock price movement.
Γ = option gamma.
h = time interval between hedge readjustments.
$x_i$ = the number of standard deviations the stock price moves during period i.

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 13, pp. 431.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem RVRDHMM1.** Imhotep is a delta-hedged market-maker who readjusts his hedges every 5 months. He hedges using the stock of Tranquil Co., which has a price of $500 per share; the standard deviation of this price is 0.03. A certain call option on Tranquil Co. stock has a gamma of 0.001. Imhotep has a *long* position in this option. During a particular 5-month period, the stock price moves by 0.23 standard deviations. Find the return to Imhotep during this time period.

**SolutionRVRDHMM1.** Since Imhotep has a long position, we use the formula
$R_{h,i} = (1/2)S^2\sigma^2\Gamma(x_i^2-1)h = (1/2)*500^2*0.03^2*0.001(0.23^2-1)(5/12) =$ $\mathbf{R_{h,i} = -0.0443953125}$

**Problem RVRDHMM2.** Imhotep is a delta-hedged market-maker who readjusts his hedges every 5 months. He hedges using the stock of Tranquil Co., which has a price of $500 per share; the standard deviation of this price is 0.03. A certain call option on Tranquil Co. stock has a gamma of 0.001. Imhotep has a *long* position in this option. During a particular 5-month period, the stock price moves by 0.23 standard deviations. Find the variance of the return to Imhotep during this time period.

**Solution RVRDHMM2.** We use the formula $\mathrm{Var}(R_{h,i}) = (1/2)(S^2\sigma^2\Gamma h)^2 = (1/2)(500^2 0.03^2 0.001*(5/12))^2 =$ **$\mathrm{Var}(R_{h,i}) = 0.0043945313$**

**Problem RVRDHMM3.** Cuauhtemoc is a delta-hedged market-maker who has a short position in a call option on the stock of Volatile Co. Cuauhtemoc readjusts his hedges every 2 months. The stock has a price of $45; the standard deviation of this price is 0.33. The gamma of the call option is 0.02. During a particular 2-month period, the stock price moves by 0.77 standard deviations. Find the return to Cuauhtemoc during this time period.

**Solution RVRDHMM3.** Since Cuauhtemoc has a short position, we use the formula

$R_{h,i} = -(1/2)S^2\sigma^2\Gamma(x_i^2-1)h = -(1/2)45^2 0.33^2 0.02(0.77^2-1)(2/12) =$ **$R_{h,i} = 0.1496245163$**

**Problem RVRDHMM4.** Cuauhtemoc is a delta-hedged market-maker who has a short position in a call option on the stock of Volatile Co. Cuauhtemoc readjusts his hedges every 2 months. The stock has a price of $45; the standard deviation of this price is 0.33. The gamma of the call option is 0.02. During a particular 2-month period, the stock price moves by 0.77 standard deviations. Find the variance of the return to Cuauhtemoc during this time period.

**Solution RVRDHMM4.** We use the formula $\mathrm{Var}(R_{h,i}) = (1/2)(S^2\sigma^2\Gamma h)^2 = (1/2)(45^2 0.33^2 0.02*2/12)^2 = \mathrm{Var}(R_{h,i}) = 0.2701676278$

**Problem RVRDHMM5.** Gilgamesh - a delta-hedged market-maker - has a short position in a call option on the stock of Mesopotamian Industries. The stock price has a standard deviation of 0.39. The call option has a gamma of 0.007. Gilgamesh readjusts his hedges daily. During a particular day, the stock price of Mesopotamian Industries changed by 0.03 standard deviations. Gilgamesh obtained a return of $1.23 as a result. What was the original stock price of Mesopotamian Industries on that day? Assume that there are 365 days in a year.

**Solution RVRDHMM5.** Since Gilgamesh has a short position, we use the formula $R_{h,i} = -(1/2)S^2\sigma^2\Gamma(x_i^2-1)h$, where we want to find S. Here, h = 1/365.

We rearrange the formula thus:
$-R_{h,i} = (1/2)S^2\sigma^2\Gamma(x_i^2-1)h$
$-2R_{h,i} = S^2\sigma^2\Gamma(x_i^2-1)h$
$[-2R_{h,i}/\sigma^2\Gamma(x_i^2-1)h] = S^2$
$S = \sqrt{[-2R_{h,i}/(\sigma^2\Gamma(x_i^2-1)h)]}$
Here, $R_{h,i} = 1.23$, $\sigma = 0.39$, $\Gamma = 0.007$, $x = 0.03$, and $h = 1/365$.
$\sqrt{[-2R_{h,i}/(\sigma^2\Gamma(x_i^2-1)h)]} = S = \sqrt{[-2*1.23/(0.39^2 0.007(0.03^2-1)(1/365))]} = \sqrt{[844095.8373]} =$
**S = $918.7468842**

# Section 51

# Exam-Style Questions on Market-Making and Delta-Hedging

**Note 51.1:** Before we solve the exam-style questions on market-making and delta-hedging, one additional formula is in order. Within the Black-Scholes framework, if a delta-hedged market-maker makes exactly zero profit during a specific time period, it can be assumed that a stock price moved by one standard deviation during that period. That is, the magnitude of the move will be $\sigma S_t\sqrt{(h)}$, where t is the time at which the original stock price existed, $S_t$ is the stock price, h is the time period during which the stock price moves, and σ is the annual standard deviation of the stock price movement.

**Note 51.2:** Furthermore, it is possible to use MS Excel to find x when given N(x). Using the function "=NormSInv(N(x))", where you can substitute in the relevant value for N(x), will accomplish this aim. On the exam, you will need to use a chart of values to get the same results, but Excel is much more convenient to use when one is attempting to learn the problem-solving procedures as efficiently as possible.

The problems in this section were designed to be similar to problems from past versions of Exam 3F / Exam MFE. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

**Problem ESQMMDH1.**

**Similar to Question 33 from the Casualty Actuarial Society's Spring 2007 Exam 3:**

Which of these statements about the role of option Greeks in market-making are correct?

(a) Hedging does not require the purchase of stock.
(b) If the gamma of a call is positive, then by writing the call, a market-maker will lose money in proportion to the square of the stock price change.
(c) If the gamma of a call is positive, then by purchasing the call, a market-maker will lose money in proportion to the square of the stock price change.
(d) If theta for a call is negative, the option buyer benefits from theta.
(e) If theta for an option is negative and gamma is positive, it is possible to benefit from theta and gamma simultaneously.

**Solution ESQMMDH1.** We examine these statements by keeping in mind the delta-gamma-theta approximation:
When a market-maker has purchased Δ shares and short-sold the call, his profit is

Profit = $-(0.5\epsilon^2\Gamma_t + \theta_t h + rh[\Delta_t S_t - C(S_t)])$

(a) is clearly untrue, because the market-maker needs to purchase Δ shares of stock for every call option he writes in order to offset the negative delta of the call option. We also note that the difference between $\Delta_t S_t$ and $C(S_t)$ implies that there exists a net cost to the market-maker from purchasing the stock. This is the interest cost of the position.

(b) is correct. If the stock changes by ϵ, the market-maker will lose $-0.5\epsilon^2\Gamma_t$ due to the effects of gamma.

For (c): For a market-maker who has purchased a call (and sold Δ shares), the sign of the profit equation will be reversed: Profit = $(0.5\epsilon^2\Gamma_t + \theta_t h + rh[\Delta_t S_t - C(S_t)])$. Thus, the market-maker will *gain* money in proportion to the square of the stock price change, and (c) is incorrect.
For (d): If theta for a call is negative, the option *writer* benefits from theta, and the buyer loses. We see that in the equation Profit = $-(0.5\epsilon^2\Gamma_t + \theta_t h + rh[\Delta_t S_t - C(S_t)])$ for the option writer, a negative theta implies that $-\theta_t h$ is positive. So (d) is incorrect.

For (e): We consider the equation Profit = $-(0.5\epsilon^2\Gamma_t + \theta_t h + rh[\Delta_t S_t - C(S_t)])$. If theta is negative, then $-\theta_t h$ is positive. If gamma is positive, then $-0.5\epsilon^2\Gamma_t$ is negative. Thus, if theta is negative and gamma is positive, the two Greeks will have the opposite effects on the market-maker profit, and it is impossible for the market-maker to benefit from gamma and theta simultaneously. So (e) is incorrect. Thus, **only (b) is correct.**

**Problem ESQMMDH2.**

**Similar to Question 24 from the Casualty Actuarial Society's Fall 2007 Exam 3:**

N50 is a market-marker who has 564 \$200-strike put options on the stock of Invulnerable Co. in his portfolio. He also has 17 shares of the stock in his portfolio. He can write \$177-strike put options on the stock. The delta and gamma values for the put options are as follows:
\$177-strike put option: $\Delta = -0.7$, $\Gamma = 0.33$

\$200-strike put option: $\Delta = -0.2$, $\Gamma = 0.54$

What would N50 need to do to delta- and gamma-neutralize his portfolio?

**Solution ESQMMDH2.**

First, we consider the current delta and gamma values of N50's portfolio:

From 17 shares: $\Delta = 17$, $\Gamma = 0$

From 564 \$200-strike put options: $\Delta = 564*-0.2 = -112.8$, $\Gamma = 0.54*564 = 304.56$

Net $\Delta = 17 - 112.8 = -95.8$

We first try to gamma-neutralize the portfolio using the \$177-strike puts; then we can delta-neutralize the remaining net delta with shares.

One \$177-strike put option has $\Gamma = 0.33$, so a gamma of -304.56 is possessed by

-304.56/0.33 = -922.90909090909 \$177-strike put options. So to gamma-neutralize the portfolio, N50 would need to write 922.90909090909 \$177-strike put options.

That changes net delta by -(-0.7*922.90909090909) = 646.03636363636 - so that the new net delta is -95.8 + 646.03636363636 = 550.236363636. To delta-neutralize this portfolio, N50 will need to sell 550.236363636 shares of stock (each with $\Delta = 1$, as is always the case for shares of stock). Thus, to delta- and gamma-neutralize his portfolio, N50 will need to **write 922.90909090909 \$177-strike put options and sell 550.236363636 shares of stock.**

**Problem ESQMMDH3.**

**Similar to Question 10 from the Society of Actuaries' May 2007 Exam MFE:**

There are two possible call options on Stock A: Call Ψ and Call Φ.

The calls have the following Greeks:
Φ: $\Delta = 0.66$, $\Gamma = 0.04$, vega = 0.05

Ψ: $\Delta = 0.32$, $\Gamma = 0.10$, vega = 0.006

Aethelred just sold 542 units of Call Φ. How many units of Stock A and Call Ψ would he need to buy or sell in order to both delta-hedge and gamma-hedge his position?

**Solution ESQMMDH3.**

We first note that vega values are completely irrelevant to this problem, so we only need to consider delta and gamma values. Selling 542 units of Call Φ means that Aethelred has a position delta of -0.66*542 = -357.2 and a position gamma of -0.04*542 = -21.68.

We first try to gamma-neutralize this portfolio. Each Call Ψ has a gamma of 0.10, so we need 21.68/0.1 = 216.8 Call Ψ options to obtain a gamma of 21.68. By buying 216.8 Call Ψ options, Aethelred will thus gamma-neutralize the portfolio. Doing so will increase the portfolio's net delta to -357.2 + 216.8*0.32 = -287.824, which leaves 287.824 in delta to be accounted for. This increase in delta can be obtained by purchasing 287.824 shares of Stock A.

Thus, Aethelred would need to **buy 216.8 Call Ψ options and buy 287.824 shares of Stock A**.

**Problem ESQMMDH4.**

**Similar to Question 19 from the Society of Actuaries' May 2007 Exam MFE:**

Assume that the Black-Scholes framework holds. The price of Blackscholesian Co. stock, which pays no dividends, is \$566. A certain put option on this stock trades for \$25. The option has a delta of -0.66 and a gamma of 0.04. Suddenly, the price of Blackscholesian Co. stock increases to \$588 per share. Find the new put option price using the delta-gamma approximation.

**Solution ESQMMDH4.** We use the delta-gamma approximation, noting that C here stands for the *put* option price:

$C(S_{t+h}) = C(S_t) + \epsilon\Delta(S_t) + (1/2)\epsilon^2\Gamma(S_t)$, where $C(S_t) = 25$, $\epsilon = 22$, $\Delta(S_t) = -0.66$, and $\Gamma(S_t) = 0.04$

Thus, $C(S_{t+h}) = 25 + 22*-0.66 + (1/2)22^2 0.04 =$ $\mathbf{C(S_{t+h}) = \$20.16}$

**Problem ESQMMDH5.**

**Similar to Question 10 from the Society of Actuaries' Sample MFE Questions and Solutions:**

Assume the Black-Scholes framework. Xenophon, a market-maker who delta-hedges his position, has sold a four-year at-the-money European call option on Stock Q, which pays no dividends. Stock Q currently trades for \$99 per share, and the annual continuously-compounded risk-free interest rate is 0.02. The delta of the call option is 0.725746935. There are 365 days in a year. After seven particular days, Xenophon has no profits or losses on his position. It is known that the annual standard deviation of the stock price movement is greater than 0.33. Determine the stock price movement during the course of these seven days.

**Solution ESQMMDH5.** As stated in Note 51.1, we can assume that the stock period moved by one one-week standard deviation during this period. So the stock price movement is $\sigma S_t\sqrt{(h)}$, where $h = 7/365$ and $S_t = 99$. All that remains to do is to find the value of σ.

Since the stock pays no dividends, $\Delta = N(d_1) = 0.725746935$.

As per Note 51.2, we use the input "=NormSInv(0.725746935)" in MS Excel to get $d_1 = 0.6$.

Thus, $0.6 = [(r + 0.5\sigma^2)T]/[\sigma\sqrt{(T)}]$, since the stock pays no dividends and S = K, so ln(S/K) = 0.

Thus, $0.6 = [(0.02 + 0.5\sigma^2)4]/[\sigma\sqrt{(4)}]$ and $0.6 = 0.04/\sigma + \sigma$, and $0.6\sigma = 0.04 + \sigma^2$, so

$\sigma^2 - 0.6\sigma + 0.04 = 0$ and, by the quadratic formula, $\sigma = 0.0763932023$ or $\sigma = 0.5236067978$. But we are given that $\sigma > 0.33$, so $\sigma = 0.5236067978$ and the stock price movement is $\sigma S_t\sqrt{(h)} =$ $0.5236067978*99\sqrt{(7/365)}$ = **\$7.178654513**

# Section 52

# Asian Options

An **Asian option** "has a payoff that is based on the average price over some period of time" (McDonald 2006, p. 444). An Asian option is thus, **path-dependent** in that the way in which the underlying stock reached its final price matters for determining the price of the option. An **Asian tail** refers to settling on the basis of the average price.

Asian options can be based on an arithmetic or a geometric average.

The formula for the arithmetic average of N stock prices is $A(T) = (1/N)_{i=1}{}^{N}\Sigma S_{ih}$

The formula for the geometric average of N stock prices is $G(T) = (S_h * S_{2h} * .... * S_{Nh})^{(1/N)}$

It is always the case that $G(T) \leq A(T)$

An *Asian average price option* occurswhen the average used is the asset price.

An *Asian average strike option* occurswhen the average used is the strike price. There are four different possible types of Asian options using arithmetic averages and four different types of Asian options using geometric averages. The following formulas hold for the payoffs of the geometric average Asian options. To find the payoffs of arithmetic average Asian options, simply replace G(T) with A(T) in the relevant equation.

Geometric average price call = max[0, G(T) - K]

Geometric average price put = max[0, K - G(T)]

Geometric average strike call = max[0, $S_T$ - G(T)]

Geometric average strike put = max[0, G(T) - $S_T$]

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 14, pp. 444-449.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem AO1.** At the end of each of the past four months, the stock of Fluctuating Co. had the following prices: $345, $435, $534, and $354. A certain 4-month Asian geometric average price call on this stock has a strike price of $400. It expires today, and its payoff is computed on the basis of the geometric average of the stock prices given above. What is the payoff of this option?

**Solution AO1.** We first find $G(T) = (345*435*534*354)^{(1/4)}$ = 410.405543. The payoff of the option is thus Geometric average price call = max[0, G(T) - K] = 410.405543 - 400 = **$10.405543**

**Problem AO2.** At the end of each of the past four months, the stock of Fluctuating Co. had the following prices: $345, $435, $534, and $354. A certain 4-month Asian arithmetic average price call on this stock has a strike price of $400. It expires today, and its payoff is computed on the basis of the arithmetic average of the stock prices given above. What is the payoff of this option?

**Solution AO2.** We first find A(T) = (345 + 435 + 534 + 354)/4= 417. The payoff of the option is thus Arithmetic average price call = max[0, A(T) - K] = 417 - 400 = **$17**

**Problem AO3.** A highly peculiar Asian geometric average strike put on the stock of Lucrative Co. has a life of 6 months and is computed based on an average of monthly strikes. The option's payoff is based on the following strike prices: $4 in month 1, $1500 in month 2, $5 in month 3, $1400 in month 4, $23.4 in month 5, $1322 in month 6. At the end of month 6, the stock of Lucrative Co. trades at $44 per share. What is the payoff of this option?

**Solution AO3.** We first find $G(T) = (4*1500*5*1400*23.4*1322)^{(1/6)}$ = 104.4598586.

The payoff of the option is thus Geometric average strike put = max[0, G(T) - $S_T$] = 104.4598586 - 44 = **$60.45985865**

**Problem AO4.** The stock price of Predictable Co. is $1 at the end of month 1 and increases by $1 every month without exception. A 99-month Asian arithmetic average price put on Predictable Co. stock has a strike price of $56 and a payoff that is computed based on an average of monthly prices. The option expires at the end of month 99. What is the payoff of this option?

**Solution AO4.** We recall that 1 + 2 + 3 +... + n = n(n+1)/2.

To find A(T), we need to find (1 + 2 + 3 + ... + 99)/99 = (99*100/2)/99 = 50.

To find the payoff, we use the formula Arithmetic average price put = max[0, K - A(T)] = 56 - 50 = **$6**

**Problem AO5.** The stock price of Predictable Co. is $1 at the end of month 1 and increases by $1 every month without exception. A 10-month Asian geometric average price call on Predictable Co. stock has a strike price of $2 and a payoff that is computed based on an average of monthly prices. The option expires at the end of month 10. What is the payoff of this option?

**Solution AO5.** Here, $G(T) = (1*2*3*...*10)^{(1/10)} = (10!)^{(1/10)}$ = 4.528728688

We find the payoff by using the formula Geometric average price call = max[0, G(T) - K] = 4.528728688 - 2 = **$2.528728688**

# Section 53

# Barrier Options

The payoff for a **barrier option** depends on whether the underlying asset price ever reaches a specified level - or barrier - during the life of the option. Barrier puts or calls are always no more expensive than otherwise equivalent regular puts or calls, because the payoff on the former can never be larger than the payoff on the latter.

Three types of barrier options are

**Knock-out options:** Options go out of existence when the asset price reaches the barrier. These options are called **down-and-out** when the price has to decline to reach the barrier. They are called **up-and-out** when the price has to increase to reach the barrier.

**Knock-in options:** Options come into existence when asset price reaches the barrier. These options are called **down-and-in** when the price has to decline to reach the barrier. They are called **up-and-in** when the price has to increase to reach the barrier.

**Rebate options:** Options make a fixed payment if the asset price reaches the barrier. The payment can be made at the time the barrier is reached or at the time the option expires. If the latter is true, then the option is deferred rebate. **Up rebate** options occur when the barrier is above the current price. **Down rebate** options occur when the barrier is below the current price.

For barrier options, the following parity relationship holds:

Knock-in option + Knock-out option = Ordinary option

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 14, pp. 449-451.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem BO1.** The stock of Tradable Co. once traded for \$100 per share. Several barrier option contracts were then written on the stock. Suddenly, the stock price increased to \$130 per share - which is the barrier for the options. Which of these barrier options would have a positive payoff in this case? More than one answer may be possible.

(a) Up-and-out put with a strike price of \$120
(b) Up-and-out call with a strike price of \$120
(c) Up-and-in put with a strike price of \$120
(d) Up-and-in call with a strike price of \$120
(e) Rebate option that pays a rebate of \$12

**Solution BO1.** Since the barrier has been reached, the properties of all these options have been triggered. The up-and-out options go *out* of existence when the barrier is reached, so (a) and (b) are now worthless. The up-and-in put in (c) went into existence when the barrier was reached, but the stock price is above its strike price, so it is also worthless. The up-and-in call in (d), however, now has a payoff of 130-120 = $10. The rebate option in (e) has also been triggered and will now pay $12 to the holder. Thus, **(d) and (e)** will have positive payoffs.

**Problem BO2.** An ordinary call option on Lucrative Co. stock with a strike price of $50 and time to expiration of 1 year trades for $4. An otherwise identical up-and-in call option on Lucrative Co. stock with a barrier of $55 trades for $2.77. Find the price of an up-and-out call option on Lucrative Co. stock with a barrier of $55, a strike price of $50, and time to expiration of 1 year.

**Solution BO2.** We use the formula

Knock-in option + Knock-out option = Ordinary option

We are given that

Ordinary option = 4 and Knock-in option = 2.77. Thus,

Knock-out option = 4 - 2.77 = **$1.23**

**Problem BO3.** You own a portfolio of the following options on the stock of Imperious LLC:

- An up-and-in call with strike = $63 and barrier = $66

- An up-and-out put with strike = $78 and barrier = $65

- An up rebate option with rebate = $14 and barrier = $61

- An up-and-out call with strike = $35 and barrier = $61

Originally, the stock traded at $59 per share. Right before the options expired, the stock began to trade at $63 per share, its record high. What is your total payoff on this portfolio?

**Solution BO3.** The barrier for the up-and-in call with strike = $63 and barrier = $66 was not hit, so this option is worthless. The barrier for the up-and-out put with strike = $78 and barrier = $65 was not hit, so this option is still in existence and has the payoff of a regular put: 78 - 63 = 15. The barrier for the up rebate option was reached, so this option pays its rebate of $14. The barrier for the up-and-out call with strike = $35 and barrier = $61 was reached, so this option is worthless. Thus, the total payoff on this portfolio is

15 + 14 = **$29.**

**Problem BO4.** Hildebrand owns a portfolio of the following options on the stock of Imperious LLC:

- An up-and-in call with strike = \$63 and barrier = \$66

- An up-and-out put with strike = \$78 and barrier = \$65

- An up rebate option with rebate = \$14 and barrier = \$61

- An up-and-out call with strike = \$35 and barrier = \$61

Originally, the stock traded at \$59 per share. Right before the options expired, the stock began to trade at \$67 per share, its record high. What is Hildebrand's total payoff on this portfolio?

**Solution BO4.** The barrier for the up-and-in call with strike = \$63 and barrier = \$66 was reached, so this call now has a payoff of 67 - 63 = 4. The barrier for the up-and-out put with strike = \$78 and barrier = \$65 has been reached, so this option is now worthless. The barrier for the up rebate option with rebate = \$14 and barrier = \$61 has been reached, so the option pays its rebate of \$14. The barrier for the up-and-out call with strike = \$35 and barrier = \$61 has been reached, so the option goes out of existence and is worthless. Thus, Hildebrand's total payoff is 4 + 14 = **\$18**

**Problem BO5.** The stock of Tractable Co. pays no dividends. The stock currently trades at \$54 per share. An up-and-in call with strike = \$55 and barrier = \$60 has a price of \$3.04, and an up-and-out call with strike = \$55 and barrier = \$60 has a price of \$1.32. The options expire in 2 months, and the annual continuously compounded interest rate is 0.03. Find the price of one ordinary put option on the stock of Tractable Co. with strike price of \$55 and time to expiration of 2 months.

**Solution BO5.** First we use the formula

Knock-in option + Knock-out option = Ordinary option, from which we find that the price of the ordinary call for this stock is 3.04 + 1.32 = 4.36

Now we use put-call parity: $C - P = S - Ke^{-rT}$, where r = 0.03, T = 1/6, K = 55, C = 4.36, and S = 54. Thus, $P = C - S + Ke^{-rT} = 4.36 - 54 + 55e^{-0.03/6}$ = **P = \$5.085686356**

# Section 54

# Compound Options

An option on an option is called a **compound option.** A compound option has two strikes and two expirations associated with it - one each for the underlying option and for the compound option itself.

At a compound call option's expiration, the payoff of the compound call option on a call option (a CallOnCall option) is

$\max[C(S_{t\_1}, K, T - t_1) - x , 0]$

**Meaning of Variables:**
$C(S_{t\_1}, K, T - t_1)$ = underlying call option price.

x = compound option's strike price.

T = time to expiration of underlying call option.

$t_1$ **=** time to expiration of compound call option. Note: $t_1 < T$

K = strike price of underlying call option.

$S_{t\_1}$ = underlying stock price at time $t_1$.

Now we let S* be the *critical stock price* above which the compound option gets exercised. Then it follows that $C(S^*, K, T - t_1) = x$, and the option gets exercised for all prices $S_{t\_1} > S^*$.

Two conditions must exist for a compound CallOnCall option to be valuable ultimately:

1. $S_{t\_1} > S^*$ at time $t_1$.

2. $S_T > K$ at time T.

There four types of compound options: CallOnCall, CallOnPut, PutOnCall, and PutOnPut. The naming convention for these options is XOnY, where Y is the underlying option type and X is the type of the option on the underlying option.

**Compound Option Parity**

The following relationship holds among prices of CallOnCall, PutOnCall, and ordinary call options (as priced by the Black-Scholes formula):

CallOnCall(S, K, x, σ, r, $t_1$, $t_2$, δ) - PutOnCall(S, K, x, σ, r, $t_1$, $t_2$, δ) + $xe^{-rt\_1}$ =

BSCall(S, K, σ, r, $t_2$, δ)

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 14, pp. 453-454.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem CO1.** The stock of Vicious Co. currently trades for $53 per share. You (but only you) have perfect knowledge that the stock will trade for $66 per share in 6 months. (The rest of the market does not have this knowledge.) A certain call option on the stock of Vicious Co. has a strike price of $58 at time to expiration of 6 months. You have perfect knowledge that this option will trade in 3 months at one-fifth of what you know its ultimate payoff will be. A CallOnCall option on this call option has a strike price of $1.33 and time to expiration of 3 months. What will the payoff on this CallOnCall option be in 3 months?

**Solution CO1.** The payoff of the CallOnCall option is max[C($S_{t\_1}$, K, T - $t_1$) - x , 0], where x = 1.33, T

= ½, and $t_1$ = ¼. We know that C($S_{t\_1}$, K, T - $t_1$) = (1/5)(66-58) = 1.6. Thus, the payoff on the CallOnCall option will be 1.6 - 1.33 = **$0.27**

**Problem CO2.** The Black-Scholes price of Call Option Q - which expires is 2 years - is $44. The annual continuously-compounded risk-free interest rate is 0.03. The price of a PutOnCall option on Option Q with a strike price of $50 and expiring in 1 year is $10. What is the price of a CallOnCall option on Option Q with a strike price of $50 and expiring in 1 year?

**Solution CO2.** We use the formula

CallOnCall(S, K, x, σ, r, $t_1$, $t_2$, δ) - PutOnCall(S, K, x, σ, r, $t_1$, $t_2$, δ) + $xe^{-rt\_1}$ =

BSCall(S, K, σ, r, $t_2$, δ), where PutOnCall = 10, BSCall = 44, x = 50, r = 0.03, and $t_1$ = 1.

We rearrange the formula thus:

CallOnCall = BSCall + PutOnCall - $xe^{-rt\_1}$ = 44 + 10 - $50e^{-0.03}$ =

CallOnCall = **$5.477723323**

**Problem CO3.** The annual continuously-compounded risk-free interest rate is 0.12. A CallOnCall option on Call Option Ÿ has a price of $53. A PutOnCall option on Option Ÿ has a price of $44. Both compound options have a strike price of $100 and time to expiration of 2 years. Find the Black-Scholes price of the underlying Option Ÿ.

**Solution CO3.** We use the formula

CallOnCall(S, K, x, σ, r, $t_1$, $t_2$, δ) - PutOnCall(S, K, x, σ, r, $t_1$, $t_2$, δ) + $xe^{-rt\_1}$ =

BSCall(S, K, σ, r, $t_2$, δ), where CallOnCall = 53, PutOnCall = 44, x = 100, r = 0.12, and $t_1$ = 2.

Thus, BSCall = 53 - 44 + $100e^{-0.12*2}$ = **BSCall = $87.66278611**

**Problem CO4.** The Black-Scholes price of Call Option Ï is $12. A CallOnCall option on Call Option Ï has a price of $5. A PutOnCall option on Option Ï has a price of $3.33. Both compound options have a strike price of $14 and expire in 3 years. Find the annual continuously-compounded interest rate.

**Solution CO4.** We use the formula

CallOnCall(S, K, x, σ, r, $t_1$, $t_2$, δ) - PutOnCall(S, K, x, σ, r, $t_1$, $t_2$, δ) + $xe^{-rt\_1}$ =

BSCall(S, K, σ, r, $t_2$, δ), where BSCall = 12, CallOnCall = 5, PutOnCall = 3.33, x = 14, and $t_1$ = 3. We rearrange the formula thus: $xe^{-rt\_1}$ = BSCall - CallOnCall + PutOnCall, so

$14e^{-3r}$ = 12 - 5 + 3.33 and $e^{-3r}$ = 0.7378571429 and r = -ln(0.7378571429)/3 = **r = 0.1013350155**

**Problem CO5.** Call Option Q on Stock F expires in 1 year and has a strike price of $66. A CallOnCall option on Call Option Q expires in 6 months. The critical stock price for this option is $45. For which of these combinations of stock prices $S_{1/2}$ 6 months from now and $S_1$ 1 year from now will the compound CallOnCall option be ultimately valuable? More than one answer is possible.

(a) $S_{1/2}$ = 43, $S_1$ = 79
(b) $S_{1/2}$ = 46, $S_1$ = 65
(c) $S_{1/2}$ = 46, $S_1$ = 79
(d) $S_{1/2}$ = 67, $S_1$ = 47
(e) $S_{1/2}$ = 99, $S_1$ = 79

**Solution CO5.** Two conditions must exist for a compound CallOnCall option to be valuable:

1. $S_{t\_1}$ > S* at time $t_1$.

2. $S_T$ > K at time T.

Here, S* = 45, and K = 66. Thus, in order for the option to be ultimately valuable, it must be that $S_{1/2}$ > 45 and $S_1$ > 66. The requirement that $S_{1/2}$ > 45 rules out choice (a).

The requirement that $S_1$ > 66 rules out choices (b) and (d). The remaining choices, (c) and (e), fulfill the requirements that $S_{1/2}$ > 45 and $S_1$ > 66. Thus, **(c) and (e)** are correct answers.

# Section 55

# Pricing Options on Dividend-Paying Stocks

We consider American options where the underlying stock pays a dividend D at time $t_1$ (the expiration time of the compound option). We can either exercise the option at the stock price right before the dividend is paid ($S_{t\_1}$ + D) or we can hold the option until expiration. The underlying option will be priced on the basis of the stock price *after* the dividend is paid ($S_{t\_1}$).

The regular call option payoff at time $t_1$ is max[C($S_{t\_1}$, T - $t_1$), $S_{t\_1}$ + D - K].

The value of the unexercised call is
C($S_{t\_1}$, T - $t_1$) = P($S_{t\_1}$, T - $t_1$) + $S_{t\_1}$ - $Ke^{-r(T-t\_1)}$.

The call option payoff at time $t_1$ can thus be written as
$S_{t\_1}$ + D - K + max[P($S_{t\_1}$, T - $t_1$) - D + K(1 - $e^{-r(T-t\_1)}$) , 0]

The current value of the call option is the present value of this amount.

We note that it is possible to express P($S_{t\_1}$, T - $t_1$) - D + K(1 - $e^{-r(T-t\_1)}$) above as the value of a compound CallOnPut option with strike price D - K(1 - $e^{-r(T-t\_1)}$).

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 14, p. 455.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem PODPS1.** 4 months from now, the stock of Ludicrous Co. will pay a dividend of $2 per share. A certain Call Option F on Ludicrous Co. stock has a strike price of $33 and time to expiration of 6 months. You know with certainty that the stock will have a price of $35 4 months from now after the dividend is paid and that Call Option F will have a price of $3 4 months from now after the dividend is paid. What will be the value of Call Option F 4 months from now?

**Solution PODPS1.** The value of the call option will be
max[C($S_{t\_1}$, T - $t_1$), $S_{t\_1}$ + D - K]. We know that C($S_{t\_1}$, T - $t_1$) = 3, $S_{t\_1}$= 35, D = 2, K = 33. Thus, the value of the call option will be max(3, 35 + 2 - 33) = max(3, 4) = **$4**.

**Problem PODPS2.** 4 months from now, the stock of Ludicrous Co. will pay a dividend of $2 per share. A certain Call Option F on Ludicrous Co. stock has a strike price of $33 and time to expiration of 6 months. You know with certainty that the stock will have a price of $35 4 months from now after the dividend is paid and that Call Option F will have a price of $3 4 months from now after the dividend is paid. The annual continuously compounded risk-free interest rate r is 0.03. If the call option is unexercised, what will be the value 4 months from now of a put option on Ludicrous Co. stock with a strike price of $33 and time to expiration of 6 months?

**Solution PODPS2.** The value of the unexercised call is
$C(S_{t\_1}, T - t_1) = P(S_{t\_1}, T - t_1) + S_{t\_1} - Ke^{-r(T-t\_1)}$.

We know from the given information that $C(S_{t\_1}, T - t_1) = 3$, $S_{t\_1} = 35$, $K = 33$, $r = 0.03$, $T = ½$, $t_1 = 1/3$. We rearrange the formula:
$P = C - S_{t\_1} + Ke^{-r(T-t\_1)} = P = 3 - 35 + 33e^{-0.03(1/2-1/3)} =$ **$P = \$0.8354118134$**

**Problem PODPS3.** You have perfect knowledge that 1 year from now, the stock of Imperious LLC will pay a dividend of $10 per share. Right after it pays the dividend, the stock will be worth $100 per share. A certain put option on this stock has a strike price of $102, time to expiration of 2 years, and will have a price of $12 in 1 year if unexercised. The annual continuously compounded risk-free interest rate r is 0.08. What will be the price in 1 year (after the dividend is paid) of a call option on this stock with a strike price of $102 and time to expiration of 2 years?

**Solution PODPS3.** We use the call option payoff expression
$S_{t\_1} + D - K + \max[P(S_{t\_1}, T - t_1) - D + K(1 - e^{-r(T-t\_1)}) , 0]$, where $S_{t\_1} = 100$, $D = 10$, $K = 102$, $T = 2$, $t_1 = 1$, $r = 0.08$, and $P = 12$. $P(S_{t\_1}, T - t_1) - D + K(1 - e^{-r(T-t\_1)}) =$
$12 - 10 + 102(1 - e^{-0.08}) = 9.842132669 > 0$. Thus, the call option price will be
$100 + 10 - 102 + 9.842132669 =$ **$17.84213267**

**Problem PODPS4.** You have perfect knowledge that 1 year from now, the stock of Imperious LLC will pay a dividend of $10 per share. Right after it pays the dividend, the stock will be worth $100 per share. A certain put option on this stock has a strike price of $102, time to expiration of 2 years, and will have a price of $12 in 1 year if unexercised. The annual continuously compounded risk-free interest rate r is 0.08. You can use a compound CallOnPut option on the Imperious LLC put to determine the price in 1 year (after the dividend is paid) of a call option on this stock with a strike price of $102 and time to expiration of 2 years. What is the strike price of such a CallOnPut option?

**Solution PODPS4.** The strike price of such a CallOnPut option is $D - K(1 - e^{-r(T-t\_1)}) = 10 - 102(1 - e^{-0.08}) =$ **$2.157867331**.

**Problem PODPS5.** You have perfect knowledge that 1 year from now, the stock of Imperious LLC will pay a dividend of $10 per share. Right after it pays the dividend, the stock will be worth $100 per share. A certain put option on this stock has a strike price of $102, time to expiration of 2 years, and will have a price of $12 in 1 year if unexercised. The annual continuously compounded risk-free interest rate r is 0.08. You can use a compound CallOnPut option on the Imperious LLC put to determine the price in 1 year (after the dividend is paid) of a call option on this stock with a strike price of $102 and time to expiration of 2 years. What will be the payoff in 1 year on such a compound CallOnPut option on the Imperious LLC put?

**Solution PODPS5.** The payoff on this CallOnPut option is

$P(S_{t\_1}, T - t_1) - D + K(1 - e^{-r(T-t\_1)}) = P(S_{t\_1}, T - t_1) - \text{Strike}$. It is given that $P = 12$ and Strike = 2.157867331. Thus, $P(S_{t\_1}, T - t_1) - \text{Strike} = 12 - 2.157867331 =$ **$9.842132669**

# Section 56

# Gap Options

A **gap option** pays the difference between the asset price and the strike price when the price of the underlying asset exceeds a *trigger price* different from the strike price.

A **gap call** payoff is $S - K_1$ when $S > K_2$. Otherwise, the gap call pays nothing.

A **gap put** payoff is $K_1 - S$ when $S < K_2$. Otherwise, the gap put pays nothing.

We note that in both cases, S is the stock price, $K_1$ is the strike price, and $K_2$is the trigger price.

A modified Black-Scholes formula can be used to price a gap call:

$C(S, K_1, K_2, \sigma, r, T, \delta) = Se^{-\delta T}N(d_1) - K_1e^{-rT}N(d_2)$, where

$d_1 = (\ln[(Se^{-\delta T})/(K_2e^{-rT})] + 0.5\sigma^2T)/(\sigma\sqrt{(T)})$ and $d_2 = d_1 - \sigma\sqrt{(T)}$

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 14, pp. 457-458.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem GO1.** You own three different gap calls on Obsequious Co. stock. Call A pays S - 43 when the stock price S is above 56. Call B pays S - 80 when the stock price S is above 75. Call C pays S - 30 when the stock price S is above 70. At the expiration of all three of these gap calls, Obsequious Co. stock trades for $76 per share. What is your total payoff on all three options?

**Solution GO1.** 76 > 75 > 70 > 56, so the trigger prices of all three gap calls have been exceeded. Call A will thus pay 76 - 43 = 33. Call B will pay 76 - 80 = -4. (We note that negative payoffs are sometimes possible with gap options.) Call C will pay 76 - 30 = 46. So your total payoff is 33 - 4 + 46 = **$75.**

**Problem GO2.** The payoff on a gap put is currently 1/5 the price of the underlying stock. The gap put's trigger price is $60, and the stock currently trades at $40. What is the strike price of the gap put?

**Solution GO2.** The gap put pays $K_1$ - S when S < 60. $K_1$ is the desired strike price. The gap put payoff is given as (1/5)40 = 8. Thus, 8 = $K_1$ - 40 and so $K_1$ = 48.

**Problem GO3.** Assume that the Black-Scholes framework holds. A gap call option on Bombastic LLC stock has a trigger price of $55, a strike price of $50, and a time to expiration of 2 years. Bombastic LLC stock currently trades for $53 per share and pays dividends with a continuously compounded annual yield of 0.03. The annual continuously compounded risk-free interest rate is 0.09, and the relevant price volatility for the Black-Scholes formula is 0.33. Find $d_1$ in the Black-Scholes formula for the price of this gap call.

**Solution GO3.** $d_1 = (\ln[(Se^{-\delta T})/(K_2e^{-rT})] + 0.5\sigma^2T)/(\sigma\sqrt{(T)})$, where $K_2$, the trigger price, is 55. Thus, $d_1 = (\ln[(53e^{-0.03*2})/(55e^{-0.09*2})] + 0.5*0.33^2*2)/(0.33\sqrt{(2)})$ = $\mathbf{d_1 = 0.4111048722}$

**Problem GO4.** Assume that the Black-Scholes framework holds. A gap call option on Bombastic LLC stock has a trigger price of $55, a strike price of $50, and a time to expiration of 2 years. Bombastic LLC stock currently trades for $53 per share and pays dividends with a continuously compounded annual yield of 0.03. The annual continuously compounded risk-free interest rate is 0.09, and the relevant price volatility for the Black-Scholes formula is 0.33. Find $d_2$ in the Black-Scholes formula for the price of this gap call.

**Solution GO4.** $d_2 = d_1 - \sigma\sqrt{(T)}$, where $d_1$, from Solution GO3, is 0.4111048722. Thus,

$d_2 = 0.4111048722 - 0.33\sqrt{(2)}$ = $\mathbf{d_2 = -0.0555856034}$

**Problem GO5.** Assume that the Black-Scholes framework holds. A gap call option on Bombastic LLC stock has a trigger price of $55, a strike price of $50, and a time to expiration of 2 years. Bombastic LLC stock currently trades for $53 per share and pays dividends with a continuously compounded annual yield of 0.03. The annual continuously compounded risk-free interest rate is 0.09, and the relevant price volatility for the Black-Scholes formula is 0.33. Find the Black-Scholes price of this gap call.

**Solution GO5.**

In MS Excel, we enter "=NormSDist(0.4111048722)" to find $N(d_1) = 0.659502181$

In MS Excel, we enter "=NormSDist(-0.0555856034)" to find $N(d_2) = 0.477835967$

Now we can use the formula $C(S, K_1, K_2, \sigma, r, T, \delta) = Se^{-\delta T}N(d_1) - K_1e^{-rT}N(d_2)$, noting that $K_1 = 50$. Thus, $C = 53e^{-0.03*2}0.659502181 - 50e^{-0.09*2}0.477835967$ = **C = $12.96196803**

# Section 57

# Exchange Options

An **exchange option** is also known as an **outperformance option.** It only has a payoff if the underlying asset performs better than some other **benchmark asset.** The payoff to an exchange option is max(0, $S_T$ - $K_T$), where T is the option's time to expiration, $S_T$ is the price of the underlying asset at time T, and $K_T$ is the price of the benchmark asset at time T.

The price of a European exchange option can be expressed using a generalized Black-Scholes formula:
$C(S, K, \sigma, \delta_S, \delta_K, T) = Se^{-(\delta\_S)T}N(d_1) - Ke^{-(\delta\_K)T}N(d_2)$, where
$d_1 = [\ln(Se^{-(\delta\_S)T}/Ke^{-(\delta\_K)T}) + 0.5\sigma^2T]/[\sigma\sqrt{(T)}]$, $d_2 = d_1 - \sigma\sqrt{(T)}$, and $\sigma = \sqrt{[\sigma_S{}^2 + \sigma_K{}^2 - 2\rho\sigma_S\sigma_K]}$

**Meanings of variables:**
$\sigma_S$ = underlying asset annual price volatility.
$\sigma_K$ = benchmark asset annual price volatility.
ρ = the correlation between the continously compounded returns on the two assets.
$\delta_S$ = the annual continuously compounded dividend yield on the underlying asset.
$\delta_K$ = the annual continuously compounded dividend yield on the benchmark asset.

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 14, pp. 459-460.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem EO1.** Amenhotep owns an exchange call with one share of the stock of Upper Egyptian Co. as the underlying asset, and 2.5 shares of the stock of Lower Egyptian Co. as the benchmark asset. At expiration of the option, Upper Egyptian Co. trades for $531 per share, and Lower Egyptian Co. trades for $121 per share. What is Amenhotep's payoff on the option?

**Solution EO1.** At expiration, the benchmark asset is worth 2.5*121 = $302.5, while the underlying asset is worth $531. Thus, Amenhotep's payoff is 531 - 302.5 = **$228.50**

**Problem EO2.** One share of the stock of Comical Co. is used as the underlying asset on an exchange option, for which the benchmark asset is one share of the stock of Tragic Co. Currently, Comic Co. trades for $321 per share, and Tragic Co. trades for $300 per share. Comic Co. has an annual price volatility of 0.34 and pays dividends at an annual continuously compounded yield of 0.22. Tragic Co. has an annual price volatility of 0.66 and pays dividends at an annual continuously compounded yield of 0.02. The correlation between the continously compounded returns on the two assets is 0.84. The exchange option expires in 4 years. Find σ in the generalized Black-Scholes formula for the price of this option.

**Solution EO2.** We use the formula $\sigma = \sqrt{[\sigma_S{}^2 + \sigma_K{}^2 - 2\rho\sigma_S\sigma_K]}$ =

$\sqrt{[0.34^2 + 0.66^2 - 2*0.84*0.34*0.66]}$ = **σ = 0.4173823187**

**Problem EO3.** One share of the stock of Comical Co. is used as the underlying asset on an exchange option, for which the benchmark asset is one share of the stock of Tragic Co. Currently, Comic Co. trades for $321 per share, and Tragic Co. trades for $300 per share. Comic Co. has an annual price volatility of 0.34 and pays dividends at an annual continuously compounded yield of 0.22. Tragic Co. has an annual price volatility of 0.66 and pays dividends at an annual continuously compounded yield of 0.02. The correlation between the continosly compounded returns on the two assets is 0.84. The exchange option expires in 4 years. Find $d_1$ in the generalized Black-Scholes formula for the price of this option.

**Solution EO3.** From Solution EO2, we know that σ = 0.4173823187. We use the formula $d_1$ = $[\ln(Se^{-(\delta\_S)T}/Ke^{-(\delta\_K)T}) + 0.5\sigma^2T]/[\sigma\sqrt{(T)}]$ =
$[\ln(321e^{-0.22*4}/300e^{-0.02*4}) + 0.5*0.4173823187^2*4]/[0.4173823187\sqrt{(4)}]$ = $\mathbf{d_1 = -0.4599204785}$

**Problem EO4.** One share of the stock of Comical Co. is used as the underlying asset on an exchange option, for which the benchmark asset is one share of the stock of Tragic Co. Currently, Comic Co. trades for $321 per share, and Tragic Co. trades for $300 per share. Comic Co. has an annual price volatility of 0.34 and pays dividends at an annual continuously compounded yield of 0.22. Tragic Co. has an annual price volatility of 0.66 and pays dividends at an annual continuously compounded yield of 0.02. The correlation between the continosly compounded returns on the two assets is 0.84. The exchange option expires in 4 years. Find $d_2$ in the generalized Black-Scholes formula for the price of this option.

**Solution EO4.** From Solution EO2, we know that σ = 0.4173823187. From Solution EO3, we know that $d_1$ = -0.4599204785. We use the formula $d_2 = d_1 - \sigma\sqrt{(T)}$ =
$-0.4599204785 - 0.4173823187\sqrt{(4)}$ = $\mathbf{d_2 = -1.294685116}$

**Problem EO5.** One share of the stock of Comical Co. is used as the underlying asset on an exchange option, for which the benchmark asset is one share of the stock of Tragic Co. Currently, Comic Co. trades for $321 per share, and Tragic Co. trades for $300 per share. Comic Co. has an annual price volatility of 0.34 and pays dividends at an annual continuously compounded yield of 0.22. Tragic Co. has an annual price volatility of 0.66 and pays dividends at an annual continuously compounded yield of 0.02. The correlation between the continosly compounded returns on the two assets is 0.84. The exchange option expires in 4 years. Find the Black-Scholes price of this option.

**Solution EO5.** From Solution EO3, we know that $d_1$ = -0.4599204785. From Solution EO4, we know that $d_2$ = -1.294685116.
In MS Excel, we enter "=NormSDist(-0.4599204785)" to find $N(d_1)$ = 0.32278665
In MS Excel, we enter "=NormSDist(-1.294685116)" to find $N(d_2)$ = 0.097714438
Now we can use the formula $C(S, K, \sigma, \delta_S, \delta_K, T) = Se^{-(\delta\_S)T}N(d_1) - Ke^{-(\delta\_K)T}N(d_2)$ =
$321e^{-0.22*4}0.32278665 - 300e^{-0.02*4}0.097714438$ = **C = $15.91699158**

# Section 58

# Exam-Style Questions on Exotic Options

The problems in this section were designed to be similar to problems from past versions of Exam 3F / Exam MFE. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

**Problem ESQEO1.**

**Similar to Question 34 from the Casualty Actuarial Society's Spring 2007 Exam 3:**

Barrier call options on Lucrative Co. stock have the following prices:
Down-and-out: $65 Barrier: $34320
Up-and-out: $34 Barrier: $57212
Up-and-in: $55 Barrier: $57212
Down rebate: $12 Barrier: $34320
Up rebate: $150 Barrier: $57212

Calculate the price of a down-and-in barrier call option on Lucrative Co. stock with a barrier of $34320. All options mentioned have the same strike price and time to expiration.

**Solution ESQEO1.** We recall the parity relationship for barrier options:
Knock-in option + Knock-out option = Ordinary option
Here, Up-and-in + Up-and-out = Ordinary call price, so Ordinary call price = 34 + 55 = $89.
Also, Down-and-in + Down-and-out = Ordinary call price, so
Down-and-in = Ordinary call price - Down-and-out = 89 - 65 = **Down-and-in = $24**

**Problem ESQEO2.**

**Similar to Question 26 from the Casualty Actuarial Society's Fall 2007 Exam 3:**

Which of these statements about exotic options are true? More than one answer may be correct.

(a) Gap options can be priced with the Black-Scholes formula.
(b) The premium on barrier options always exceeds the premium on standard options.
(c) Exchange options can be priced with the Black-Scholes formula.
(d) Gap options always have a positive payoff.
(e) Asian options are path-dependent.
(f) Asian options can be based on three kinds of averages: arithmetic, geometric, and median.

**Solution ESQEO2.**
(a) is true. We saw in Section 56 how to price gap options with the Black-Scholes formula.

(b) is false. The premium on barrier options is always less than or equal to the premium on standard options.

(c) is true. We saw in Section 57 how to price exchange options with the Black-Scholes formula.

(d) is false. It is possible for gap some options to have a negative payoff, as examples in Section 56 indicate.

(e) is true. Asian option prices depend on the prices of the underlying asset at the end of many time periods, as opposed to just one.

(f) is false. The median is not used as a basis for computing payoffs on Asian options.

So **(a), (c), and (e) are correct answers**.

**Problem ESQEO3.**

**Similar to Question 27 from the Casualty Actuarial Society's Fall 2007 Exam 3:**

Geoffrey owns an 11-month geometric average price option on the stock of Rapid Rabid Rabbits Co. The option has a strike price of $44.5, and the option's payoff is based on evaluating the stock's price at the end of each month. The stock prices for these 11 months turn out to be as follows:

End of month 1: $50

End of month 2: $43

End of month 3: $47

End of month 4: $35

End of month 5: $67

End of month 6: $77

End of month 7: $43

End of month 8: $89

End of month 9: $100

End of month 10: $23

End of month 11: $50

What is ultimate payoff of the option?

**Solution ESQEO3.** The option payoff is Geometric average - Strike. We find the geometric average thus: $(50*43*47*35*67*77*43*89*100*23*50)^{1/11} = 52.31322233$

Thus, the payoff is 52.31322233 - 44.5 = **$7.813222334**

**Problem ESQEO4.**

**Similar to Question 28 from the Casualty Actuarial Society's Fall 2007 Exam 3:**

The annual continuously compounded risk-free interest rate is 0.1. The stock of Devious Co. currently trades for $100 per share and follows perfectly the assumptions of the two-period binomial model. In one year, the stock will trade for either $S_H = 134$ or $S_L = 69$. The stock pays no dividends. In two years, the stock will trade for either $S_{HH} = 179.56$, $S_{HL} = 92.46$, or $S_{LL} = 47.61$. American put options are written on this stock; these options expire 2 years from now and have a strike price of $105. Compound CallOnPut options are written on these puts, expiring 1 year from now and with a strike price of $5. Use a two-period binomial model to calculate the value of one such CallOnPut option.

**Solution ESQEO4.** First, we consider the values of the put option at various nodes of the binomial tree. Using the formula introduced in Section 16, we find $p^* = (e^{(r-\partial)h} - d)/(u - d)$. Here, u = 1.34, d = 0.69, h = 1, r = 0.1, and δ = 0. Thus, $p^* = (e^{0.1} - 0.69)/(1.34 - 0.69) = p^* = 0.6387244893$

We consider the possible put values at the put's expiration:

$P_{HH} = 0$, since 179.56 > 105

$P_{HL} = 105 - 92.46 = 12.54$

$P_{LL} = 105 - 47.61 = 57.39$

Now we try to find $P_H$:

$P_H = e^{-rh}[p^*P_{HH} + (1 - p^*)P_{HL}] = e^{-0.1}[0 + (1 - 0.6387244893)12.54] = 4.099270827.$

4.099270827 > 0, the value of exercising the put at h = 1 year, when the stock price will be greater than the put's strike price, so indeed, $P_H = 4.099270827$

This puts the value of the compound option at expiration if the stock price increases at $CP_H = 0$, since the put price is not as high as the CallOnPut strike of $5.

Now we try to find $P_L$:

$P_L = e^{-rh}[p^*P_{HL} + (1 - p^*)P_{LL}] = e^{-0.1}[0.6387244893*12.54 + (1 - 0.6387244893)57.39] = 26.00792889$. But if the option is exercised at h = 1 year, the payoff can be 105 - 69 = 36 > 26.00792889, so $P_L = 36$ and $CP_L = 36 - 5 = 31$

Thus, the value of the compound CallOnPut today is

$CP = e^{-rh}[p^*CP_H + (1 - p^*)CP_L] = e^{-0.1}[0 + (1 - 0.6387244893)31] =$ **CP = $10.13376361**

**Problem ESQEO5.**

**Similar to Question 17 from the Society of Actuaries' May 2007 Exam MFE:**

At time 0, the price of Functional Co. stock is S(0) = $500. A European gap option on this stock has expiration T > 0. If the stock price at time T is greater than $600, the payoff for the option is S(T) - 550. If this is not the case, the payoff is 0. You know that the price of a European call option with expiration date T and a strike price of $600 is $39. The delta of this call option is 0.43. Find the price of the gap option.

**Solution ESQEO5.**

We recall the Black-Scholes formula for call price:
$C = Se^{-\delta T}N(d_1) - Ke^{-rT}N(d_2)$, where $\Delta = e^{-\delta T}N(d_1)$. For convenience, we can refer to $e^{-rT}N(d_2)$ as $\Xi$. Thus, $C = S\Delta - K\Xi$

For a gap option, the strike price K in the Black-Scholes formula is replaced by $K_1$, the trigger price. Thus, $C_G = S\Delta - K_1\Xi$. First, we find $\Xi$:

It is given that C = 39, S = 500, $\Delta = 0.43$, K = 600, and so

$\Xi = (S\Delta - C)/K = (500*0.43 - 39)/600 = \Xi = 0.29333333333$

It is also given that $K_1 = 550$. Thus,

$C_G = S\Delta - K_1\Xi = 500*0.43 - 550*0.29333333333 =$ **$C_G$ = $53.666666667**

# Section 59

# The Basics of Brownian Motion

A **stochastic process** is "a random process that is a function of time."

**Brownian motion** is "a stochastic process that is a random walk occurring in continuous time, with movements that are continuous rather than discrete." Brownian motion can be defined as follows:

"Let Z(t) represent the value of a Brownian motion at time t. Brownian motion is a continuous stochastic process, Z(t), with the following characteristics:"

1. $Z(0) = 0$

2. $Z(t + s) - Z(t)$ is normally distributed with mean 0 and variance s.

3. $Z(t + s_1) - Z(t)$ is independent of $Z(t) - Z(t + s_2)$ where $s_1, s_2 > 0$. In other words, nonoverlapping increments are independently distributed.

4. Z(t) is continuous (you can draw a picture of Brownian motion without lifting your pencil)."

Z(t) is a **martingale,** which is a stochastic process for which $E[Z(t+s) \mid Z(t)] = Z(t)$. Z(t) can also be called a **diffusion process.**

(All quotations are from McDonald 2006, p. 650.)

**Arithmetic Brownian motion** is a process expressed by the following equation:
$dX(t) = \alpha dt + \sigma dZ(t)$

α can be called the *instantaneous mean per unit time.* It is also the *drift factor*.

$\sigma^2$ can be called the *instantaneous variance per unit time.*

σ is the *volatility or variance factor*.

X(t) is the sum of the individual changes dX.

Arithmetic Brownian motion has the following properties, as discussed in McDonald 2006, p. 654:

"1. X(t) is normally distributed

2. The random term has been multiplied by a scale factor that enables us to change variance. Since dZ(t) has a variance of 1 per unit time, σdZ(t) will have a variance of $\sigma^2$ per unit time.

3. The αdt term introduces a nonrandom *drift* into the process. Adding αdt has the effect of adding α per unit time to X(0).

4. Drawback: there is nothing to prevent X from becoming negative, so it is a poor model for stock prices.

5. Drawback: The mean and variance of changes in dollar terms are independent of the level of the stock price. In practice, if a stock price doubles, we would expect both the dollar expected return and the dollar standard deviation of returns to approximately double."

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 20, pp. 650-654.

Learning to apply Brownian motion is highly counterintuitive and difficult. Unfortunately, most works addressing it focus on the theory and abstract derivations behind it rather than the methods via which it might be used in actual problems. This study guide will attempt to remedy this issue by means of the following approach:

1. **Gradual progression of sample problems** - so that they begin at an elementary level that can be easily understood and move gently along to higher levels of difficulty

2. **Focus on memorizing the most important formulas.** Rote memorization is highly underrated in most of today's textbooks and classrooms. However, the reality of the matter is that you need to have some content firmly in your mind before you can think about it, analyze it, and understand it. Once you memorize a few formulas, variations and extensions of them will no longer seem like gibberish on a page.

3. **A building-up of ideas and methods.** Most exam-style Brownian motion problems presuppose knowledge of the entire conceptual infrastructure established in Chapter 20 of McDonald's *Derivatives Markets.* Problems that help students learn the components of this infrastructure are, sadly, lacking. This will be remedied in the study guide.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem BBM1.** Z(t) follows a Brownian motion. What is the variance of Z(53) - Z(9)?

**Solution BBM1.** One of the qualities of Brownian motion is that Z(t + s) - Z(t) is normally distributed with mean 0 and variance s. Here, we have t = 9 and s = 53 - 9 = 44. Thus, the variance of Z(53) - Z(9) is **s = 44.**

**Problem BBM2.** Z(t) follows a Brownian motion. What is $E[Z(t + 0.9) \mid Z(t)]$?

**Solution BBM2.** If Z(t) follows a Brownian motion, then Z(t) is a martingale, which means that E[Z(t+s) | Z(t)] = Z(t), no matter what the s. Here, s = 0.9.

Thus, E[Z(t + 0.9) | Z(t)] = **Z(t)**

**Problem BBM3.** A particular arithmetic Brownian motion is as follows:

dX(t) = 0.4dt + 0.543dZ(t). What is the *drift factor* of this Brownian motion?

**Solution BBM3.** An arithmetic Brownian motion is expressible as follows:

dX(t) = αdt + σdZ(t), where α is the drift factor. Here, **α = 0.4**

**Problem BBM4.** A particular arithmetic Brownian motion is as follows:

dX(t) = 0.4dt + 0.543dZ(t). What is the *volatility or variance factor* of this Brownian motion?

**Solution BBM4.** An arithmetic Brownian motion is expressible as follows:

dX(t) = αdt + σdZ(t), where σ is the variance factor. Here, **σ = 0.543**

**Problem BBM5.** A particular arithmetic Brownian motion is as follows:

dX(t) = 0.4dt + 0.543dZ(t). For this Brownian motion, what is the *instantaneous variance per unit time*?

**Solution BBM5.** An arithmetic Brownian motion is expressible as follows:

dX(t) = αdt + σdZ(t), where $\sigma^2$ is the instantaneous variance per unit time. Here, $\sigma^2 = 0.543^2 = \boldsymbol{\sigma^2} \mathbf{= 0.294849}$

These problems were gentle and simple by deliberate design. A gradual buildup of the difficulty and the concepts involved is the only reasonable and effective way to get students to internalize both the meaning and the methods involved in Brownian motion.

Now write down the expression for arithmetic Brownian motion six times. It will help you internalize it and use it later on. At random times during the day, pause and ask yourself to recall the formula from memory.

dX(t) = αdt + σdZ(t)
dX(t) = αdt + σdZ(t)
dX(t) = αdt + σdZ(t)
dX(t) = αdt + σdZ(t)
dX(t) = αdt + σdZ(t)
dX(t) = αdt + σdZ(t)

# Section 60

# The Basics of Geometric Brownian Motion

**Geometric Brownian motion** can be expressed as follows:
$dX(t)/X(t) = \alpha dt + \sigma dZ(t)$.

This is distinct from arithmetic Brownian motion because $dX(t)$ is not equal to $\alpha dt + \sigma dZ(t)$, but rather to $dX(t) = \alpha X(t)dt + \sigma X(t)dZ(t)$.

Here, as in the arithmetic Brownian motion equation, $\alpha$ is the drift factor and $\sigma$ is the volatility factor. However, here the drift and volatility depend on the price of the stock $X(t)$. Where such dependence occurs, we have an **Ito process.**

Again, as in arithmetic Brownian motion,

$\alpha$ can be called the *instantaneous mean per unit time.* It is also the *drift factor*.

$\sigma^2$ can be called the *instantaneous variance per unit time.*

For geometric Brownian motion, "the percentage change in the asset value is normally distributed with instantaneous mean $\alpha$ and instantaneous variance $\sigma^2$."

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 20, p. 655.

If you have already memorized the expression for arithmetic Brownian motion, you can memorize the expression for geometric Brownian motion by noting that the only difference is that in geometric Brownian motion, $dX(t)$ is divided by $X(t)$.

**Problem BGBM1.** A given geometric Brownian motioncan be expressed as follows: $dX(t)/X(t) = 0.215dt + 0.342dZ(t)$. What is the instantaneous mean of the percentage change in the asset value?

**Solution BGBM1.** For geometric Brownian motion, "the percentage change in the asset value is normally distributed with instantaneous mean $\alpha$ and instantaneous variance $\sigma^2$." Here, the instantaneous mean $\boldsymbol{\alpha = 0.215}$.

**Problem BGBM2.** A given geometric Brownian motioncan be expressed as follows: $dX(t)/X(t) = 0.215dt + 0.342dZ(t)$. What is the instantaneous variance of the percentage change in the asset value?

**Solution BGBM2.** For geometric Brownian motion, "the percentage change in the asset value is normally distributed with instantaneous mean α and instantaneous variance $\sigma^2$." Here, the instantaneous variance $\sigma^2 = 0.342^2$ = $\boldsymbol{\sigma^2}$ **=0.116964.**

**Problem BGBM3.** Stock X follows a geometric Brownian motion where the drift factor is 0.93 and the variance factor is 0.55. At some particular time t, it is known that dt = 0.035, and dZ(t) = 0.43. At time t, the stock trades for $2354 per share. What is the instantaneous rate of change in the price of stock X?

**Solution BGBM3.** We can express the is the instantaneous rate of change in the price of stock X as

$dX(t) = \alpha X(t)dt + \sigma X(t)dZ(t)$, where α = 0.93, σ = 0.55, dt = 0.035, dZ(t) = 0.43, and X(t) = 2354. Thus,

dX(t) = 0.93*2354*0.035 + 0.55*2354*0.43 = dX(t) = **$633.3437** (quite an instantaneous rate of change!)

**Problem BGBM4.** Stock Φ follows a geometric Brownian motion where the drift factor is 0.0234 and the variance factor is 0.953. At some particular time t, it is known that dt = 0.531, dZ(t) = 0.136, and dX(t) - the instantaneous rate of change in the price of the stock - is 0.245. What is the stock price at time t?

**Solution BGBM4.** Geometric Brownian motioncan be expressed as follows:
$dX(t)/X(t) = \alpha dt + \sigma dZ(t)$. We want to find $X(t) = dX(t)/[\alpha dt + \sigma dZ(t)]$ =

0.245/(0.0234*0.531 + 0.953*0.136) = **X(t) = 1.724946386**

**Problem BGBM5.** Stock Ψ follows a geometric Brownian motion where the variance factor is 0.35. At some particular time t, it is known that dt = 0.0143, dZ(t) = 0.0154, and dX(t) - the instantaneous rate of change in the price of the stock - is 0.353153. The price of Stock Ψ at time t is $31.23. What is the drift factor for this Brownian motion?

**Solution BGBM5.** Geometric Brownian motioncan be expressed as follows:
$dX(t)/X(t) = \alpha dt + \sigma dZ(t)$. Thus, $\alpha dt = [dX(t)/X(t) - \sigma dZ(t)]$ and

$\alpha = [dX(t)/X(t) - \sigma dZ(t)]/dt$ = [0.353153/31.23 - 0.35*0.0154]/0.0143 = **α = 0.4138554689**

# Section 61

# The Basics of Mean-Reversion Processes

Mean reversion can be incorporated into a modified version of Brownian motion. If a value departs from the mean in either direction, it will tend to *revert* back to the mean.

The drift term of arithmetic Brownian is modified to account for this.

This is the generic equation for Brownian motion with mean reversion.

$dX(t) = \lambda[\alpha - X(t)]dt + \sigma dZ(t)$

As in arithmetic and geometric Brownian motion, $\alpha$ is the instantaneous mean per unit time and $\sigma$ is the volatility factor. $\lambda$ is the reversion factor and measures the speed of the reversion to the mean.

If $X > \alpha$, the drift is negative. If $X < \alpha$, the drift is positive.

When $\alpha = 0$, we call this equation the **Ornstein-Uhlenbeck process:**

$dX(t) = -\lambda X(t)dt + \sigma dZ(t)$

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 20, pp. 654-655.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem BMRP1.** The price of Stock T follows the following Brownian motion:

$dX(t) = 2.1[0.35 - X(t)]dt + 0.642dZ(t)$. What is the reversion factor for this Brownian motion?

**Solution BMRP1.** This is the generic equation for Brownian motion with mean reversion: $dX(t) = \lambda[\alpha - X(t)]dt + \sigma dZ(t)$

The reversion factor is $\lambda$. Here, $\boldsymbol{\lambda = 2.1}$

**Problem BMRP2.** The price of Stock R follows a mean reversion process, where the instantaneous mean per unit time is 54, and the volatility factor is 0.46. The reversion factor for this process is 3.5. At some time t, you know that dt = 0.24, dZ(t) = 0.94, and the stock price is \$46 per share. What is the rate of change in the stock price - dX(t)?

**Solution BMRP2.** This is the generic equation for Brownian motion with mean reversion: $dX(t) = \lambda[\alpha - X(t)]dt + \sigma dZ(t)$. Here, $\lambda = 3.5$, $\alpha = 54$, $\sigma = 0.46$, $dZ(t) = 0.94$, $dt = 0.24$, $X(t) = 46$. Thus, $dX(t) = 3.5[54 - 46]0.24 + 0.46*0.94$ = **dX(t) = 7.1524**

**Problem BMRP3.** The price of Stock S follows a mean reversion process, but the mean is currently unknown. You do know that at time t, $dt = 0.125$, $dZ(t) = 0.3125$, and $dX(t)$, the rate of change in the stock price, is 0.634. Also, $X(t)$, the stock price, is 613, the volatility is 0.53, and the reversion factor for this process is 0.1531. What is the instantaneous mean per unit time for this process?

**Solution BMRP3.** This is the generic equation for Brownian motion with mean reversion: $dX(t) = \lambda[\alpha - X(t)]dt + \sigma dZ(t)$. It can be rearranged as follows:

$\lambda[\alpha - X(t)]dt = dX(t) - \sigma dZ(t)$

$\alpha - X(t) = [dX(t) - \sigma dZ(t)]/[\lambda dt]$

$\alpha = [dX(t) - \sigma dZ(t)]/[\lambda dt] + X(t) = [0.634 - 0.53*0.3125]/[0.1531*0.125] + 613 =$

**$\alpha$ = 637.4741999**

**Problem BMRP4.** The price of Stock Q follows an Ornstein-Uhlenbeck process. The reversion factor for this process is 0.0906, and the volatility factor is 0.63. At some time t, $dt = 1$, $dZ(t) = 0.9356$, and $dX(t)$, the rate of change in the stock price, is 0.0215. What is the price of Stock Q at time t?

**Solution BMRP4.** This is the equation for the Ornstein-Uhlenbeck process:

$dX(t) = -\lambda X(t)dt + \sigma dZ(t)$. It can be rearranged as follows:

$\lambda X(t)dt = \sigma dZ(t) - dX(t)$

$X(t) = [\sigma dZ(t) - dX(t)]/[\lambda dt] = [0.63*0.9356 - 0.0215]/[0.0906*1]$ = **X(t) = 6.268520971**

**Problem BMRP5.** The price of Stock Q follows a mean reversion process with $\lambda = 15$, $\alpha = 53$, and $\sigma = 0.15$. For which of these values of the stock price $X(t)$ will the drift for this process be negative? More than one answer is possible.

(a) $X(t) = 0.04$
(b) $X(t) = 0.26$
(c) $X(t) = 14$
(d) $X(t) = 32$
(e) $X(t) = 75$

**Solution BMRP5.** If $X > \alpha$, the drift is negative. Otherwise, the drift is positive. Here, $\alpha = 53$. The only case here where $X > \alpha$ is $X(t) = 75$. So the only correct answer is **(e).**

# Section 62

# Basics of Ito's Lemma for Actuaries

Ito's Lemma enables us to solve some problems involving Brownian motion. Here, the focus will be on internalizing the statement of Ito's Lemma.

**Ito's Lemma:** Let C[S(t), t] be a twice differentiable function of S(t). Then the change in C, dC[S(t), t] is $dC(S, t) = \{[\alpha(S, t) - \delta(S, t)]C_S + (1/2)\sigma(S, t)^2C_{SS} + C_t\}dt + \sigma(S, t)C_SdZ(t)$

If S(t) follows a geometric Brownian motion, then

$dC(S, t) = \{[\alpha - \delta]SC_S + (1/2)\sigma^2S^2C_{SS}+ C_t\}dt + \sigma SC_SdZ(t)$

Note: $C_S$ is the partial derivative of C with respect to S. $C_{SS}$ is the second partial derivative of C with respect to S. As usual, α is the drift factor and σ is the volatility factor. δ is the instantaneous dividend yield of the stock.

When applying Ito's Lemma to mean reverting processes, replace α(S, t) with λ[α - S(t)].

Source: McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 20, p. 664.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem BIL1.** Assume that S follows an arithmetic Brownian motion: $dS(t) = \alpha dt + \sigma dZ(t)$, and $\delta(S, t) = 0$. Use Ito's Lemma to find $d(2S^2)$ in terms of S, α, σ, dt, and dZ(t).

**Solution BIL1.** Here, $C(S, t) = 2S^2$.

Taking partial derivatives, we get $C_t = 0$, $C_S = 4S$ and $C_{SS} = 4$.

By Ito's Lemma: $dC(S, t) = \{[\alpha(S, t) - \delta(S, t)]C_S + (1/2)\sigma(S, t)^2C_{SS} + C_t\}dt + \sigma(S, t)C_SdZ(t)$

$= (4S\alpha + (1/2)\sigma^2*4 + 0)dt + 4S\sigma dZ(t) =$ $\mathbf{d(2S^2) = (4S\alpha + 2\sigma^2)dt + 4S\sigma dZ(t).}$

**Problem BIL2.** Assume that S follows a mean reverting process: $dS(t) = \lambda[\alpha - S(t)]dt + \sigma dZ(t)$, and $\delta(S, t) = 0$. Use Ito's Lemma to find $d(2S^2)$ in terms of S, λ, α, σ, dt, and dZ(t).

**Solution BIL2.** Here, $C(S, t) = 2S^2$.

Taking partial derivatives, we get $C_t = 0$, $C_S = 4S$ and $C_{SS} = 4$.

By Ito's Lemma, modified to account for mean reversion:

$dC(S, t) = \{[\lambda[\alpha - S(t)] - \delta(S, t)]C_S + (1/2)\sigma(S, t)^2C_{SS} + C_t\}dt + \sigma(S, t)C_SdZ(t)$

$= (\lambda[\alpha - S]4S + (1/2)\sigma^2*4 + 0)dt + 4S\sigma dZ(t) = (\lambda[\alpha - S]4S + 2\sigma^2)dt + 4S\sigma dZ(t) =$

$\mathbf{d(2S^2) = (4S\lambda\alpha - 4S^2\lambda + 2\sigma^2)dt + 4S\sigma dZ(t)}$

**Problem BIL3.** Assume that S follows a geometric Brownian motion: $dS(t)/S(t) = \alpha dt + \sigma dZ(t)$, and $\delta(S, t) = 0$. Use Ito's Lemma to find $d(2S^2)$ in terms of S, α, σ, dt, and dZ(t).

**Solution BIL3.** Here, $C(S, t) = 2S^2$.

Taking partial derivatives, we get $C_t = 0$, $C_S = 4S$ and $C_{SS} = 4$.

The relevant form of Ito's Lemma to be applied here is

$dC(S, t) = \{[\alpha - \delta]SC_S + (1/2)\sigma^2S^2C_{SS} + C_t\}dt + \sigma SC_SdZ(t) =$

$\{4S^2\alpha + (1/2)*4\sigma^2S^2\}dt + 4S^2\sigma dZ(t) =$ $\mathbf{d(2S^2) = \{4S^2\alpha + 2\sigma^2S^2\}dt + 4S^2\sigma dZ(t)}$

**Problem BIL4.** Assume that S follows an arithmetic Brownian motion: $dS(t) = \alpha dt + \sigma dZ(t)$, and $\delta(S, t) = 0$. Use Ito's Lemma to find $d(35S^5 + 2t)$ in terms of S, α, σ, dt, and dZ(t).

Here, $C(S, t) = 35S^5 + 2t$. Taking partial derivatives, we get $C_t = 2$, $C_S = 175S^4$ and $C_{SS} = 700S^3$.

By Ito's Lemma: $dC(S, t) = \{[\alpha(S, t) - \delta(S, t)]C_S + (1/2)\sigma(S, t)^2C_{SS} + C_t\}dt + \sigma(S, t)C_SdZ(t)$

$= [175S^4\alpha + (1/2)\sigma^2700S^3 + 2]dt + 175S^4\sigma dZ(t) =$

$\mathbf{d(35S^5 + 2t) = [175S^4\alpha + 350S^3\sigma^2 + 2]dt + 175S^4\sigma dZ(t)}$

**Problem BIL5.** Assume that S follows a geometric Brownian motion: $dS(t)/S(t) = \alpha dt + \sigma dZ(t)$, and $\delta(S, t) = 0$. Use Ito's Lemma to find $d[\sin(S)]$ in terms of S, α, σ, dt, and dZ(t).

**Solution BIL5.** Here, $C(S, t) = \sin(S)$.

Taking partial derivatives, we get $C_t = 0$, $C_S = \cos(S)$ and $C_{SS} = -\sin(S)$.

The relevant form of Ito's Lemma to be applied here is

$dC(S, t) = \{[\alpha - \delta]SC_S + (1/2)\sigma^2S^2C_{SS} + C_t\}dt + \sigma SC_SdZ(t) =$

$\mathbf{d[\sin(S)] = [S\cos(S)\alpha - (1/2)\sin(S)S^2\sigma^2]dt + S\cos(S)\sigma dZ(t)}$

# Section 63

# Probability Problems Using Arithmetic Brownian Motion

We can perform probability calculations using arithmetic Brownian motion. When X(t) follows an arithmetic Brownian motion, the following equation holds:

$X(t) = X(a) + \alpha(t - a) + \sigma\sqrt{(t-a)}\xi$,

Where a is a specific time, α is the drift factor, σ is the volatility, and ξ is a standard normal variable. X is normally distributed, with mean $X(a) + \alpha(t - a)$ and standard deviation $\sigma\sqrt{(t-a)}$.

**Source:** Schaeffer, Colby. MFE Study Guide (Fall 2007). p. 12.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem PPUABM1.** X follows an arithmetic Brownian motion with a drift factor of 0.35 and a volatility of 0.43. We know that X(4) = 2. At time t = 13, we know that X is normally distributed according to the following equation:

$X(t) = X(a) + \alpha(t - a) + \sigma\sqrt{(t-a)}\xi$. What is the mean of this normal distribution?

**Solution PPUABM1.** If X follows the equation above, the mean of this normal distribution will be $X(a) + \alpha(t - a) = X(4) + 0.35(13 - 4) = 2 + 0.35(13 - 4) =$ **5.15**

**Problem PPUABM2.** X follows an arithmetic Brownian motion with a drift factor of 0.35 and a volatility of 0.43. We know that X(4) = 2. At time t = 13, we know that X is normally distributed according to the following equation:

$X(t) = X(a) + \alpha(t - a) + \sigma\sqrt{(t-a)}\xi$. What is the standard deviation of this normal distribution?

**Solution PPUABM2.** If X follows the equation above, the standard deviation of this normal distribution will be $\sigma\sqrt{(t-a)} = 0.43\sqrt{(13 - 4)} =$ **1.29.**

**Problem PPUABM3.** X follows an arithmetic Brownian motion with a drift factor of 0.35 and a volatility of 0.43. We know that X(4) = 2. What is the probability that X(13) > 9?

**Solution PPUABM3.** We want to find $P(X(13) > 9 \mid X(4) = 2)$. First, we use the formula $X(t) = X(a) + \alpha(t - a) + \sigma\sqrt{(t-a)}\xi$. Here,

$X(13) = X(4) + 0.35(13 - 4) + 0.43\sqrt{(13 - 4)}\xi =$

X(13) = 5.15 + 1.29ξ

Thus, P(X(13) > 9 | X(4) = 2) = P(5.15 + 1.29ξ > 9) = P(1.29ξ > 3.85) =

P(ξ > 2.984496124) = 1 - N(2.984496124) = N(-2.984496124). In MS Excel, we enter "=NormSDist(-2.984496124)" to get our desired probability, **0.001420229**.

**Problem PPUABM4.** Q follows an arithmetic Brownian motion such that Q(45) is 41. The drift factor of this Brownian motion is 0.153, and the volatility is 0.98. What is the probability that Q(61) < 50?

**Solution PPUABM4.** We want to find P(Q(61) < 50 | Q(45) = 41). First, we use the formula Q(t) = Q(a) + α(t - a) + σ√(t-a)ξ. Here,

Q(61) = Q(45) + 0.153(61 - 45) + 0.98√(61- 45)ξ = Q(61) = 43.448 + 3.92ξ

Thus, P(Q(61) < 50 | Q(45) = 41) = P(43.448 + 3.92ξ < 50) = P(3.92ξ < 6.552) =

P(ξ < 1.671428571) = N(1.671428571). In MS Excel, we enter

"=NormSDist(1.671428571)" to get our desired probability, **0.952681472**.

**Problem PPUAMB5.** S follows an arithmetic Brownian motion such that S(30) is 2. The drift factor of this Brownian motion is 0.435, and the volatility is 0.75. What is the probability that S(34) < 0?

**Solution PPUAMB5.** We want to find P(S(34) < 0 | S(30) = 2). First, we use the formula S(t) = S(a) + α(t - a) + σ√(t-a)ξ. Here,

S(34) = S(30) + 0.435(34 - 30) + 0.75√(34-30)ξ = S(34) = 3.74 + 1.5ξ

P(S(34) < 0 | S(30) = 2) = P(3.74 + 1.5ξ < 0) = P(1.5ξ < -3.74) = P(ξ < -2.4933333333) = N(-2.4933333333). In MS Excel, we enter

"=NormSDist(-2.4933333333)" to get our desired probability, **0.006327499**.

# Section 64

# Probability Problems Using Geometric Brownian Motion

We can perform probability calculations using geometric Brownian motion. When X(t) follows a geometric Brownian motion, the following equation holds:

$X(t) = X(a)\exp[(\alpha - 0.5\sigma^2)(t - a) + \sigma\sqrt{(t-a)}\xi]$

Where a is a specific time, α is the drift factor, σ is the volatility, and ξ is a standard normal variable. ln(X(t)/X(a)) is normally distributed, with mean $(\alpha - 0.5\sigma^2)(t - a)$ and standard deviation $\sigma\sqrt{(t-a)}$.

**Sources:** JRaven. Actuarial Outpost Discussion Forum. Post #11. August 31, 2007.

Schaeffer, Colby. MFE Study Guide (Fall 2007). p. 12.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem PPUGBM1.** X follows an geometric Brownian motion with a drift factor of 0.35 and a volatility of 0.43. We know that X(4) = 2. At time t = 13, we know that ln(X(t)/X(a)) is normally distributed according to the following equation:

$X(t) = X(a)\exp[(\alpha - 0.5\sigma^2)(t - a) + \sigma\sqrt{(t-a)}\xi]$. What is the mean of this normal distribution?

**Solution PPUGBM1.** The mean of this normal distribution is $(\alpha - 0.5\sigma^2)(t - a)$, where α = 0.35, σ = 0.43, t = 13, a = 4. Thus, the mean is $(0.35- 0.5*0.43^2)(13 - 4)$ = **2.31795**

**Problem PPUGBM2.** Q follows a geometric Brownian motion such that Q(45) is 41. The drift factor of this Brownian motion is 0.153, and the volatility is 0.98. At time t = 61, we know that ln(X(t)/X(a)) is normally distributed according to the following equation:

$X(t) = X(a)\exp[(\alpha - 0.5\sigma^2)(t - a) + \sigma\sqrt{(t-a)}\xi]$. What is the standard deviation of this normal distribution?

**Solution PPUGBM2.** The standard deviation of this normal distribution is

$\sigma\sqrt{(t-a)} = 0.98\sqrt{(61-45)}$ = **3.92**

**Problem PPUGBM3.** X follows an geometric Brownian motion with a drift factor of 0.35 and a volatility of 0.43. We know that X(4) = 2. What is the probability that X(13) > 9?

**Solution PPUGBM3.** We want to find P(X(13) > 9 │ X(4) = 2). First, we use the formula

$X(t) = X(a)\exp[(\alpha - 0.5\sigma^2)(t - a) + \sigma\sqrt{(t-a)}\xi]$

Here, $X(13) = X(4) \exp[(0.35- 0.5*0.43^2)(13 - 4) + 0.43\sqrt{(13-4)}\xi] =$

$X(13) = 2\exp[2.31795 + 1.29\xi]$

$P(X(13) > 9 \mid X(4) = 2) = P(2\exp[2.31795 + 1.29\xi] > 9) = P(\exp[2.31795 + 1.29\xi] > 4.5) =$

$P(2.31795 + 1.29\xi > \ln(4.5)) = P(2.31795 + 1.29\xi > 1.504077397) = P(1.29\xi > -0.8138726032)$ $= P(\xi > -0.6309089947) = 1 - N(-0.6309089947) = N(0.6309089947)$. In MS Excel, we enter "=NormSDist(0.6309089947)" to get our desired probability, **0.735949985.**

**Problem PPUGBM4.** Q follows a geometric Brownian motion such that Q(45) is 41. The drift factor of this Brownian motion is 0.153, and the volatility is 0.98. What is the probability that Q(61) < 50?

**Solution PPUGBM4.** We want to find P(Q(61) < 50 │ Q(45) = 41). First, we use the formula $Q(t) = Q(a)\exp[(\alpha - 0.5\sigma^2)(t - a) + \sigma\sqrt{(t-a)}\xi]$. Here,

$Q(61) = Q(45)\exp[(0.153 - 0.5*0.98^2)(61- 45) + 0.98\sqrt{(61-45)}\xi] = 41\exp[-5.2352 + 3.92\xi]$

$P(Q(61) < 50 \mid Q(45) = 41) = P(41\exp[-5.2352 + 3.92\xi] < 50) =$

$P(\exp[-5.2352 + 3.92\xi] < 50/41) = P(-5.2352 + 3.92\xi < \ln(50/41)) =$

$P(3.92\xi < \ln(50/41) + 5.2352) = P(3.92\xi < 5.433650939) = P(\xi < 1.386135444) =$ $N(1.386135444)$. In MS Excel, we enter "=NormSDist(1.386135444)" to get our desired probability, **0.917147225**.

**Problem PPUGBM5.** S follows a geometric Brownian motion such that S(30) is 2. The drift factor of this Brownian motion is 0.435, and the volatility is 0.75. What is the probability that S(34) < 1?

**Solution PPUGBM5.** We want to find P(S(34) < 1 │ S(30) = 2). First, we use the formula
$S(t) = S(a)\exp[(\alpha - 0.5\sigma^2)(t - a) + \sigma\sqrt{(t-a)}\xi]$. Here,
$S(34) = S(30)\exp[(\alpha - 0.5\sigma^2)(t - a) + \sigma\sqrt{(t-a)}\xi] =$
$S(30)\exp[(0.435- 0.5*0.75^2)(34 - 30) + 0.75\sqrt{(34-30)}\xi] =$
$2\exp[0.615 + 1.5\xi]$
Thus, $P(S(34) < 1 \mid S(30) = 2) = P(2\exp[0.615 + 1.5\xi] < 1) = P(\exp[0.615 + 1.5\xi] < 0.5) =$
$P(0.615 + 1.5\xi < \ln(0.5)) = P(1.5\xi < \ln(0.5)-0.615) = P(1.5\xi < -1.308147181) =$
$P(\xi < -0.8720981204) = N(-0.8720981204)$. In MS Excel, we enter
"=NormSDist(-0.8720981204)" to get our desired probability, **0.191577426.**

# Section _

# Section 65

# Sharpe Ratios of Assets Following Geometric Brownian Motions

We recall from Section 43 that the Sharpe ratio of any asset is $(\alpha - r)/\sigma$, where

$\alpha$ = expected annual continuously compounded return on the underlying asset (most often a stock).
$r$ = annual continuously compounded risk-free interest rate.
$\sigma$ = annual asset price volatility.

When two non-dividend-paying stocks are perfectly correlated (i.e., are driven by the same dZ), their Sharpe ratios are equal. Thus, if we have two stocks following Geometric Brownian motions so that these equations hold -

$dS_1 = \alpha_1 S_1 dt + \sigma_1 S_1 dZ$
$dS_2 = \alpha_2 S_2 dt + \sigma_2 S_2 dZ$,

- then it is the case that $(\alpha_1 - r)/\sigma_1 = (\alpha_2 - r)/\sigma_2$, where r is the risk-free interest rate, $\alpha$ is the expected return on the stock, and $\sigma$ is stock price volatility - all usually expressed on an annual continuously compounded basis.

**Sources:** McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 20, p. 659.

Schaeffer, Colby. MFE Study Guide (Fall 2007). p. 13.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem SRAFGBM1.** Non-dividend-paying Stock Y follows a geometric Brownian motion such that $dY/Y = 0.5dt + 0.354dZ$. The annual continuously compounded risk-free interest rate is 0.034. What is the Sharpe ratio of stock Y?

**Solution SRAFGBM1.** The Sharpe ratio is $(\alpha - r)/\sigma$, where $\alpha = 0.5$, $r = 0.034$, and $\sigma = 0.354$, so the Sharpe ratio is $(\alpha - r)/\sigma = (0.5 - 0.034)/0.354 =$ **1.316384181**

**Problem SRAFGBM2.** Non-dividend-paying Stocks W and X are perfectly correlated and follow these geometric Brownian motions:
$dW/W = 0.453dt + 0.884dZ$.

$dX/X = 0.3567dt + HdZ$.
The annual continuously compounded risk-free interest rate is 0.1. What is H?

**Solution SRAFGBM2.** Let W be $S_1$ and X be $S_2$. We know that
$(\alpha_1 - r)/\sigma_1 = (\alpha_2 - r)/\sigma_2$, where $\alpha_1 = 0.453$, $r = 0.1$, $\sigma_1 = 0.884$, $\alpha_2 = 0.3567$, and $\sigma_2 = H$ Thus,
$(0.453 - 0.1)/0.884 = (0.3567 - 0.1)/H$ and
$0.399321267H = 0.2567$, so $\mathbf{H = 0.6428407932}$

**Problem SRAFGBM3.** Non-dividend-paying Stocks U and V are perfectly correlated and follow these geometric Brownian motions:

$dU/U = 0.1111dt + 0.3443dZ$.
$dV/V = Gdt + 0.53dZ$.

The annual continuously compounded risk-free interest rate is 0.0231. What is G?

**Solution SRAFGBM3.** Let U be $S_1$ and V be $S_2$. We know that
$(\alpha_1 - r)/\sigma_1 = (\alpha_2 - r)/\sigma_2$, where $\alpha_1 = 0.1111$, $r = 0.0231$, $\sigma_1 = 0.3443$, $\alpha_2 = G$, and $\sigma_2 = 0.53$ Thus,
$(\alpha_1 - r)/\sigma_1 = (\alpha_2 - r)/\sigma_2$ implies that
$\sigma_2(\alpha_1 - r)/\sigma_1 = (\alpha_2 - r)$ and $\alpha_2 = \sigma_2(\alpha_1 - r)/\sigma_1 + r = 0.53(0.1111 - 0.0231)/0.3443 + 0.0231 = \mathbf{G = 0.1585632588}$

**Problem SRAFGBM4.** Non-dividend-paying Stocks F and G are perfectly correlated and follow these geometric Brownian motions:
$dF/F = 0.152dt + 0.251dZ$.
$dG/G = 0.2dt + 0.351dZ$.
What is the annual continuously compounded risk-free interest rate?

**Solution SRAFGBM4.** Let F be $S_1$ and G be $S_2$. We know that
$(\alpha_1 - r)/\sigma_1 = (\alpha_2 - r)/\sigma_2$, where $\alpha_1 = 0.152$, $\sigma_1 = 0.251$, $\alpha_2 = 0.2$, and $\sigma_2 = 0.351$.
Thus, $(0.152 - r)/0.251 = (0.2 - r)/0.351$ and
$1.398406375(0.152 - r) = 0.2 - r$
$0.2125577689 - 1.398406375r = 0.2 - r$
$0.398406375r = 0.012557689$
$\mathbf{r = 0.03152}$

**Problem SRAFGBM5.** Two non-dividend-paying stocks $S_1$ and $S_2$ are perfectly correlated such that $\sigma_1 = 6\sigma_2$ and $\alpha_1 = 0.2$. What is the smallest value of $\alpha_2$ required if the annual continuously compounded risk-free interest rate is to be nonnegative?

**Solution SRAFGBM5.** We use the formula $(\alpha_1 - r)/\sigma_1 = (\alpha_2 - r)/\sigma_2$. By our given information,
$(\alpha_1 - r)/6\sigma_2 = (\alpha_2 - r)/\sigma_2$, so $(\alpha_1 - r)/6 = (\alpha_2 - r)$ and $\alpha_1/6 - r/6 = \alpha_2 - r$

Thus, $\alpha_1/6 - \alpha_2 = -5r/6$ so $\alpha_2 - \alpha_1/6 = 5r/6$. We see that r gets larger as $\alpha_2$ gets larger.

The smallest value of $\alpha_2$ required if r is nonnegative is the value of $\alpha_2$ such that r = 0. Thus, it is the value such that $\alpha_2 - \alpha_1/6 = 5*0/6 = 0$ and $\alpha_2 = \alpha_1/6$. Since we are given that $\alpha_1 = 0.2$, the desired smallest value of $\alpha_2$ is 0.2/6 = **0.03333333333**

# Section 66

# Another Form of Ito's Lemma for Geometric Brownian Motion

Section 62 gave a form of Ito's Lemma for dC(S, t) in terms of dt and dZ(t). It also helps to have a form of Ito's Lemma in terms of dS and dt. This form of Ito's Lemma applies where S(t) follows a geometric Brownian motion.

$dC(S, t) = C_S dS + (1/2)C_{SS}(dS)^2 + C_t dt$

In this section, we will focus on learning this version of Ito's Lemma and applying it to basic situations. In the following sections, we will put the groundwork we have built up in Sections 61-66 to good use in solving more advanced problems.

**Sources:** McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 20, p. 664.

Schaeffer, Colby. MFE Study Guide (Fall 2007). p. 12.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

For all the problems below, S(t) follows a geometric Brownian motion.

**Problem AFIL1.** Given $C(S, t) = 25St + 3412t^2 + 43\sin(S)$, use Ito's Lemma to find dC(S, t) in terms of dS, dt, S, and t.

**Solution AFIL1.** By Ito's Lemma, $dC(S, t) = C_S dS + (1/2)C_{SS}(dS)^2 + C_t dt$

Here, $C_S = 25t + 43\cos(S)$, $C_{SS} = -43\sin(S)$, and $C_t = 25S + 6824t$

Thus, $dC(S, t) = [25t + 43\cos(S)]dS + (1/2)[-43\sin(S)](dS)^2 + [25S + 6824t]dt =$

$\mathbf{dC(S, t) = [25t + 43\cos(S)]dS + (-43/2)\sin(S)](dS)^2 + [25S + 6824t]dt}$

**Problem AFIL2.** Given $G(S, t) = S^4 + S^3 + S^2 + S + 1 + t$, use Ito's Lemma to find dG(S, t) in terms of dS, dt, S, and t.

**Solution AFIL2.** By Ito's Lemma, $dG(S, t) = G_S dS + (1/2)G_{SS}(dS)^2 + G_t dt$

Here, $G_S = 4S^3 + 3S^2 + 2S + 1$, $G_{SS} = 12S^2 + 6S + 2$ and $G_t = 1$

Thus, $dG(S, t) = [4S^3 + 3S^2 + 2S + 1]dS + (1/2)[12S^2 + 6S + 2](dS)^2 + dt$

$\mathbf{dG(S, t) = [4S^3 + 3S^2 + 2S + 1]dS + [6S^2 + 3S + 1](dS)^2 + dt}$

**Problem AFIL3.** Given $Q(S, t) = \ln(S^4) + 2e^t$, use Ito's Lemma to find $dQ(S, t)$ in terms of dS, dt, S, and t.

**Solution AFIL3.** By Ito's Lemma, $dQ(S, t) = Q_S dS + (1/2)Q_{SS}(dS)^2 + Q_t dt$

Here, $Q_S = 4S^3/S^4 = 4S^{-1}$, $Q_{SS} = -4S^{-2}$,and $Q_t = 2e^t$.

Thus, $dQ(S, t) = 4S^{-1}dS + (1/2)[-4S^{-2}](dS)^2 + 2e^t dt$

$\mathbf{dQ(S, t) = 4S^{-1}dS + [-2S^{-2}](dS)^2 + 2e^t dt}$

**Problem AFIL4.** Given $H(S, t) = S^a$, where a is a constant, use Ito's Lemma to find $dH(S, t)$ in terms of dS, dt, S, and t.

**Solution AFIL4.** By Ito's Lemma, $dH(S, t) = H_S dS + (1/2)H_{SS}(dS)^2 + H_t dt$

Here, $H_S = aS^{a-1}$, $H_{SS} = a(a-1)S^{a-2}$,$H_t = 0$

Thus, $\mathbf{dH(S, t) = aS^{a-1}dS + (1/2)a(a-1)S^{a-2}(dS)^2}$

**Problem AFIL5.** Given $F(S, t) = S^2 + S$, use Ito's Lemma to find $dF(S, t)$ when $dS = 3$, $dt = 1$, and $S = 99$.

**Solution AFIL5.** By Ito's Lemma, $dF(S, t) = F_S dS + (1/2)F_{SS}(dS)^2 + F_t dt$

Here, $F_S = 2S + 1$, $F_{SS} = 2$,$F_t = 0$.

Thus, $dF(S, t) = (2S + 1)dS + (1/2)2(dS)^2 = (2S + 1)dS + (dS)^2 = (2*99 + 1)*3 + 9 =$

$\mathbf{dF(S, t) = 606}$

# Section 67

# Multiplication Rules and Exam-Style Questions for Brownian Motion and Ito's Lemma

In this section, we will finally put the tools we learned in Sections 59-66 to use in solving some challenging exam-style questions on Brownian motion and Ito's Lemma. But first, there are a few convenient and easily memorized rules left to learn.

Sometimes when doing problems involving Ito's Lemma and Brownian motion, you will encounter situations where you need to multiply dZ by itself, dt by itself, or dZ by dt. The following *multiplication rules* make this multiplication a matter of routine:

$(dt)^2 = 0$

$dt * dZ = 0$

$(dZ)^2 = dt$

$dZ * dZ' = \rho * dt$, where Z' is a different Brownian motion from Z, and ρ is the correlation between the underlying assets driven by these different Brownian motions.

**Sources:** McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 20, pp. 658-659.

Schaeffer, Colby. MFE Study Guide (Fall 2007). p. 12.

Some of the problems in this section were designed to be similar to problems from past versions of Exam 3F / Exam MFE. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem MRESQBMIL1.** For asset S, which follows a Brownian motion Z, find the simplest possible equivalent expression to (34dZ + 45dt)(42dZ + 3dt).

**Solution MRESQBMIL1.** Did you try to expand the expression (34dZ + 45dt)(42dZ + 3dt)? If you did, then know that this was unnecessary. We know by our multiplication rules that $(dt)^2 = 0$ and dt * dZ = 0, so only the $(dZ)^2$ term of the expansion will have a nonzero value. So we only need to multiply 34 by 42 to get the dt coefficient, which is 1428. Thus, the desired answer is **1428dt.**

**Problem MRESQBMIL2.** Asset X follows a Brownian motion Z, while Asset Y follows a geometric Brownian motion Z'. The assets have a correlation ρ = 0.365. Find the simplest possible equivalent expression to (95dZ + 424dZ')(2dZ + 241dt).

**Solution MRESQBMIL2.** Again, a thorough expansion of the given expression is unnecessary. However, it is important to note that here, two terms will have nonzero values - the $(dZ)^2$ term and the dZ'*dZ term, so we will need to find their coefficients. $(dZ)^2$ = dt, and the relevant coefficient is 95*2 = 190. dZ'*dZ = ρ = 0.365, which we multiply by the product of the coefficients 424 and 2, which is 848. So 0.365*848 = 309.52 - a constant.

We add 190dt to 296.8 to get our desired answer: **190dt + 309.52**

**Problem MRESQBMIL3.** Asset F follows a Brownian motion Z, while Asset G follows a Brownian motion Z'. The assets have a correlation ρ = 0.

Find the simplest possible equivalent expression to (4dZ + dt)(2dZ + dt)(dZ' + dt).

**Solution MRESQBMIL3.** We can first recognize that the expression (4dZ + dt)(2dZ + dt) can, without expanding, be simplified to 4dZ*2dZ = $8(dZ)^2$ = 8dt, so the given expression can be simplified to 8dt(dZ' + dt), which is equal to zero, since dZ'*dt and $(dt)^2 = 0$. Thus, the desired answer is **0.**

**Problem MRESQBMIL4.**

**Similar to Question 12 from the Society of Actuaries' May 2007 Exam MFE:**

You are given the following information.

(i) S(t) is the value of one Yap piece of stone (YPS) in terms of Golden Hexagons (GH) at time t.

(ii) dS(t)/S(t) = 0.7dt + 0.25dZ(t)

(iii) The YPS-demoninated annual continuously compounded risk-free interest rate r* is 0.22.

(iv) The GH-demoninated annual continuously compounded risk-free interest rate r is 0.17.

(v) Q(t) = S(t)exp[(r-r*)(T-t)] is the forward price in Golden Hexagons per YPS, and T is the maturity time of the currency forward contract.

Use Ito's Lemma to find a stochastic differential equation satisfied by G(t) such that

$dQ(t) = Q(t)[xdt + ydZ(t)]$, where x and y are constants.

**Solution MRESQBMIL4.** By Ito's Lemma, $dQ(S, t) = Q_S dS + (1/2)Q_{SS}(dS)^2 + Q_t dt$

Here, $Q_S = \exp[(r-r^*)(T-t)]$, $Q_{SS} = 0$, and $Q_t = -(r - r^*)S(t)\exp[(r-r^*)(T-t)] = -(r - r^*)Q(t)$

By (ii), $[0.7dt + 0.25dZ(t)]S(t) = dS(t)$, so $Q_S dS = \exp[(r-r^*)(T-t)][0.7dt + 0.25dZ(t)]S(t) =$

$[0.7dt + 0.25dZ(t)]S(t)\exp[(r-r^*)(T-t)]$, and $Q_S dS = [0.7dt + 0.25dZ(t)]Q(t)$

Since $Q_{SS} = 0$, $(1/2)Q_{SS}(dS)^2 = 0$.
$Q_t dt = -(r - r^*)Q(t)dt = -(0.17 - 0.22)Q(t)dt = [0.05dt]Q(t)$

So $dQ(S, t) = [0.7dt + 0.25dZ(t)]Q(t) + [0.05dt]Q(t) =$ **$dQ(S, t) = Q(t)[0.75dt + 0.25dZ(t)]$**

**Problem MRESQBMIL5.**

**Similar to Question 18 from the Society of Actuaries' May 2007 Exam MFE:**

Two assets, B and W, are driven by the same Brownian motion Z. They satisfy the following equations:

$dB(t)/B(t) = 0.44dt + 0.29dZ(t)$

$dW(t)/W(t) = Cdt + KdZ(t)$, where C and K are constants

You are given the following information:

(i) The annual continuously compounded risk-free interest rate is 0.09

(ii) $d(\ln[W(t)]) = 0.33dt + \sigma dZ(t)$

(iii) $\sigma < 0.37$

Find C.

**Solution MRESQBMIL5.** This problem involves applications of Ito's Lemma and the Sharpe Ratio.

From Section 65, we know that $(\alpha_1 - r)/\sigma_1 = (\alpha_2 - r)/\sigma_2$, where B is Asset 1 and W is Asset 2. It is given that $r = 0.09$, $\alpha_1 = 0.44$, $\sigma_1 = 0.29$, $\alpha_2 = C$, $\sigma_2 = K$. Thus, $(0.44-0.09)/0.29 = (C - 0.09)/K$ and $1.206896552 = (C - 0.09)/K$.

We note that $d(\ln[W(t)]) = 0.06dt + \sigma dZ(t)$ must have had Ito's Lemma applied to it. We try to see what that application must have looked like. Let $Q(w, t) = \ln[W(t)]$

By Ito's Lemma, $dQ(W, t) = Q_W dW + (1/2)Q_{WW}(dW)^2 + Q_t dt$.

Here, $Q_W = (1/W(t))$, $Q_{WW} = (1/W(t)) = (-1/W(t)^2)$, $Q_t = 0$.

Thus, $dQ(W, t) = (1/W(t))dW + (1/2)(-1/W(t)^2)(dW)^2$.

But we are given that $dQ(W, t) = 0.33dt + \sigma dZ(t)$

and that $dW(t)/W(t) = Cdt + KdZ(t)$, so $dW = W(t)[Cdt + KdZ(t)]$

Now, we make the following substitution for dW:

$dQ(W, t) = (1/W(t))W(t)[Cdt + KdZ(t)] + (1/2)(-1/W(t)^2)(W(t)[Cdt + KdZ(t)])^2$

$dQ(W, t) = Cdt + KdZ(t) +(-1/2)[Cdt + KdZ(t)]^2$

$dQ(W, t) = Cdt + KdZ(t) +(-1/2)[C^2dt^2 + 2CKdtdZ(t) + K^2dZ(t)^2]$

Here, we will need to apply our multiplication rules. The expression is thereby dramatically simplified:

$dQ(W, t) = Cdt + KdZ(t) +(-1/2)[C^2 0 + 2CK*0 + K^2dt]$

$dQ(W, t) = Cdt + (-1/2)K^2dt + KdZ(t)$

We note that $K = \sigma$ and $C - (1/2)K^2 = 0.33$, since $dQ(W, t) = 0.33dt + \sigma dZ(t)$.

From our analysis of the Sharpe ratios, we know that $1.206896552 = (C - 0.09)/K$ and so

$1.206896552K = C - 0.09$ and $C = 1.206896552K + 0.09$

Thus, $- (1/2)K^2 + 1.206896552K + 0.09 = 0.33$ and $- (1/2)K^2 + 1.206896552K - 0.24 = 0$.

By the quadratic formula, $K = 0.2186661703$ or $K = 2.195126934$. But $K = \sigma$, and we are given that $\sigma < 0.37$, so $K = 0.2186661703$ and so $C = 1.206896552K + 0.09 =$

$1.206896552*0.2186661703 + 0.09 =$ **$C = 0.353907447$**

# Section 68

# Conceptual Questions on Brownian Motion

The problems in this section will prepare you to solve problems 11 and 12 of the Society of Actuaries' Sample MFE Questions and Solutions. It will also equip you to deal with similar multiple choice questions, which require conceptual analysis of Brownian motion instead of extensive uses of computational techniques.

It is useful to learn some additional information about Brownian motion.

The Black-Scholes option pricing framework is based on the assumption that the underlying asset follows a geometric Brownian motion:
$dS(t)/S(t) = \alpha dt + \sigma dZ(t)$.

When S(t) follows a geometric Brownian motion, ln[S(t)] follows an arithmetic Brownian motion for all $t \geq 0$.

For any arithmetic Brownian motion X(t), the random variable [X(t + h) - X(t)] is normally distributed for all $t \geq 0$, $h > 0$, and has a mean of $X(t) + \alpha h$ and a variance of $\sigma^2 h$.

Let Z(t) be the Standard Brownian Motion. (That is, dZ(t) has a variance of 1 per unit time and Z(t + s) - Z(t) is normally distributed with mean 0 and variance s.) Then for any arithmetic Brownian motion {X(t)} where $dX(t) = \alpha dt + \sigma dZ(t)$, it is the case that

$[X(t + h) - X(t)] = \sigma[Z(t + h) - Z(t)]$

The **quadratic variation** of a Brownian process is expressible as

${}_{n\to\infty}\lim_{i=1}{}^{n}\Sigma(Z[ih] - Z[(i-1)h])^2 = T$ from time 0 to time T.

Here, Z is the Standard Brownian Motion.

The quadratic variation is "the sum of the squared increments to the process" (McDonald 2006, p. 652).

For Z(t), the Standard Brownian Motion, $Var[dZ(t) | Z(t)]$, the conditional variance of dZ(t) when Z(t) is known, is dt. This follows from the multiplication rules discussed in Section 67.

Remember that constants within a variance expression can be taken out of the variance expression, but to do so you will need to square the constant when you take it outside. For instance, $Var[\sigma dZ(t) | Z(t)] = \sigma^2 Var[dZ(t) | Z(t)]$

Furthermore, for any geometric Brownian motion S(t), $Var[S(t + dt) | S(t)] = Var[dS(t) | S(t)]$

**Sources:** McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 20, pp. 649-654.

Society of Actuaries. Sample MFE Questions and Solutions. Questions 11-12.

**Problem CQBM1.** Which of these expressions will follow an arithmetic Brownian motion? More than one correct answer is possible.

(a) ln[S(t)], $t \geq 0$, where S(t) follows an arithmetic Brownian motion.

(b) ln[S(t)], $t \geq 0$, where S(t) follows a geometric Brownian motion.

(c) ln[$S(t)^2$], $t \geq 0$, where S(t) follows a geometric Brownian motion.

(d) ln[$S(t)^2$], $t \geq 0$, where $S(t)^2$ follows a geometric Brownian motion.

(e) S(t), where $dS(t) = \alpha dt + \sigma dZ(t)$.

(f) S(t), where $dS(t)/S(t) = \alpha dt + \sigma dZ(t)$.

**Solution CQBM1.** The rule for determining whether an expression in terms of S(t) follows an arithmetic Brownian motion is that, in order to do so, the expression must be a natural logarithm of some expression that follows a geometric Brownian motion. If S(t) follows a geometric Brownian motion, then ln[S(t)] follows an arithmetic Brownian motion, and (b) is correct. If $S(t)^2$ follows a geometric Brownian motion, then ln[$S(t)^2$] follows an arithmetic Brownian motion, and (d) is correct. (e) simply indicates that S(t) follows the formula for arithmetic Brownian motion.
(c) is correct because $\ln(S(t)^2) = 2*\ln(S(t))$. Since S(t) follows a geometric Brownian motion, ln(S(t)) follows an arithmetic Brownian motion. An arithmetic Brownian motion multiplied by a constant, such as 2, is still an arithmetic Brownian motion .

Thus, **(b), (c), (d), and (e) are correct answers.**

**Problem CQBM2.** Given an arithmetic Brownian motion Y(t) where $dY(t) = 0.34dt + 0.124dZ(t)$, find Var[X(t + 30) - X(t)].

**Solution CQBM2.** For any arithmetic Brownian motion X(t), the random variable [X(t + h) - X(t)] is normally distributed for all $t \geq 0$, $h > 0$, and has a mean of $X(t) + \alpha h$ and a variance of $\sigma^2 h$. Here, $\sigma = 0.124$, and h = 30. Thus, Var[X(t + 30) - X(t)] = $0.124^2 30$ = **0.46128**

**Problem CQBM3.** Given an arithmetic Brownian motion Y(t) where $dY(t) = 0.662dt + 0.344dZ(t)$, find

$_{n\to\infty}\lim_{i=1}{}^{n}\Sigma(Y[ih] - Y[(i-1)h])^2$, where the time period T under consideration is 33 units.

**Solution CQBM3.** Since Y is an arithmetic Brownian motion, it is the case that

[Y(t + h) - Y(t)] = σ[Z(t + h) - Z(t)], where Z is the standard Brownian motion. Here, T = 33 and

σ = 0.344. We can thus deduce that

$_{n\to\infty}\lim_{i=1}{}^{n}\Sigma(Y[ih] - Y[(i-1)h])^2 = {}_{n\to\infty}\lim_{i=1}{}^{n}\Sigma(\sigma Z[ih] - \sigma Z[(i-1)h])^2 =$

$\sigma^2{}_{n\to\infty}\lim_{i=1}{}^{n}\Sigma(Z[ih] - Z[(i-1)h])^2 = \sigma^2 T = 0.3444^2 33 =$ **3.91417488**

**Problem CQBM4.** S(t) follows a geometric Brownian motion where dS(t)/S(t) = 0.36dt + 0.66dZ(t). Find Var[dS(t)/S(t) | S(t)], the variance of dS(t)/S(t) when S(t) is known.

**Solution CQBM4.**

By the equation for our given Brownian motion,

Var[dS(t)/S(t) | S(t)] = Var[0.36dt + 0.66dZ(t) | S(t)] = Var[0.66dZ(t) | S(t)]

You might be wondering where the 0.36dt term went. Why was its elimination legitimate? We recall from Section 64 that the following expression holds for any Geometric Brownian motion: $X(t) = X(a)\exp[(\alpha - 0.5\sigma^2)(t - a) + \sigma\sqrt{(t-a)}\xi]$, where ξ is a standard normal random variable and the standard deviation is $\sigma\sqrt{(t-a)}$, so the variance is $\sigma^2(t-a)$. The variance, we note has nothing to do with the term involving α and everything to do with the term involving σ and the standard normal random variable. We also note that when the variance depends solely on the term dZ(t), then our answer will not change whether S(t) or Z(t) is known. Thus, we can assert the following equality: Var[0.66dZ(t) | S(t)] = Var[0.66dZ(t) | Z(t)]. Now we can take 0.66 out of the expression: Var[0.66dZ(t) | Z(t)] = $0.66^2$Var[dZ(t) | Z(t)]. We know that Var[dZ(t) | Z(t)] = dt, so our desired answer is $0.66^2$Var[dZ(t) | Z(t)] = **0.4356dt**.

**Problem CQBM5** S(t) follows a geometric Brownian motion where dS(t)/S(t) = 0.4365dt + 0.226dZ(t). Find Var[S(t + dt) | S(t)], the variance of dS(t)/S(t) when S(t) is known and is equal to 75.

**Solution CQBM5.**

We use the quality Var[S(t + dt) | S(t)] = Var[dS(t) | S(t)]

By the procedure in Solution CQBM4, we can do the following:

Var[dS(t) | S(t)] = Var[S(t)(0.4365dt + 0.226dZ(t)) | S(t)] =

Var[75(0.4365dt + 0.226dZ(t)) | S(t)] = Var[75*0.226dZ(t) | S(t)] = Var[75*0.226dZ(t) | Z(t)] =

Var[16.95dZ(t) | Z(t)] = $16.95^2$Var[dZ(t) | Z(t)] = $16.95^2$dt = **287.3025dt**.

# Section 69

# More Exam-Style Questions on Ito's Lemma and Brownian Motion

When you are given a geometric Brownian motion {X(t)} in terms of {Z(t)} - some other Brownian motion - and asked to find the drift of this motion, or whether this motion has zero drift, the suggested procedure is to apply Ito's Lemma to find dX(t) and express dX(t) as some Kdt + JdZ(t). If K = 0, then {X(t)} has zero drift. If K is not zero, then {X(t)} has drift Kdt.

By Ito's Lemma, $dX(Z, t) = X_Z dZ + (1/2)X_{ZZ}(dZ)^2 + X_t dt$, but $(dZ)^2 = dt$ by our multiplication rules from Section 67, so $dX(Z, t) = X_Z dZ + [(1/2)X_{ZZ} + X_t]dt$ and

$dX(t) = [(1/2)X_{ZZ} + X_t]dt + X_Z dZ(t)$

Some of the problems in this section were designed to be similar to problems from past versions of Exam 3F / Exam MFE. They use original exam questions as their inspiration – and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem MESQILBM1.**

**Similar to Question 13 from the Society of Actuaries' Sample MFE Questions and Solutions:**

C and W are two non-dividend-paying assets, both driven by the Brownian motion {Z(t)}. The following equations hold:

dC(t)/C(t) = 0.23dt + 0.46dZ(t)

dW(t)/W(t) = Adt + BdZ(t), where A and B are constants.

You also know that

(i) d[ln(W(t)] = μdt + 0.363dZ(t)

(ii) The annual continuously compounded risk-free interest rate is 0.06.

Find A.

**Solution MESQILBM1.** This problem involves applications of Ito's Lemma and the Sharpe Ratio.

From Section 65, we know that $(\alpha_1 - r)/\sigma_1 = (\alpha_2 - r)/\sigma_2$, where C is Asset 1 and W is Asset 2. It is given that r = 0.06, $\alpha_1 = 0.23$, $\sigma_1 = 0.46$, $\alpha_2 = A$, $\sigma_2 = B$. Thus, (0.23-0.06)/0.46 = (A - 0.06)/B and 0.3695652174 = (A - 0.06)/B.

We note that d(ln[W(t)]) = μdt + 0.363dZ(t) must have had Ito's Lemma applied to it. We try to see what that application must have looked like. Let Q(w, t) = ln[W(t)]

By Ito's Lemma, $dQ(W, t) = Q_W dW + (1/2)Q_{WW}(dW)^2 + Q_t dt$.

Here, $Q_W = (1/W(t))$, $Q_{WW} = (1/W(t)) = (-1/W(t)^2)$, $Q_t = 0$.

Thus, $dQ(W, t) = (1/W(t))dW + (1/2)(-1/W(t)^2)(dW)^2$.

But we are given that dQ(W, t) = μdt + 0.363dZ(t)

and that dW(t)/W(t) = Adt + BdZ(t), so dW = W(t)

Now, we make the following substitution for dW:

$dQ(W, t) = (1/W(t))W(t) )^2)(W(t))^2$

$dQ(W, t) = {}^2$

$dQ(W, t) = Adt + BdZ(t) +(-1/2)2dt^2 + 2ABdtdZ(t) + B^2dZ(t)^2]$

Here, we will need to apply our multiplication rules. The expression is thereby dramatically simplified:

$dQ(W, t) = Adt + BdZ(t) +(-1/2)20 + 2AB*0 + B^2dt]$

$dQ(W, t) = Adt + (-1/2)B^2dt + BdZ(t) = \mu dt + 0.363dZ(t)$

Thus, because the dZ(t) term is both B and 0.363, we know that B = 0.363.
We can use this in the equality we derived from the Sharpe ratios:

0.3695652174 = (A - 0.06)/B

0.3695652174 = (A - 0.06)/0.363

0.3695652174*0.363 + 0.06 = **A = 0.1941521739**

**Note:** The reasoning for this solution and for Solution MRESQBMIL5 in Section 67 can save you a lot of intermediate work when encountering problems of this type in the future. Here is a conclusion that we can draw:

**Rule 69.1.** Whenever we are given

dW(t)/W(t) = Adt + BdZ(t), where A and B are constants.

and d[ln(W(t)] = μdt + σdZ(t), it is the case that B = σ. When the value of σ is given, this is especially convenient, because - when we are also given r and some specific Brownian motion for another asset C, we can use Sharpe ratios, knowing the expression for $\sigma_2$, to arrive at the value of A without needing to apply Ito's Lemma every time. Let us try this approach in the next problem.

**Problem MESQILBM2.**

**Similar to Question 13 from the Society of Actuaries' Sample MFE Questions and Solutions:**

X and Y are two non-dividend-paying assets, both driven by the Brownian motion {Z(t)}. The following equations hold:

dX(t)/X(t) = 0.333dt + 0.366dZ(t)

dY(t)/Y(t) = Adt + BdZ(t), where A and B are constants.

You also know that

(i) d[ln(Y(t)] = μdt + 0.998dZ(t)

(ii) The annual continuously compounded risk-free interest rate is 0.22.

Find A.

**Solution MESQILBM2.** By Rule 69.1, we know that B = 0.998.

Now we use Sharpe Ratios.

From Section 65, we know that $(\alpha_1 - r)/\sigma_1 = (\alpha_2 - r)/\sigma_2$, where X is Asset 1 and Y is Asset 2. It is given that r = 0.22, $\alpha_1 = 0.333$, $\sigma_1 = 0.366$, $\alpha_2 = A$, $\sigma_2 = 0.998$.

Thus, $\sigma_2(\alpha_1 - r)/\sigma_1 = (\alpha_2 - r)$ and $\alpha_2 = \sigma_2(\alpha_1 - r)/\sigma_1 + r$,

so $\alpha_2$ = A = 0.998(0.333 - 0.22)/0.366 + 0.22 = **A = 0.5281256831**

Problems 3-5 will prepare you to answer Question 14 of the Society of Actuaries' Sample MFE Questions and Solutions.

**Problem MESQILBM3.** Given that {Z(t)} is some Brownian motion, you know that X(t) = 19Z(t) + 96. Find the drift of the Brownian process {X(t)}.

**Solution MESQILBM3.** We use the form of Ito's Lemma developed in this section to find
$dX(t) = [(1/2)X_{ZZ} + X_t]dt + X_Z dZ(t)$

Here, $X_Z = 19$, $X_{ZZ} = 0$, and $X_t = 0$. Thus,
$dX(t) = [0X_{ZZ} + 0]dt + 19dZ(t)$, so $dX(t) = 19dZ(t)$, and the dt term is 0.

Thus, **{X(t)} has a drift of 0**.

**Problem MESQILBM4.** Given that {Z(t)} is some Brownian motion, you know that $Y(t) = 14[Z(t)^2] - 8t$ Find the drift of the Brownian process {Y(t)}.

**Solution MESQILBM4.** We use the form of Ito's Lemma developed in this section to find

$dY(t) = [(1/2)Y_{ZZ} + Y_t]dt + Y_Z dZ(t)$

Here, $Y_Z = 28Z(t)$, $Y_{ZZ} = 28$, and $Y_t = -8$. Thus,

$dY(t) = [(1/2)28 - 8]dt + 28Z(t)dZ(t)$ and so the dt term is $[(1/2)28 - 8]dt = 6dt$, and the drift of {Y(t)} is **6dt.**

**Problem MESQILBM5.** Given that {F(t)} is some Brownian motion, you know that

$F(t) = 4t^2Z(t) - 10{}_0{}^t\int sZ(s)ds$. Find the drift of the Brownian process {F(t)}.

**Solution MESQILBM5.**

The differential of F(t) can be expressed as the differential of $4t^2Z(t) - 10{}_0{}^t\int sZ(s)ds$ or, by the addition rule for derivatives, as $d[4t^2Z(t)] - d[10{}_0{}^t\int sZ(s)ds]$. Thus,
$dF(t) = d[4t^2Z(t)] - d[10{}_0{}^t\int sZ(s)ds]$. We try to find
$d[4t^2Z(t)]$ by Ito's Lemma. Let $4t^2Z(t) = Q(t)$

Then $dQ(t) = [(1/2)Q_{ZZ} + Q_t]dt + Q_Z dZ(t)$, where
$Q_Z = 4t^2$, $Q_{ZZ} = 0$, and $Q_t = 8tZ(t)$

Thus, $dQ(t) = 8tZ(t)dt + 4t^2dZ(t)$

Now we try to find $d[10{}_0{}^t\int sZ(s)ds]$. This is the differential of an integral in terms of t, which is simply 10tZ(t)dt. Thus,
$dF(t) = d[4t^2Z(t)] - d[10{}_0{}^t\int sZ(s)ds] = 4t^2dZ(t) + 8tZ(t)dt - 10tZ(t)dt$ and
$dF(t) = 4t^2dZ(t) - 2tZ(t)dt$. Thus, the drift of {F(t)} is **-2tZ(t)dt.**

# Section 70

# The Vasicek Interest-Rate Model

The Vasicek interest-rate model is based on an arithmetic Brownian motion with mean reversion:

$dr = a(b - r)dt + \sigma dz$

Thus, the interest rate follows an Ornstein-Uhlenbeck process.

The way to solve this equation for the price P[t, T, r(t)] at time t of $1 paid with certainty at time T, where $t \leq T$ and P(T, T, r) = 1, is

$P[t, T, r(t)] = A(t, T)e^{-B(t, T)r(t)}$, where

$A(t, T) = \exp[\bar{r}(B(t, T) + t - T) - B(t, T)^2\sigma^2/4a]$

$B(t, T) = (1 - e^{-a(T-t)})/a$, and

$\bar{r} = b + \sigma\varphi/a - 0.5\sigma^2/a^2$.

**Meanings of some variables:**

$\varphi$ = Sharpe ratio (assumed to be constant).

$\bar{r}$ = yield to maturity on an infinitely lived bond.

$\sigma$ = volatility factor.

a = drift factor.

b = the mean around which mean reversion occurs.

r = the short-term interest rate.

Often, the following expression for P[t, T, r(t)] in the Vasicek model will be more useful:

$P[t, T, r(t)] = \exp(-[\alpha(T - t) + \beta(T - t)r])$, where $\alpha(T - t)$ and $\beta(T - t)$ are constants that stay the same whenever the difference between T and t is the same, even if the values of T and t are different.

**Sources:** McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 24, pp. 786-787.

There are very few practice problems available on the Vasicek model, and most of the problems presuppose full knowledge of the model - instead of teaching the model step by step. We shall attempt to remedy this shortcoming in this section.

Some of the problems in this section were designed to be similar to problems from past versions of Exam 3F / Exam MFE. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem VIRM1.** A Vasicek model holds for an interest rate r such that

dr = 0.34(0.09 - r)dt + 0.26dz. The relevant Sharpe ratio φ is 0.88. Find the yield to maturity on an infinitely lived bond, $r^-$.

**Solution VIRM1.** We use the formula $r^- = b + \sigma\varphi/a - 0.5\sigma^2/a^2$, where b = 0.09, a = 0.34, σ = 0.26, and φ = 0.88. Thus, $r^- = 0.09 + 0.26(0.88/0.34) - 0.5*0.26^2/0.34^2 =$

**$r^-$ = 0.4705536332**

**Problem VIRM2.** A Vasicek model holds for an interest rate r such that

dr = 0.34(0.09 - r)dt + 0.26dz. The relevant Sharpe ratio φ is 0.88. Find B(2, 3) in this Vasicek model.

**Solution VIRM2.** We use the formula $B(t, T) = (1 - e^{-a(T-t)})/a$, where T = 3, t = 2, and a = 0.34. Thus, $B(2, 3) = (1 - e^{-0.34(3-2)})/0.34 =$ **B(2, 3) = 0.8477343448**

**Problem VIRM3.** A Vasicek model holds for an interest rate r such that

dr = 0.34(0.09 - r)dt + 0.26dz. The relevant Sharpe ratio φ is 0.88. Find A(2, 3) in this Vasicek model.

**Solution VIRM3.** We use the formula $A(t, T) = \exp[r^-(B(t, T) + t - T) - B(t, T)^2\sigma^2/4a]$ From Solution VIRM1, we know that $r^-$ = 0.4705536332. From Solution VIRM2, we know that

B(2, 3) = 0.8477343448. Also, T = 3, t = 2, σ = 0.26, and a = 0.34.

Thus, $A(2, 3) = \exp[0.4705536332(0.8477343448+ 2 - 3) - 0.8477343448^2 0.26^2/4*0.34] =$ **A(2, 3) = 0.8981928627**

**Problem VIRM4.** A Vasicek model holds for an interest rate r such that dr = 0.34(0.09 - r)dt + 0.26dz. The relevant Sharpe ratio φ is 0.88. Find the price

P[2, 3, 0.11] in this Vasicek model.

**Solution VIRM4.** We use the formula P[t, T, r(t)] = A(t, T)$e^{-B(t, T)r(t)}$, where, from Solution VIRM2, B(2, 3) = 0.8477343448, and, from Solution VIRM3,

A(2, 3) = 0.8981928627. We are also given that r(t) = 0.11. Thus,

P[2, 3, 0.11] = 0.8981928627$e^{-0.8477343448*0.11}$ = **P[2, 3, 0.11] = 0.8182222807**

**Problem VIRM5.**

**Similar to Question 13 from the Society of Actuaries' May 2007 Exam MFE:**

Let P(t, T, r) be the price at time t of $1 paid with certainty at time T, t ≤ T, if the short-term interest rate at time t is equal to r.

Assume that a Vasicek model holds. You know that

P(0.08, 4, 9) = 0.5556

P(0.10, 7, 12) = 0.5051

P(r*, 14, 19) = 0.4432. Find r*

**Solution VIRM5.** Here, it is useful to employ the formula

P[t, T, r(t)] = exp(-[α(T - t) + β(T - t)r]), where α(T - t) and β(T - t) are constants that stay the same whenever the difference between T and t is the same. We note that T - t is 5 for all three given cases, so this formula will hold. Thus,

P(0.08, 4, 9) = 0.5556 = exp(-[5α + 5*0.08β])

P(0.10, 7, 12) = 0.5051 = exp(-[5α + 5*0.1β])

P(r*, 14, 19) = 0.4432 = exp(-[5α + 5*r*β])

We can take the natural logarithm of both sides of each of these expressions:

(i) -0.5869869847 = -[5α + 5*0.08β]

(ii) -0.6829988495 = -[5α + 5*0.1β]

(iii) -0.8137341435 = -[5α + 5*r*β]

We can subtract (ii) from (i) to get

0.0960118648 = 5*0.02β = 0.1β

Thus, β = 0.960118648

Now we substitute for β in (i):

-0.5869869847 = -[5α + 5*0.08*0.960118648]

0.5869869847 = 5α + 5*0.08*0.960118648

5α = 0.2029395255

α = 0.0405879051

Now we can substitute for both α and β in (iii):

-0.8137341435 = -[5*0.0405879051 + 5*0.960118648*r*]

0.8137341435 = 5*0.0405879051 + 5*0.960118648*r*

4.80059324r* = 0.610794618

**r* = 0.1272331538**

# Section 71

# Exam-Style Questions on the Vasicek Interest-Rate Model

In this section, we will work through multiple instances of one particular highly involved exam-style question regarding the Vasicek interest rate model. This is one of the only examples of prior test questions available, and mastering it will get you a long way toward being able to apply the model.

The recommended way of doing these problems is to see the way either Problem ESQVIRM1 or the exam question on which it is based are solved. Then, using the same methods, try to solve the subsequent problems on your own and check your work. Going through the same procedure multiple times should render it sufficiently firm in your mind.

The problems in this section were designed to be similar to problems from past versions of Exam 3F / Exam MFE. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem ESQVIRM1.**

**Similar to Question 16 from the Society of Actuaries' Sample MFE Questions and Solutions:**

In a particular Vasicek model, the short-term interest rate is modeled by the following Brownian process: $dr(t) = 0.77[b - r(t)]dt + \sigma dZ(t)$.

For $t \leq T$, we can let $P[r, t, T]$ be the price at time t of a zero-coupon bond paying $1 at time T.

The short-term interest rate at time t is r.

The price of every zero-coupon bond in the Vasicek model behaves according to this Ito process:

$dP[r(t), t, T]/P[r(t), t, T] = \alpha[r(t), t, T]dt + q[r(t), t, T]dZ(t)$, for $t \leq T$.

You know that $\alpha(0.11, 5, 9) = 0.1878$

Find $\alpha(0.15, 10, 19)$.

**Solution ESQVIRM1.** In the Vasicek model, the Sharpe ratio φ is assumed to be constant.

We note that in the geometric Brownian motion

dP[r(t), t, T]/P[r(t), t, T] = α[r(t), t, T]dt + q[r(t), t, T]dZ(t), α is the drift factor and q is the volatility factor. So the Sharpe ratio can be expressed as

φ = (α[r(t), t, T] - r)/q[r(t), t, T]

In our case, we can express φ as

φ = (α[0.11, 5, 9] - 0.11)/q[0.11, 5, 9] = (α[0.15, 10, 19] - 0.15)/q[0.15, 10, 19]

Furthermore, since α(0.11, 5, 9) = 0.1878,

(0.1878 - 0.11)/q[0.11, 5, 9] = (α[0.15, 10, 19] - 0.15)/q[0.15, 10, 19]

0.0778/q[0.11, 5, 9] = (α[0.15, 10, 19] - 0.15)/q[0.15, 10, 19]

Now we try to find an expression for q[r(t), t, T].

By Ito's Lemma,

dP[r(t), t, T] = $P_r$dr + (1/2)$P_{rr}$(dr)$^2$ + $P_t$dt

For a general Vasicek model, dr(t) = a[b - r(t)]dt + σdZ(t). Thus,

dP[r(t), t, T] = $P_{rrr}$($^2$ + $P_t$dt

dP[r(t), t, T] = a[b - r(t)]$P_r$dt + $P_r$σdZ(t) + (1/2)$P_{rr}$(σ$^2$)dt + $P_t$dt by the multiplication rules in Section 67.

dP[r(t), t, T] = (a[b - r(t)]$P_r$ + (1/2)$P_{rr}$(σ$^2$) + $P_t$)dt + $P_r$σdZ(t)

Fortunately, it is the coefficient of dZ(t) that concerns us here.

q[r(t), t, T] is the coefficient of dZ(t) in the expression for dP[r(t), t, T]/P[r(t), t, T]. To get the value of q[r(t), t, T] using $P_r$σ, we simply need to divide both sides of the above equation by P[r(t), t, T] in order to get

dP[r(t), t, T]/P[r(t), t, T] = (a[b - r(t)]$P_r$ + (1/2)$P_{rr}$(σ$^2$) + $P_t$)dt/P[r(t), t, T] + ($P_r$σ/P[r(t), t, T])dZ(t)

Thus, we have q[r(t), t, T] = $P_r$σ/P[r(t), t, T]

$P_r$/P[r(t), t, T] is the same as d[lnP[r(t), t, T]]/dr. Try to differentiate the latter expression with respect to r, and you will see that the results are the same.

We recall from Section 70 that in the Vasicek model, $P[t, T, r(t)] = A(t, T)e^{-B(t, T)r(t)}$.

So $\ln P[r(t), t, T] = \ln^{-B(t, T)r(t)}] = \ln$

Since, with respect to r, ln

$d[\ln P[r(t), t, T]]/dr = - B(t, T)$

Therefore, $q[r(t), t, T] = P_r\sigma/P[r(t), t, T] = - \sigma B(t, T)$

We recall from Section 70 that in the Vasicek model, $B(t, T) = (1 - e^{-a(T-t)})/a$

Thus, $- \sigma B(t, T) = -\sigma(1 - e^{-a(T-t)})/a = q[r(t), t, T]$

*Please do not go through this derivation every time you solve a problem of this sort!* It was intended to help you learn *why* the answer we will arrive at is correct; it was not intended to be replicated during the exam. Unless you enjoy pain and suffering, you should do with this result the same thing you do with results like the solution for the indefinite integral of $\sec^3(\theta)d\theta$: *memorize it!* (More precisely, memorize one of the equations 71.1 or 71.2 later in this section.)

In the general case, $q[r(t), t, T] = -\sigma(1 - e^{-a(T-t)})/a$

Because σ and a are the same for all values of t and T, they will cancel out in our equation for the Sharpe ratio. The negative sign will also drop out.

We examine our equation derived above:

$0.0778/q[0.11, 5, 9] = (\alpha[0.15, 10, 19] - 0.15)/q[0.15, 10, 19]$

$0.0778/q[0.11, 5, 9] = (\alpha[0.15, 10, 19] - 0.15)/q[0.15, 10, 19]$

We can put the appropriate $1 - e^{-a(T-t)}$ in the denominator of each side of the equation.

For the left side, since T = 9 and t = 5, $1 - e^{-a(T-t)} = 1 - e^{-4a}$

For the left side, since T = 19 and t = 10, $1 - e^{-a(T-t)} = 1 - e^{-9a}$

Thus, $0.0778/(1 - e^{-4a}) = (\alpha[0.15, 10, 19] - 0.15)/(1 - e^{-9a})$

Here, a = 0.77

Thus, $0.0778/(1 - e^{-4*0.77}) = (\alpha[0.15, 10, 19] - 0.15)/(1 - e^{-9*0.77})$

$0.0778(1 - e^{-9*0.77})/(1 - e^{-4*0.77}) + 0.15 =$ $\mathbf{\alpha[0.15, 10, 19] = 0.231468126}$

To save you time and mental anguish, here is a procedure for solving problems where you have the following short-term interest rate process in the Vasicek model:

$dr(t) = a[b - r(t)]dt + \sigma dZ(t)$ and

$dP[r(t), t, T]/P[r(t), t, T] = \alpha[r(t), t, T]dt + q[r(t), t, T]dZ(t)$, for $t \leq T$.

You are given some $\alpha(r_1, t_1, T_1)$ and are asked to find $\alpha(r_2, t_2, T_2)$.

All you need to do is to solve the following equation:

**Equation 71.1**

$$[\alpha(r_1, t_1, T_1) - r_1]/(1 - \exp[-a(T_1 - t_1)]) = [\alpha(r_2, t_2, T_2) - r_2]/(1 - \exp[-a(T_2 - t_2)])$$

Better yet,

**Equation 71.2**

$$\alpha(r_2, t_2, T_2) = (1 - \exp[-a(T_2 - t_2)])[\alpha(r_1, t_1, T_1) - r_1]/(1 - \exp[-a(T_1 - t_1)]) + r_2$$

Now you just need to substitute in the relevant values.

Now that we have learned how to save time, let us proceed to the other four problems of this study guide.

**Problem ESQVIRM2.**

**Similar to Question 16 from the Society of Actuaries' Sample MFE Questions and Solutions:**

In a particular Vasicek model, the short-term interest rate is modeled by the following Brownian process: $dr(t) = 0.34[b - r(t)]dt + \sigma dZ(t)$.

For $t \leq T$, we can let $P[r, t, T]$ be the price at time t of a zero-coupon bond paying $1 at time T.

The short-term interest rate at time t is r.

The price of every zero-coupon bond in the Vasicek model behaves according to this Ito process:

$dP[r(t), t, T]/P[r(t), t, T] = \alpha[r(t), t, T]dt + q[r(t), t, T]dZ(t)$, for $t \leq T$.

You know that $\alpha(0.01, 4, 11) = 0.0333$

Find $\alpha(0.04, 3, 8)$.

**Solution ESQVIRM2.**

We use Equation 71.2:
$\alpha(r_2, t_2, T_2) = (1 - \exp[-a(T_2 - t_2)])[\alpha(r_1, t_1, T_1) - r_1]/(1 - \exp[-a(T_1 - t_1)]) + r_2$
Here, $a = 0.34$, $t_1 = 4$, $T_1 = 11$, $r_1 = 0.01$, $t_2 = 3$, $T_2 = 8$, $r_2 = 0.04$.
$\alpha(0.04, 3, 8) = (1 - \exp[-0.34(8 - 3)])[0.0333 - 0.01]/(1 - \exp[-0.34(11 - 4)]) + 0.04$
$\alpha(0.04, 3, 8) = (1 - \exp[-0.34*5])[0.0233]/(1 - \exp[-0.34*7]) + 0.04$
**$\alpha(0.04, 3, 8) = 0.0609857138$**

**Problem ESQVIRM3.**

**Similar to Question 16 from the Society of Actuaries' Sample MFE Questions and Solutions:**

In a particular Vasicek model, the short-term interest rate is modeled by the following Brownian process: $dr(t) = 0.29[b - r(t)]dt + \sigma dZ(t)$.

For $t \leq T$, we can let $P[r, t, T]$ be the price at time t of a zero-coupon bond paying $1 at time T.

The short-term interest rate at time t is r.

The price of every zero-coupon bond in the Vasicek model behaves according to this Ito process:

$dP[r(t), t, T]/P[r(t), t, T] = \alpha[r(t), t, T]dt + q[r(t), t, T]dZ(t)$, for $t \leq T$.

You know that $\alpha(0.31, 122, 334) = 0.44$

Find $\alpha(0.1, 4, 15)$.

**Solution ESQVIRM3.**

We use Equation 71.2:

$\alpha(r_2, t_2, T_2) = (1 - \exp[-a(T_2 - t_2)])[\alpha(r_1, t_1, T_1) - r_1]/(1 - \exp[-a(T_1 - t_1)]) + r_2$

Here, $a = 0.29$, $t_1 = 122$, $T_1 = 334$, $r_1 = 0.31$, $t_2 = 4$, $T_2 = 15$, $r_2 = 0.1$.

$\alpha(0.1, 4, 15) = (1 - \exp[-0.29(15 - 4)])[0.44 - 0.31]/(1 - \exp[-0.29(334 - 122)]) + 0.1$

$\alpha(0.1, 4, 15) = (1 - \exp[-0.29*11])[0.13]/(1 - \exp[-0.29*212]) + 0.1$

**$\alpha(0.1, 4, 15) = 0.2246476568$**

**Problem ESQVIRM4.**

**Similar to Question 16 from the Society of Actuaries' Sample MFE Questions and Solutions:**

In a particular Vasicek model, the short-term interest rate is modeled by the following Brownian process: $dr(t) = 0.86[b - r(t)]dt + \sigma dZ(t)$.

For $t \leq T$, we can let $P[r, t, T]$ be the price at time t of a zero-coupon bond paying $1 at time T.

The short-term interest rate at time t is r.

The price of every zero-coupon bond in the Vasicek model behaves according to this Ito process:

$dP[r(t), t, T]/P[r(t), t, T] = \alpha[r(t), t, T]dt + q[r(t), t, T]dZ(t)$, for $t \leq T$.

You know that $\alpha(0.04, 0, 15) = 0.09$

Find $\alpha(0.04, 10, 15)$.

**Solution ESQVIRM4.**

We use Equation 71.2:

$\alpha(r_2, t_2, T_2) = (1 - \exp[-a(T_2 - t_2)])[\alpha(r_1, t_1, T_1) - r_1]/(1 - \exp[-a(T_1 - t_1)]) + r_2$

Here, $a = 0.86$, $t_1 = 0$, $T_1 = 15$, $r_1 = 0.04$, $t_2 = 10$, $T_2 = 15$, $r_2 = 0.04$.

$\alpha(0.04, 10, 15) = (1 - \exp[-0.86(15 - 10)])[0.09 - 0.04]/(1 - \exp[-0.86(15 - 0)]) + 0.04$

$\alpha(0.04, 10, 15) = (1 - \exp[-0.86*5])[0.05]/(1 - \exp[-0.86*15]) + 0.04$

**$\alpha(0.04, 10, 15)= 0.0893216953$**

**Problem ESQVIRM5.**

**Similar to Question 16 from the Society of Actuaries' Sample MFE Questions and Solutions:**

In a particular Vasicek model, the short-term interest rate is modeled by the following Brownian process: $dr(t) = 0.12[b - r(t)]dt + \sigma dZ(t)$.

For $t \leq T$, we can let $P[r, t, T]$ be the price at time t of a zero-coupon bond paying $1 at time T.

The short-term interest rate at time t is r.

The price of every zero-coupon bond in the Vasicek model behaves according to this Ito process:

$dP[r(t), t, T]/P[r(t), t, T] = \alpha[r(t), t, T]dt + q[r(t), t, T]dZ(t)$, for $t \leq T$.

You know that $\alpha(0.19, 0, 30) = 0.23$

Find $\alpha(0.29, 0, 30)$.

**Solution ESQVIRM5.**

We use Equation 71.2:

$\alpha(r_2, t_2, T_2) = (1 - \exp[-a(T_2 - t_2)])[\alpha(r_1, t_1, T_1) - r_1]/(1 - \exp[-a(T_1 - t_1)]) + r_2$

Here, $a = 0.12$, $t_1 = 0$, $T_1 = 30$, $r_1 = 0.19$, $t_2 = 0$, $T_2 = 30$, $r_2 = 0.29$.

$\alpha(0.29, 0, 30) = (1 - \exp[-0.12(30 - 0)])[0.23 - 0.19]/(1 - \exp[-0.12(30 - 0)]) + 0.29$

$\alpha(0.29, 0, 30) = 0.04 + 0.29 =$ **$\alpha(0.29, 0, 30) = 0.33$**

# Section 72

# The Cox-Ingersoll-Ross (CIR) Interest-Rate Model

In the Cox-Ingersoll-Ross (CIR) interest rate model, the short-term interest rate follows this Brownian process:

$dr = a(b - r)dt + \sigma\sqrt{(r)}dZ$

**Meaning of some variables:**

$\sigma$ = volatility factor.

a = drift factor.

b = the mean around which mean reversion occurs.

r = the short-term interest rate.

The way to solve this equation for the price P[t, T, r(t)] at time t of $1 paid with certainty at time T, where $t \leq T$ and P(T, T, r) = 1, is

$P[t, T, r(t)] = A(t, T)e^{-B(t, T)r(t)}$, as in the Vasicek Model. The values of A(t, T) and B(t, T) are calculated differently, however, and the formulas for doing so are much more complicated than those in the Vasicek Model. The chance you will need to use these formulas on the exam will be minuscule.

It is more instructive to compare the CIR model to the Vasicek model. The following matters stand out in particular.

1. In the Vasicek model, interest rates can be negative. In the CIR model, negative interest rates are impossible. If r = 0 in the CIR model, the drift factor a(b - r) becomes ab > 0, while the volatility factor is $\sigma\sqrt{(0)} = 0$, so the interest rate will increase. As one's time horizon increases, the likelihood of interest rates becoming negative in the Vasicek model greatly increases as well.

2. In the Vasicek model, the volatility of the short-term interest rate is constant. In the CIR model, the volatility of the short-term interest rate increases as the short-term interest rate increases.

3. As in the Vasicek model, the short-term interest rate in the CIR model exhibits mean reversion, where b is the mean.

4. In both the Vasicek and the CIR model, the delta and gamma Greeks for a zero-coupon bond are based on the change in the short-term interest rate.

Learning Objective A2 on the syllabus for Exam 3F / Exam MFE asks students to "explain why the time-zero yield curve in the Vasicek and Cox-Ingersoll-Ross bond price models cannot be exogenously prescribed."

What does this mean, and why is it true?

An exogenous prescription of the time-zero yield curve would mean that you know-empirically, or from some source external to the models - the data regarding the yields to maturity for many different time horizons, where currently, t = 0. Then you would be able to put that data into the Vasicek or CIR model and get consistent results as well as the ability to predict yields to maturity for other time horizons. Alas, this is not the case.

For instance, you might know empirically that for

t = 0, T = 1: r = 0.02

t = 0, T = 2: r = 0.03

t = 0, T = 3: r = 0.04

t = 0, T = 4: r = 0.05

t = 0, T = 5: r = 0.08

Why can this data not be used consistently with a Vasicek or CIR model? Both models are based on four parameters: a, b, σ, and r. But if you have four or more empirical data points, it is possible (indeed, likely) for a 4-parameter model to give conclusions inconsistent with one or more of those data points.

We can think of this via an analogy.

Let us say we have the following model: x + 2y = 5.

If we are just given x, we can solve for y consistently with the model.

If we are given both x and y, however, we will not always be able to do so. For instance, if x = 3 and y = 4, there is no way for x + 2y to equal 5. If you have even more externally prescribed data, the likelihood of the model working decreases even further.

So a model based on m parameters can only be used consistently with exogenously prescribed data consisting of m-1 or fewer data points.

**Sources:** Cross, Bill. Post on Actuarial Outpost Discussion Forum. April 9, 2007.

McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 24, pp. 787-789.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem CIRIRM1.**

Assume the CIR model holds. A particular interest rate follows this Brownian motion:

$dr = 0.22(0.06 - r)dt + 0.443\sqrt{(r)}dZ$

At some particular time t, r = 0.11. Then, r suddenly becomes 0.02. What is the resulting change in the volatility?

**Solution CIRIRM1.** The volatility in the CIR model is $\sigma\sqrt{(r)}$. For r = 0.11, and $\sigma = 0.443$, the volatility is $0.443\sqrt{(0.11)}$. For r = 0.02, and $\sigma = 0.443$, the volatility is $0.443\sqrt{(0.02)}$. The change in volatility is thus $0.443\sqrt{(0.02)} - 0.443\sqrt{(0.11)} =$ **-0.08427668174**

**Problem CIRIRM2.** Assume the CIR model holds. The following values are true with regard to a particular interest rate at some specific time t: r = 0.04, a = 0.88, b = 0.09, $\sigma = 0.23$, dt = 1, dZ = 0.4. Find dr in the CIR model.

**Solution CIRIRM2.**

We use the formula $dr = a(b - r)dt + \sigma\sqrt{(r)}dZ = 0.88(0.09 - 0.04)1 + 0.23\sqrt{(0.04)}0.4 =$
**dr = 0.0624**

**Problem CIRIRM3.** Which of the following statements are true? More than one answer may be correct.

(a) The Vasicek model incorporates mean reversion.
(b) The CIR model incorporates mean reversion.
(c) In the Vasicek model, interest rates can be negative.
(d) In the CIR model, interest rates can be negative.
(e) In the Vasicek model, volatility is a function of the interest rate.

(f) In the CIR model, volatility is a function of the interest rate.

**Solution CIRIRM3.** Both the Vasicek and CIR models incorporate mean reversion, so (a) and (b) are correct. Interest rates can be negative in the Vasicek model, but not in the CIR model, so (c) is true and (d) is false. In the Vasicek model, volatility is constant, so (e) is false But in the CIR model, volatility = $\sigma\sqrt{(r)}$, so (f) is true. Thus, **(a), (b), (c), and (f) are the correct answers.**

**Problem CIRIRM4.** Assume that the CIR model holds. When a particular interest rate is 0, the Brownian motion it follows is dr = 0.55dt.

You know that a + b = 0.99 and a > 1. What is the drift factor of the Brownian motion in this model when r = 0.05?

**Solution CIRIRM4.** The drift factor in the CIR model is a(b - r). When r = 0, this is equal to ab. We are thus given that ab = 0.55 and a + b = 1.7. Thus,
b = 0.55/a and so a + 0.55/a = 1.7 and $a^2 + 0.55 = 1.7a$. Thus,
$a^2 - 1.7a + 0.55 = 0$. By the quadratic formula, a = 1.265331193 or a = 0.4346688069. Since a > 1, a = 1.265331193.
Thus, b = 1.7 - a = 1.7 - 1.265331193 = b = 0.4346688069.
When r = 0.05, a(b - r) = 1.265331193(0.4346688069 - 0.05) =
**The drift factor = 0.4867334405.**

**Problem CIRIRM5.** For which of these exogenously prescribed data sets regarding yields to maturity for various time horizons can the Vasicek and CIR models be used? More than one answer may be correct.

Set A:
t = 0, T = 1: r = 0.22
t = 0, T = 2: r = 0.26
t = 0, T = 3: r = 0.28

Set B:
t = 0, T = 0.5: r = 0.04
t = 0, T = 5: r = 0.05

Set C:
t = 0, T = 1: r = 0.12
t = 0, T = 2: r = 0.09
t = 0, T = 3: r = 0.18
t = 0, T = 6: r = 0.21
t = 0, T = 7: r = 0.03
t = 0, T = 8: r = 0.44

Set D:
t = 0, T = 1: r = 0.11
t = 0, T = 2: r = 0.31
t = 0, T = 7: r = 0.34
t = 0, T = 8: r = 0.54

**Solution CIRIRM5.** The time-zero yield curve for an interest rate model can only be interpreted consistently with the model (in most cases) when the number of data points in the time-zero yield curve is less than the number of parameters in the model. The Vasicek and CIR models each have 4 parameters: a, b, σ, and r. So only data sets with 3 points or fewer can have the models consistently applied to them. Thus, only **Sets A and B**, with 3 and 2 data points respectively, can have the models applied to them.

# Section 73

# The Black Formula for Pricing Options on Bonds

The Black formula for pricing a European call option on a zero-coupon bond is

**$C[F, P(0, T), \sigma, T] = P(0, T)[FN(d_1) - KN(d_2)]$, where**

**$d_1 = [\ln(F/K) + 0.5\sigma^2 T]/[\sigma\sqrt{(T)}]$ and**

**$d_2 = d_1 - \sigma\sqrt{(T)}$**

To find the price of an otherwise equivalent European put option on the bond, you can use the formula **$P[F, P(0, T), \sigma, T] = P(0, T)[KN(-d_2) - FN(-d_1)]$**.

Or you can use put-call parity, with the relationship derived as follows:

$C[F, P(0, T), \sigma, T] - P[F, P(0, T), \sigma, T] = P(0, T)[FN(d_1) - KN(d_2)] - P(0, T)[KN(-d_2) - FN(-d_1)]$

$C[F, P(0, T), \sigma, T] - P[F, P(0, T), \sigma, T] = P(0, T)[FN(d_1) - KN(d_2) - KN(-d_2) + FN(-d_1)]$

$C[F, P(0, T), \sigma, T] - P[F, P(0, T), \sigma, T] = P(0, T)[FN(d_1) - KN(d_2) - K[1 - N(d_2)] + F[1 - N(d_1)]]$

$C[F, P(0, T), \sigma, T] - P[F, P(0, T), \sigma, T] = P(0, T)[F - K]$

**$P[F, P(0, T), \sigma, T] = C[F, P(0, T), \sigma, T] - P(0, T)[F - K]$**

**Meaning of Variables:**

$F$ = the bond forward price $F_{0,T}[P(T, T + s)]$.

$P(0, T)$ = the bond price at time 0 for a bond expiring at time T and paying $1 at time T.

$K$ = the strike price of the option.

$\sigma$ = volatility of the bond forward price.

$T$ = time to the bond's maturity.

$s$ = some time interval greater than 0.

Note: The Black formula assumes that the bond forward price is *lognormally* distributed and has a constant volatility σ.

The initially difficult part of using the Black formula is knowing how to arrive at the value for F.

You need to be given two prices, P(0, T + s) and P(0, T), of two zero-coupon bonds - paying $1 at times T + s and T respectively. Then

$F = F_{0,T}[P(T, T+s)] = P(0, T+s)/P(0, T)$. This simple act of division solves for F.

McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 24, pp. 790-791.

**Problem BFPOB1.** Zero-coupon bonds paying $1 at maturity have the following values at t = 0

Time to maturity of 2 years: Current price of bond is 0.8853

Time to maturity of 5 years: Current price of bond is 0.6657

A European call option that expires in 2 years enables you to purchase a 3-year bond at expiration at a purchase price of 0.7799. The price of the bond forward is lognormally distributed and has a volatility of 0.33.

Find the value of F in the Black formula for the price of this call option.

**Solution BFPOB1.** We use the formula

$F = F_{0,T}[P(T, T+s)] = P(0, T+s)/P(0, T)$.

Here, T = 2 and s = 3. We are given that P(0, 5) = 0.6657 and P(0, 2) = 0.8853. Thus,

$F = F_{0,2}[P(2, 5)] = P(0, 5)/P(0, 2) = 0.6657/0.8853 =$ **F = 0.751948492**

**Problem BFPOB2.** Zero-coupon bonds paying $1 at maturity have the following values at t = 0

Time to maturity of 2 years: Current price of bond is 0.8853

Time to maturity of 5 years: Current price of bond is 0.6657

A European call option that expires in 2 years enables you to purchase a 3-year bond at expiration at a purchase price of 0.7799. The price of the bond forward is lognormally distributed and has a volatility of 0.33.

Find the value of $d_1$ in the Black formula for the price of this call option.

**Solution BFPOB2.** We use the formula $d_1 = [\ln(F/K) + 0.5\sigma^2T]/[\sigma\sqrt{(T)}]$. Here, from Solution BFPOB1, F = 0.751948492. Also, K = 0.7799, σ = 0.33, T = 2. Thus,

$d_1 = [\ln(0.751948492/0.7799) + 0.5*0.33^2 2]/[0.33\sqrt{(2)}]$ = $\mathbf{d_1 = 0.1551394863}$

**Problem BFPOB3.** Zero-coupon bonds paying $1 at maturity have the following values at t = 0

Time to maturity of 2 years: Current price of bond is 0.8853

Time to maturity of 5 years: Current price of bond is 0.6657

A European call option that expires in 2 years enables you to purchase a 3-year bond at expiration at a purchase price of 0.7799. The price of the bond forward is lognormally distributed and has a volatility of 0.33.

Find the value of $d_2$ in the Black formula for the price of this call option.

**Solution BFPOB3.** We use the formula $d_2 = d_1 - \sigma\sqrt{(T)}$. From Solution BFPOB2,

$d_1 = 0.1551394863$. Thus, $d_2 = 0.1551394863 - 0.33\sqrt{(2)}$ = $\mathbf{d_2 = -0.3115509893}$

**Problem BFPOB4.** Zero-coupon bonds paying $1 at maturity have the following values at t = 0

Time to maturity of 2 years: Current price of bond is 0.8853

Time to maturity of 5 years: Current price of bond is 0.6657

A European call option that expires in 2 years enables you to purchase a 3-year bond at expiration at a purchase price of 0.7799. The price of the bond forward is lognormally distributed and has a volatility of 0.33.

Use the Black formula to find the price of this call option.

**Solution BFPOB4.** From Solutions BFPOB1-3, we know that $d_1 = 0.1551394863$,

$d_2 = -0.3115509893$, F = 0.751948492. Also, P(0, 2) = 0.8853 and K = 0.7799.

In MS Excel, using the input "=NormSDist(0.1551394863)", we find that $N(d_1) = 0.561644338$

In MS Excel, using the input "=NormSDist(-0.3115509893)", we find that $N(d_2) = 0.377690958$

Thus, by the Black formula, $C[F, P(0, T), \sigma, T] = P(0, T)[FN(d_1) - KN(d_2)]$ =

0.8853[0.751948492*0.561644338 - 0.7799*0.377690958] = **C = 0.1131116248**

**Problem BFPOB5.** Zero-coupon bonds paying $1 at maturity have the following values at t = 0

Time to maturity of 2 years: Current price of bond is 0.8853

Time to maturity of 5 years: Current price of bond is 0.6657

A European put option that expires in 2 years enables you to sell a 3-year bond at expiration at a price of 0.7799. The price of the bond forward is lognormally distributed and has a volatility of 0.33.

Assume that the framework of the Black formula holds. Find the price of this put option.

**Solution BFPOB5.** We use put-call parity:

$P[F, P(0, T), \sigma, T] = C[F, P(0, T), \sigma, T] - P(0, T)[F - K]$

We showed in Solution BFPOB4 that C = 0.1131116248. Furthermore, from Solution BFPOB1, F = 0.751948492. Moreover, K = 0.7799 and P(0, 2) = 0.8853.

Thus, P = 0.1131116248 - 0.8853[0.751948492 - 0.7799] = **P = 0.1378570948**

# Section 74

# Forward Rate Agreements and Caplets

The non-annualized forward interest rate from time T to time T + s, $R_0(T, T + s)$, can be expressed as

$R_0(T, T + s) = P(0, T)/P(0, T + s) - 1$

If you invest \$1 at time T, you will have $[1 + R_0(T, T + s)]$ after s time periods elapse.

A **forward rate agreement (FRA)** has a payoff at time T + s of the difference of the forward rate $R_T(T, T + s)$ and the forward rate $R_0(T, T + s)$.

Thus, Payoff to FRA = $R_T(T, T + s) - R_0(T, T + s)$

We note that $R_T(T, T + s) = P(T, T)/P(T, T + s) - 1$ and $P(T, T) = 1$ (since a bond paying \$1 at expiration will have a value of \$1 right when it expires).

Thus, $R_T(T, T + s) = 1/P(T, T + s) - 1$

And **Payoff to FRA = $1/P(T, T + s) - P(0, T)/P(0, T + s)$**

A **caplet** is a call option on an FRA and has a strike price of $K_R$. At time T + s, a caplet has

**Payoff to caplet = $\max[0, R_T (T, T + s) - K_R]$**

A caplet can also be settled at time T, in which case it would pay

**Payoff to caplet = $[1/(1+ R_T (T, T + s)]\max[0, R_T (T, T + s) - K_R]$**

An equivalent expression for this is

**Payoff to caplet = $(1 + K_R)\max[0, 1/(1 + K_R) - 1/(1+ R_T (T, T + s))]$**

**Source:** McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 24, p. 791.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem FRAC1.** You are given the prices of zero-coupon bonds paying \$1 at expiration. A bond paying \$1 in 6 years has a price of 0.6464 today. A bond paying \$1 in 8 years has a price of 0.5461 today. What is the non-annualized forward interest rate from time t = 6 years to time t= 8 years?

**Solution FRAC1.** We use the formula $R_0(T, T + s) = P(0, T)/P(0, T + s) - 1$.

Here, T = 6 and s = 2. P(0, 6) = 0.6464 and P(0, 8) = 0.5461. Thus,

$R_0(6, 8) = 0.6464/0.5461 - 1 =$ **$R_0(6, 8) = 0.1836659952$**

**Problem FRAC2.** Alcibiades has entered into a forward rate agreement which expires in 4 years and pays him the difference between two non-annualized forward interest rates - each from time t = 2 years to time t= 4 years - one which rates occurs at time t = 0 and the other of which occurs at time t = 2. A zero-coupon bond paying $1 in 4 years has a price of 0.5523 today. A zero-coupon bond paying $1 in 2 years has a price of 0.8868 today. Now assume that two years have passed and a zero-coupon bond paying $1 in yet another 2 years has a price of 0.7769. What will be Alcibiades' payoff to the forward rate agreement?

**Problem FRAC2.** We use the formula Payoff to FRA = 1/P(T, T + s) - P(0, T)/P(0, T + s). Here,

T = 2 and s = 2. We are given that P(0, 2) = 0.8868, P(0, 4) = 0.5523, P(2, 4) = 0.7769

Thus, Payoff to FRA = 1/0.7769 - 0.8868/0.5523 = **-0.3184821582**, so Alcibiades will lose money.

**Problem FRAC3.** Alcibiades has bought a caplet which expires in 4 years and has a strike rate of 0.2334. A zero-coupon bond paying $1 in 4 years has a price of 0.5523 today. A zero-coupon bond paying $1 in 2 years has a price of 0.8868 today. Now assume that two years have passed and a zero-coupon bond paying $1 in yet another 2 years has a price of 0.7769. What will be Alcibiades' payoff to the caplet at time t = 4?

**Solution FRAC3.** We use the formula Payoff to caplet = $\max[0, R_T (T, T + s) - K_R]$.
Here, $K_R = 0.2334$.
$R_T(T, T + s) = 1/P(T, T + s) - 1$, where we are given that T = 2, s = 2, and P(2, 4) = 0.7769
Thus, $R_T(2, 4) = 1/0.7769 - 1 = 0.2871669456$
And $R_T (2, 4) - K_R = 0.2871669456 - 0.2334 = 0.0537669456 > 0$.
Thus, Payoff to caplet = $\max[0, R_T (2, 4) - K_R] =$ **0.0537669456**

**Problem FRAC4.** Alcibiades has bought a caplet which expires in 4 years and has a strike rate of 0.2334. A zero-coupon bond paying $1 in 4 years has a price of 0.5523 today. A zero-coupon bond paying $1 in 2 years has a price of 0.8868 today. Now assume that two years have passed and a zero-coupon bond paying $1 in yet another 2 years has a price of 0.7769. What will be Alcibiades' payoff to the caplet at time t = 2?

**Solution FRAC4.** We use the formula

Payoff to caplet = $[1/(1+ R_T (T, T + s)]\max[0, R_T (T, T + s) - K_R]$

From Solution FRAC3, $R_T(2, 4) = 1/0.7769 - 1 = 0.2871669456$ and

$\max[0, R_T(2, 4) - K_R] = 0.0537669456$

Thus, Payoff to caplet = [1/1.2871669456]0.0537669456 = **0.04177154**

**Problem FRAC5.** A caplet with a certain strike price has a payoff of 0.6556 at expiration 10 years and a payoff of 0.3434 in 3 years. The caplet is based on the forward rate $R_3$ (3, 10). What is the strike price for this caplet?

**Solution FRAC5.** We know that

At t = 10, Payoff to caplet = $\max[0, R_3(3, 10) - K_R] = 0.6556$

At t = 3, Payoff to caplet = $[1/(1+ R_3(3, 10)]\max[0, R_3(3, 10) - K_R] = 0.3434$

Thus, $[1/(1+ R_3(3, 10)] = 0.3434/0.6556$ and

$1+ R_3(3, 10) = 0.6556/0.3434$.

Thus, $R_3(3, 10) = 0.6556/0.3434 - 1 = 0.9091438556$

At t = 10, Payoff to caplet = $\max[0, R_3(3, 10) - K_R] = 0.6556$

Since 0.6556 > 0, the payoff at t = 10 is $R_3(3, 10) - K_R = 0.6556$

Thus, $K_R = R_3(3, 10) - 0.6556 = 0.9091438556 - 0.6556 =$ **$K_R$ = 0.2535438556**

# Section 75

# Interest Rate Caps and Pricing Caplets Using the Black Formula

A collection of caplets is called an **interest rate cap.** A at each time $t_{i+1}$, a cap makes the following payment: Cap payment at time $t_{i+1}$ = max[0, $R_{t\_i}(t_i, t_{i+1})$ - $K_R$].

A cap on a floating rate loan that makes interest payments at integer times t = 1 through t = n would have a value equal to the sum of the individual caplet values.

The Black formula can be used to price caplets. The Black formula for the price of a caplet which expires in T years and pays in T + s years is

$C[F, P(0, T+s), \sigma, T+s] = P(0, T+s)[FN(d_1) - KN(d_2)]$, where
$d_1 = [\ln(F/K) + 0.5\sigma^2 T]/[\sigma\sqrt{(T)}]$ and
$d_2 = d_1 - \sigma\sqrt{(T)}$

But, in the case of a caplet, what are the relevant values of P(0, T+s) and F?

P(0, T+s) is the (T+s)-year discount factor or the value at t = 0 of a zero-coupon bond paying $1 at time T+s.

F is P(0, T)/P(0, T + s) - 1 or the non-annualized forward rate from time T to time T + s.

**Sources:** "Interest Rate Cap and Floor." Wikipedia, the Free Encyclopedia.

McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 24, p. 792.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem IRCPCUBF1.** An interest rate cap has a strike price of 0.0345. The interest rate cap makes payments at times t = 3, t = 4, and t = 5.

The forward rates for these time periods are as follows:

Forward rate from t = 2 to t = 3: 0.04343
Forward rate from t = 3 to t = 4: 0.09661
Forward rate from t = 4 to t = 5: 0.01342
Forward rate from t = 5 to t = 6: 0.04361

Okonkwo owns such an interest rate cap. What will be the sum of the payments he receives from it?

**Solution IRCPCUBF1.** We use the formula Cap payment at time $t_{i+1} = \max[0, R_{t\_i}(t_i, t_{i+1}) - K_R]$.
Since the cap makes its first payment at t = 3 and its last payment at t = 4, we need to consider forward rates from t = 2 to t = 3, from t = 3 to t = 4, and from t = 4 to t = 5. We are given that $K_R = 0.0345$. Thus, the total sum Okonkwo will receive from this cap is
max[0, 0.04343 - 0.0345] + max[0, 0.09661 - 0.0345] + max[0, 0.01342 - 0.0345] =
0.00893 + 0.06211 + 0 = **0.07104**

**Problem IRCPCUBF2.** Alcibiades has bought a caplet which expires in 2 years and pays in 4 years and has a strike rate of 0.2334. A zero-coupon bond paying $1 in 4 years has a price of 0.5523 today. A zero-coupon bond paying $1 in 2 years has a price of 0.8868 today. The annual interest rate volatility is 0.24. In the Black formula for the price of a caplet, what is the value of F?

**Solution IRCPCUBF2.** F = P(0, T)/P(0, T + s) - 1, where T = 2 and s = 2. We are given that P(0, 2) = 0.8868 and P(0, 4) = 0.5523, so F = 0.8868/0.5523 - 1 = **F = 0.6056491037**

**Problem IRCPCUBF3.** Alcibiades has bought a caplet which expires in 4 years and has a strike rate of 0.2334. A zero-coupon bond paying $1 in 4 years has a price of 0.5523 today. A zero-coupon bond paying $1 in 2 years has a price of 0.8868 today. The annual interest rate volatility is 0.24. In the Black formula for the price of a caplet, what is the value of $d_1$?

**Solution IRCPCUBF3.** We use the formula $d_1 = [\ln(F/K) + 0.5\sigma^2 T]/[\sigma\sqrt{(T)}]$, where, from Solution IRCPCUBF2, F = 0.6056491037. We are also given that K = 0.2334, σ = 0.24, and

T = 2. Thus, $d_1 = [\ln(0.6056491037/0.2334) + 0.5*0.24^2*2]/[0.24\sqrt{(2)}]$ = $\mathbf{d_1 = 2.979120601}$

**Problem IRCPCUBF4.** Alcibiades has bought a caplet which expires in 4 years and has a strike rate of 0.2334. A zero-coupon bond paying $1 in 4 years has a price of 0.5523 today. A zero-coupon bond paying $1 in 2 years has a price of 0.8868 today. The annual interest rate volatility is 0.24. In the Black formula for the price of a caplet, what is the value of $d_2$?

**Solution IRCPCUBF4.** We use the formula $d_2 = d_1 - \sigma\sqrt{(T)}$, where, from Solution IRCPCUBF3, $d_1 = 2.979120601$. Thus, $d_2 = 2.979120601- 0.24\sqrt{(2)}$ = $\mathbf{d_2 = 2.639709346}$

**Problem IRCPCUBF5.** Alcibiades has bought a caplet which expires in 4 years and has a strike rate of 0.2334. A zero-coupon bond paying $1 in 4 years has a price of 0.5523 today. A zero-coupon bond paying $1 in 2 years has a price of 0.8868 today. The annual interest rate volatility is 0.24. Use the Black formula to find the price of this caplet.

**Solution IRCPCUBF5.** From Solutions IRCPCUBF2-4, we know that F = 0.6056491037, $d_1 = 2.979120601$, and $d_2 = 2.639709346$. Furthermore, K = 0.2334 and P(0, T + s) = P(0, 4) = 0.5523.
In MS Excel, using the input "=NormSDist(2.979120601)", we find that $N(d_1) = 0.998554546$
In MS Excel, using the input "=NormSDist(2.639709346)", we find that $N(d_2) = 0.995851102$
We use the formula $C[F, P(0, T+s), \sigma, T+s] = P(0, T+s)[FN(d_1) - KN(d_2)]$ =

0.5523[0.6056491037*0.998554546 - 0.2334*0.995851102] = **C = 0.2056444969**

# Section 76

# Binomial Interest-Rate Models

A binomial interest rate model can be used to price interest rate caps. Binomial interest rate trees can be recombining or non-recombining. Here, we will use recombining trees. Given the risk-neutral probability that an interest rate will increase in a given time period, it is possible to find the price of an interest rate cap by discounting the expected value of cap payments for every time period.

The procedure for working with an n-period binomial interest model for pricing interest rate caps is as follows:

- Determine interest rates in the event of increases and decreases: $r_u$, $r_{d,}$ $r_{uu}$, $r_{du}$, and $r_{dd}$.
- Determine cap payments for each movement of interest rates. Remember to multiply each payment by the amount of the loan - or the *notional amount* - if this amount is anything other than $1.
- Use the risk-neutral probability p* and the appropriate term in the expansion of $[p^* + (1-p^*)]^n$ to compute the expected value of a given cap payment.
- Discount the expected value of the cap payment using interest rates for the number of time periods until that payment occurs. If the payment occurs at t = n, your discount factor should be 1/([1 + rate from t = 0 to t = 1][1 + rate from t = 1 to t = 2]...[1 + rate from t = n-1 to t = n]).

Be very careful: discounting is path-dependent!

- Do the same for all scenarios where the expected cap payment is positive and add the results to get the price of the cap.

The best way to illustrate how binomial interest rate models work is by going through several examples.

Some of the problems in this section were designed to be similar to problems from past versions of Exam 3F / Exam MFE. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem BIRM1.** An interest rate cap on a $1 three-year loan has a strike rate of 0.0564. The cap makes annual payments. That is, every year that the one-year interest rate r is above 0.0564, the cap pays the difference r - 0.0564. Otherwise, the cap pays nothing. Each year, the interest rate can change by a factor of 1.6 or by a factor of 0.8. The initial one-year interest rate is 0.069. The risk-neutral probability of the interest rate increasing in any year is 0.5. What is the current value of the expected value of the cap payment that will be made after two years have elapsed? (That is the cap payment will be based on the *one-year interest rate one year from now!!*) Use a binomial interest rate tree where one time period is equal to one year. All interest rates are annual effective rates.

**Solution BIRM1.** First, we need to find $r_u$ and $r_d$, the rates in the events of increase and decrease. Since u = 1.6, $r_u$ = 0.069*1.6 = 0.1104
Since d = 0.8, $r_d$ = 0.069*0.8 = 0.0552

Since $r_d$< 0.0564, the cap will pay nothing if the rate goes down in 1 year.

If the rate goes up in 1 year, the cap will pay 0.1104 - 0.0564 = 0.054

Because the risk-neutral probability of rate increase, p*, is 0.5, the expected value of the cap payment *when it is made* will be 0.5*0.054 = 0.027.

However, we need to discount this expected value by the interest rate from year 1 to year 2 - which is 0.1104 - and also by the interest rate from year 0 to year 1 - which is 0.069.

Thus, our desired answer is 0.027/(1.1104*1.069) = **0.0227460923**

(Since the amount of the loan is $1, we do not need to multiply the answer by anything. If the amount of the loan is $K, we need to multiply the value obtained using the procedure above by K.)

**Problem BIRM2.** An interest rate cap on a $1 three-year loan has a strike rate of 0.0564. The cap makes annual payments. That is, every year that the one-year interest rate r is above 0.0564, the cap pays the difference r - 0.0564. Otherwise, the cap pays nothing. Each year, the interest rate can change by a factor of 1.6 or by a factor of 0.8. The initial one-year interest rate is 0.069. The risk-neutral probability of the interest rate increasing in any year is 0.5. What is the current value of the expected value of the cap payment that will be made after three years have elapsed? (That is the cap payment will be based on the *two-year interest rate one year from now!!*) Use a binomial interest rate tree where one time period is equal to one year. All interest rates are annual effective rates.

**Solution BIRM2.** We need to find $r_{uu}$, $r_{du}$, and $r_{dd}$.

Since u = 1.6, $r_{uu}$ = 0.069*$1.6^2$ = 0.17664.
Cap payment in that case will be 0.17664 - 0.0564 = 0.12024.

Since p* = 0.5, the probability of two increases in the interest rate is 0.5*0.5 = 0.25.

So the expected value of this situation at the time of payment is 0.12024*0.25 = 0.03006

Now we need to discount this payment by the t = 0 to t = 1, the t = 1 to t = 2, and the

t = 2 to t = 3 interest rates for this scenario:
0.03006/(1.069*1.1104*1.17664) = 0.0215222767. This is the current value of the expected value of this scenario.

Since u = 1.6 and d = 0.8, $r_{du}$ = 0.069*1.6*0.8 = 0.08832.

Cap payment in that case will be 0.08832 - 0.0564 = 0.03192.

Since p* = 0.5, the probability of one increase and one decrease in the interest rate is 2*0.5*0.5 = 0.5. (Think of the binomial expansion of $[p^* + (1-p^*)]^2$.)

So the expected value of this situation at the time of payment is 0.03192*0.5 = 0.01596.
Now we need to discount this payment by the t = 0 to t = 1, the t = 1 to t = 2, and the t = 2 to t = 3 interest rates for this scenario. We need to be careful when discounting here, because the discounting is path-dependent. Half of the payments we expect in this scenario occurred from the interest rate going down and then going up. So we would need to discount that half thus:
0.5*0.01596/(1.069*1.0552*1.08832) = 0.006500309

The other half of the expected payments occurred from the interest rate going up and then going down. So we would need to discount that half thus:

0.5*0.01596/(1.069*1.1104*1.08832) = 0.0061771639

We add 0.006500309 + 0.0061771639 = 0.0126774729

This is the current value of the expected value of this scenario.

Since $r_d$< 0.0564, it follows that $r_{dd}$< 0.0564, so cap payment in that event will be 0.

To get our desired answer, we add the current values of the expected values of the payments for the two scenarios where such payments will actually occur:

0.0215222767 + 0.0126774729 = **0.0341997496**

**Problem BIRM3.** An interest rate cap on a $1 three-year loan has a strike rate of 0.0564. The cap makes annual payments. That is, every year that the one-year interest rate r is above 0.0564, the cap pays the difference r - 0.0564. Otherwise, the cap pays nothing. Each year, the interest rate can change by a factor of 1.6 or by a factor of 0.8. The initial one-year interest rate is 0.069. The risk-neutral probability of the interest rate increasing in any year is 0.5. Use a two-period binomial interest rate model to find the price of this cap. All interest rates are annual effective rates.

**Solution BIRM3.** This cap will make three payments - at time t = 1, at time t = 2, and at time t = 3. In Solution BIRM1, have found the current value of the expected value of the t = 2 payment to be 0.0227460923. In Solution BIRM2, have found the current value of the expected value of the t = 3 payment to be 0.0341997496. The time t = 1 payment will be the difference r - K, where r is the t = 0 to t = 1 interest rate, or 0.069, and K is the strike rate, or 0.0564. Thus, r - K = 0.0126. To find the price of the cap, we just need to add these three values: 0.0126 + 0.0227460923 + 0.0341997496 = **0.0695458419**

**Problem BIRM4.**

**Similar to Question 9 from the Society of Actuaries' May 2007 Exam MFE:**

The current one-year interest rate is 0.07. In one year, the one-year interest rate will be 0.105 or 0.049. In two years, the one-year interest rate will be 0.1575, 0.0735, or 0.0343. All interest rates are annual effective rates. The risk-neutral probability of a rate going up in any year is 0.6. Loan payments are made annually on a three-year $50 loan. An interest rate cap on this loan has a strike rate of 0.09. Use a two-period binomial model to find the price of this cap.

**Solution BIRM4.** Since the current one-year interest rate is below the strike rate, the cap will make no payment at t = 1.

At t = 2, the cap will only make a payment if r = 0.105, in which case the cap will pay 0.105 - 0.09 = 0.015 - with probability 0.6, so the expected value of this payment is 0.009 per dollar of the loan.
We discount this expected value thus: 0.009/(1.105*1.07) = 0.0076119592. Since the amount of the loan is $50, we multiply 0.0076119592*50 to get 0.3805979617.

At t = 3, the cap will only make a payment if r = 0.1575, in which case the payment will be 0.1575 - 0.09 = 0.0675 - with probability $0.6^2$ = 0.36, so the expected value of this payment is 0.0243 per dollar of the loan.

We discount this expected value thus: 0.0243/(1.1575*1.105*1.07) = 0.017755758. Since the amount of the loan is $50, we multiply 0.017755758*50 to get 0.887787902.

We add these current values of the expected values of the possible payments to get the current cap price: 0.3805979617 + 0.887787902 = **$1.268385864**

**Problem BIRM5.**

**Similar to Question 9 from the Society of Actuaries' May 2007 Exam MFE:**

The current one-year interest rate is 0.12. In one year, the one-year interest rate will be 0.216 or 0.096. In two years, the one-year interest rate will be 0.3888, 0.1728, or 0.0768. All interest rates are annual effective rates. The risk-neutral probability of a rate going up in any year is 0.3. Loan payments are made annually on a three-year $300 loan. An interest rate cap on this loan has a strike rate of 0.15. Use a two-period binomial model to find the price of this cap.

**Solution BIRM5.** Since the current one-year interest rate is below the strike rate, the cap will make no payment at t = 1.

At t = 2, the cap will only make a payment if r = 0.216, in which case the cap will pay 0.216 - 0.15 = 0.066 - with probability 0.3, so the expected value of this payment is 0.0198 per dollar of the loan.
We discount this expected value thus: 0.0198/(1.216*1.12) = 0.0145382989. Since the amount of the loan is $300, we multiply 0.0145382989*300 to get $4.361489662.

At t = 3, the cap will pay if r = 0.3888 or if r = 0.1728.

If r = 0.3888, the cap will pay 0.3888 - 0.15 = 0.2388 - with probability $0.3^2$ = 0.09, so the expected value of this payment is 0.021492 per dollar of the loan.
We discount this expected value thus: 0.021492/(1.3888*1.216*1.12) = 0.0113628043. Since the amount of the loan is $300, we multiply 0.0113628043*300 to get $3.408841286.

If r = 0.1728, the cap will pay 0.1728 - 0.15 = 0.0228 - with probability 2*0.3*0.7 = 0.42, so the expected value of this payment is 0.009576 per dollar of the loan. Half of this expected payment occurs from the interest rate going up and then down. We discount this expected value thus: 0.5*0.009576/(1.1728*1.216*1.12) = 0.0029976339. Since the amount of the loan is $300, we multiply 0.0029976339*300 to get $0.8992901603.

The other half of this expected payment occurs from the interest rate going down and then up. We discount this expected value thus:

0.5*0.009576/(1.1728*1.096*1.12) = 0.003325842. Since the amount of the loan is $300, we multiply 0.003325842*300 to get $0.9977526.

We add all the current values of the expected values of the future cap payments to get the current price of the cap: 4.361489662 + 3.408841286 + 0.8992901603 + 0.9977526 = **$9.667373708**

# Section 77

# Basics of the Black-Derman-Toy (BDT) Interest-Rate Model

**Calibration** is the matching of a model to fit observed data. The models thus far examined - the Vasicek and CIR models - often do not match data exogenous to the models, because the models are based on four parameters while real-world data sets may have more than four members. The Black-Derman-Toy (BDT) model attempts to incorporate an element of calibration to allow applications to real-world data.

The BDT model assumes that the short-term interest rate is lognormally distributed. It uses a recombining binomial interest rate tree with two parameters, $R_{ih}$ and $\sigma_i$. $R_{ih}$ is the *rate level parameter*, and $\sigma_i$ is the *volatility parameter*. Both parameters are used to match the tree with observed data. (McDonald 2006, p. 799).

Note: All interest rates in the BDT are *annual effective*, not continously compounded, rates.

For each time period h, the nodes in the binomial BDT interest rate differ from one another by a factor of $\exp[2\sigma_i\surd(h)]$. So if $r_d = R_h$, then $r_u = R_h\exp[2\sigma_i\surd(h)]$. In the BDT model, i is the number of time periods that have elapsed since time 0, and h is the length of each time period.

We can let P[h, T, r(h)] be the time-h price of a zero-coupon bond maturing at T. Here, r(h) is is the time-t short-term interest rate. Then the yield of the bond is

$y[h, T, r(h)] = P[h, T, r(h)]^{-1/(T-h)} - 1.$

The yield volatility is Yield volatility = $0.5\ln(y[h, T, r_u]/y[h, T, r_d])$

The nodes in a BDT binomial tree are not necessarily determined by the nodes of the previous time period. Rather, they depend on actual observed data.

If we are given some number of nodes for a particular time period while the others are left blank, it may be possible to fill in the rest of the nodes, as the ratio between every two adjacent nodes in a single time period is the same. For instance, if we are given $r_{uu}$ and $r_{ud}$, we can find $r_{dd}$ via the following multiplication: $r_{dd} = r_{ud}(r_{ud}/ r_{uu})$. We know this is true since $(r_{ud}/ r_{uu}) = (r_{dd}/ r_{ud})$.

In general, at time i for $r_{(u^\wedge n)d^\wedge(n-i)}$, where n is the number of times the interest rate has gone up and (i-n) is the number of times the interest rate has gone down, we have the formula

$\mathbf{r_{(u^n)d^{(i-n)}} = r_{d^{(i)}}\exp[2n\sigma_i\sqrt{(h)}]}$.

**Important:** In the Black-Derman-Toy model, the risk-neutral probability of an interest rate going up in any time period is always **0.5**.

**Sources:** McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 24, pp. 798-804.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem BBDTIRM1.** You are given the following values of a Black-Derman-Toy binomial tree: $r_{ud} = 0.085$ and $r_{dd} = 0.034$. Find $r_{uu}$.

**Solution BBDTIRM1.** The ratio between the nodes in a BDT tree is constant.

Thus, $r_{uu} = r_{ud}\,(r_{ud} / r_{dd}) = 0.085*(0.085/0.034)$ = $\mathbf{r_{uu} = 0.2125}$.

**Problem BBDTIRM2.** You are given the following values of a Black-Derman-Toy binomial tree: $r_{uu} = 0.1525$ and $r_{dd} = 0.0884$. Find $r_{ud}$.

**Solution BBDTIRM2.** The ratio between the nodes in a BDT tree is constant. Thus,

$r_{uu}/ r_{ud} = r_{ud}/ r_{dd}$ and $(r_{ud})^2 = r_{dd}\, r_{uu}$ , so $r_{ud} = \sqrt{(r_{dd}\, r_{uu})} = \sqrt{(0.0884*0.1525)}$ = $\mathbf{r_{ud} = 0.1161077086}$.

**Problem BBDTIRM3.** In a BDT model, you are given that $r_{ddddd} = 0.0586$, and $\sigma_5 = 0.21$. Each time period in the binomial tree is 2 years. Find $r_{uuudd}$.

**Solution BBDTIRM3.** We use the formula $r_{(u^n)d^{(i-n)}} = r_{d^{(i)}}\exp[2n\sigma_i\sqrt{(h)}]$, which in this case translates to $r_{uuudd} = r_{ddddd}\exp[2*3\sigma_5\sqrt{(h)}]$, since here, n = 3 and i = 5. Furthermore, $\sigma_5 = 0.21$ and h = 2. Thus, $r_{uuudd} = 0.0586\exp[2*3*0.21\sqrt{(2)}]$ = $\mathbf{r_{uuudd} = 0.3841536082}$

**Problem BBDTIRM4.** In a BDT model, you are given that $r_{uudddd} = 0.1185$, and $\sigma_6 = 0.43$. Each time period in the binomial tree is 1 year. Find $r_{uuuuuu}$.

**Solution BBDTIRM4.** We use the formula $r_{(u^n)d^{(i-n)}} = r_{d^{(i)}}\exp[2n\sigma_i\sqrt{(h)}]$, which in this case translates to $r_{uuuuuu} = r_{dddddd}\exp[2*6\sigma_6\sqrt{(h)}]$. However, we are given $r_{uudddd}$, not $r_{dddddd}$.

We know that $r_{uudddd} = r_{dddddd}\exp[2*2\sigma_6\sqrt{(h)}]$, so $r_{dddddd} = r_{uudddd}/\exp[2*2\sigma_6\sqrt{(h)}]$ and thus

$r_{uuuuuu} = r_{uudddd}\exp[2*4\sigma_6\sqrt{(h)}]$, where h = 1, $\sigma_6 = 0.43$, and $r_{uudddd} = 0.1185$. Thus,

$r_{uuuuuu} = 0.1185\exp[2*4*0.43\sqrt{(1)}]$ = $\mathbf{r_{uuuuuu} = 3.695654543}$

(Yes, that is an interest rate of over 369%.)

**Problem BBDTIRM5.** A 1-year caplet (paying at time t = 2 years) has a strike rate of 0.07. For a given BDT model, $r_u$ = 0.15 and $r_d$ = 0.08. Also, $r_0$ = 0.06. Find the value of the caplet at time t = 0.

**Solution BBDTIRM5.** This problem is used to illustrate how expected caplet payments are discounted in the BDT model. The risk-neutral probability of the short-term interest rate moving up in any period is 0.5, so we can expect that half of the time, the caplet's payoff after one year will be $(0.15 - 0.07)/(1+ r_u) = 0.08/1.15 = 0.0695652174$, while the other half of the time, the caplet's payoff after one year will be $(0.08 - 0.07)/(1+ r_d) = 0.01/1.08 = 0.0092592593$. To find the value of the caplet at t = 0, we need to discount the average of these two values by the interest rate from t = 0 to t = 1, i.e., $r_0$ = 0.06. We do the discounting as follows: 0.5(0.0695652174 + 0.0092592593)/1.06 = **0.0371813569,** which is the time t = 0 value of this caplet.

# Section 78

# Pricing Caplets Using the Black-Derman-Toy (BDT) Interest-Rate Model

We can use multiple-period binomial trees in the Black-Derman-Toy model to price caplets. Remember that the risk-neutral probability on an up movement in the short-term interest rate is always 0.5 in the BDT model. Thus, if $P_x$ is the expected value of a caplet at some node in a BDT binomial tree, where x is some product of u's and d's (up and down movements in the interest rates), then we can obtain $P_x$ using the following formula:

$P_x = 0.5(P_{xu} + P_{xd})/(1 + r_x)$, where $r_x$ is the interest rate that corresponds with an expected caplet value of $P_x$.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem PCUBDTIRM1.** In a particular Black-Derman-Toy binomial tree, you are given $r_{uuu}$ = 0.1, $r_{uuuu}$ = 0.11, and $r_{uuud}$ = 0.09. A particular caplet has a strike rate of 0.08 and a notional amount of $1. What is the expected value of a caplet at node uuu of the BDT binomial tree? That is, find $P_{uuu}$.

**Solution PCUBDTIRM1.** First, we find $P_{uuuu}$ = (0.11 - 0.08)/1.11 = 0.027027027. Next, we find $P_{uuud}$ = (0.09 - 0.08)/1.09 = 0.0091743119. Now we perform discounting:
$P_{uuu} = 0.5(P_{uuuu} + P_{uuud})/(1 + r_{uuu}) = 0.5(0.027027027 + 0.0091743119)/(1.1) =$

$\mathbf{P_{uuu} = 0.0164551541}$

**Problem PCUBDTIRM2.** The price of a caplet today is $3. The caplet has a strike rate of 0.06 a notional amount of $100. The one-year short-term interest rate today is 0.05. In one year, the one-year short-term interest rate will be either 0.1 or Q. Find Q using a one-period BDT binomial tree.

**Solution PCUBDTIRM2.** We first find $P_u = 100(0.1 - 0.06)/1.1 = P_u = 3.636363636$.

We also know that $P_0 = 0.5(P_u + P_d)/(1 + r_0)$ and that $P_0 = 3$ and $r_0 = 0.05$. We can use this to solve for $P_d$. $P_0(1 + r_0) = 0.5(P_u + P_d)$ and $2P_0(1 + r_0) - P_u =$

$P_d = 2*3(1.05) - 3.636363636 = P_d = 2.663636364$

We also know that, by definition, $P_d = 100(r_d - 0.06)/(1 + r_d) = 100(Q - 0.06)/(1 + Q)$

Thus, 2.663636364(1 + Q) = 100Q - 6 and

8.663636364 = 97.33636364Q. So **Q = 0.0890071916**

**Problem PCUBDTIRM3.** You are given the following short-term interest rates in a Black-Derman-Toy binomial tree: $r_0 = 0.04$, $r_d = 0.02$, $r_u = 0.11$, $r_{dd} = 0.037$, $r_{du} = 0.074$, $r_{uu} = 0.148$.

A 2-year caplet (which actually pays at time t = 3 years) has a strike rate of 0.07. The caplet has a notional amount of \$100. What is the current value of this caplet?

**Solution PCUBDTIRM3.**

First, we find the possible year-2 expected values of payoffs to this caplet.

If we have $r_{uu} = 0.148$, then the year-2 payoff to the caplet will be 100(0.148 - 0.07)/1.148 =

$P_{uu} = 6.794425087$

If we have $r_{du} = 0.074$, then the year-2 payoff to the caplet will be 100(0.074 - 0.07)/1.074 =

$P_{du} = 0.3724394786$

Since $r_{dd} = 0.037 < 0.07$, the caplet will pay nothing if the interest rate in 2 years is 0.037.

So $P_{dd} = 0$.

Now we do discounting to find $P_u$ and $P_d$.

$P_u = 0.5(P_{uu} + P_{ud})/(1 + r_u) = 0.5(6.794425087 + 0.3724394786)/(1 + 0.11) = 3.228317372$

$P_d = 0.5(P_{du} + P_{dd})/(1 + r_d) = 0.5(0.3724394786 + 0)/(1 + 0.02) = 0.1825683719$

Now we do discounting to find $P_0$, the current value of the caplet.

$P_0 = 0.5(P_u + P_d)/(1 + r_0) = 0.5(3.228317372 + 0.1825683719)/(1.04) =$ $\mathbf{P_0 = 1.639848915}$

**Problem PCUBDTIRM4.** You are given the following short-term interest rates in a Black-Derman-Toy binomial tree: $r_0 = 0.09$, $r_d = 0.03$, $r_u = 0.13$, $r_{dd} = 0.023$, $r_{du} = 0.08$.

A 2-year caplet (which actually pays at time t = 3 years) has a strike rate of 0.05. The caplet has a notional amount of \$100. What is the current value of this caplet?

**Solution PCUBDTIRM4.** This problem is slightly different from Problem PCUBDTIRM3 in that $r_{uu}$ is left out and needs to be calculated. $r_{uu} = r_{du}\,(r_{du} / r_{dd}) = 0.08(0.08/0.023) = r_{uu} =$ 0.2782608696

Now we find the possible year-2 expected values of payoffs to this caplet.

If we have $r_{uu}$ = 0.2782608696, then the year-2 payoff to the caplet will be

$P_{uu}$ =100(0.2782608696 - 0.05)/1.2782608696 = $P_{uu}$ = 17.85714286

If we have $r_{du}$ = 0.08, then the year-2 payoff to the caplet will be 100(0.08 - 0.05)/1.08 =

$P_{du}$ = 2.777777778

Since $r_{dd}$ = 0.023 < 0.05, the caplet will pay nothing if the interest rate in 2 years is 0.023.

So $P_{dd}$ = 0.

Now we do discounting to find $P_u$ and $P_d$.

$P_u = 0.5(P_{uu} + P_{ud})/(1 + r_u)$ = 0.5(17.85714286 + 2.777777778)/(1.13) = $P_u$ = 9.127967424

$P_d = 0.5(P_{du} + P_{dd})/(1 + r_d)$ = 0.5(2.777777778 + 0)/(1.03) = $P_d$ = 1.348435814

Now we do discounting to find $P_0$, the current value of the caplet.

$P_0 = 0.5(P_u + P_d)/(1 + r_0)$ = 0.5(9.127967424 + 1.348435814)/(1.09) = $\mathbf{P_0 = 4.805689559}$

**Problem PCUBDTIRM5.**

**Similar to Question 15 from the Society of Actuaries' Sample MFE Questions and Solutions:**

You are given the following short-term interest rates in a Black-Derman-Toy binomial tree: $r_0$ = 0.05, $r_d$ = 0.04, $r_u$ = 0.09, $r_{dd}$ = 0.06, $r_{uu}$ = 0.18, $r_{ddd}$ = 0.04, $r_{uud}$ = 0.16. A 3-year caplet (which actually pays at time t = 4 years) has a strike rate of 0.07. The caplet has a notional amount of \$100. What is the current value of this caplet?

**Solution PCUBDTIRM5.**

This is a three-period BDT model problem, where we need to find a lot of the nodes that are not given. We know that $r_{uu} / r_{du} = r_{du} / r_{dd}$, so $(r_{du})^2 = r_{uu}r_{dd}$ and $r_{du} = \sqrt{(r_{uu}r_{dd})} = \sqrt{(0.18*0.06)}$ =

$r_{du}$ = 0.1039230485.

We also know that $r_{uud} / r_{udd} = r_{udd} / r_{ddd}$, so $r_{udd} = \sqrt{(r_{uud}r_{ddd})} = \sqrt{(0.16*0.04)}$ = $r_{udd}$ = 0.08

Moreover, $r_{uuu} = r_{uud}(r_{uud}/ r_{udd})$ = 0.16(0.16/0.08) = $r_{uuu}$ = 0.32

Now we find the possible year-3 expected values of payoffs to this caplet.

$P_{uuu}$ =100(0.32 - 0.07)/1.32 = $P_{uuu}$ = 18.9393939393

$P_{uud}$ =100(0.16 - 0.07)/1.16 = $P_{uud}$ = 7.75862069

$P_{udd}$ =100(0.08 - 0.07)/1.08 = $P_{udd}$ = 0.9259259259

Since $r_{ddd} = 0.04 < 0.07$, $P_{ddd} = 0$.

Now we do discounting to find $P_{uu}$,$P_{ud}$,and $P_d$.

$P_{uu} = 0.5(P_{uuu} + P_{uud})/(1 + r_{uu})$ = 0.5(18.9393939393 + 7.75862069)/(1.18) = $P_{uu}$ = 11.31271806

$P_{ud} = 0.5(P_{uud} + P_{udd})/(1 + r_{ud})$ = 0.5(7.75862069 + 0.9259259259)/(1.1039230485) =

$P_{ud}$ = 3.93349275

$P_{dd} = 0.5(P_{udd} + P_{ddd})/(1 + r_{dd})$ = 0.5(0.9259259259 + 0)/(1.06) = $P_{dd}$ = 0.4367575122

Now we do discounting to find $P_u$ and $P_d$.

$P_u = 0.5(P_{uu} + P_{ud})/(1 + r_u)$ = 0.5(11.31271806 + 3.93349275)/(1.09) = $P_u$ = 6.993674683

$P_d = 0.5(P_{du} + P_{dd})/(1 + r_d)$ = 0.5(3.93349275 + 0.4367575122)/(1.04) = $P_d$ = 2.101081857

Now we do discounting to find $P_0$, the current value of the caplet.

$P_0$ = 0.5(6.993674683 + 2.101081857)/(1.05) = **$P_0$ = 4.330836448**

# Section 79

# Determining Yield Volatilities and the Basics of Constructing Binomial Trees in the Black-Derman-Toy (BDT) Interest-Rate Model

This section will focus on determining yield volatilities in the BDT model. Some of the following information and formulas were already given in Section 77.

We can let P[h, T, r(h)] be the time-h price of a zero-coupon bond maturing at T. Here, r(h) is is the time-t short-term interest rate. Then the yield of the bond is

$y[h, T, r(h)] = P[h, T, r(h)]^{-1/(T-h)} - 1.$

The yield volatility is Yield volatility = $0.5\ln(y[h, T, r_u]/y[h, T, r_d]) =$

$0.5\ln([P[h, T, r_u]^{-1/(T-h)} - 1]/[P[h, T, r_d]^{-1/(T-h)} - 1])$

It may be possible for one to be asked on the exam to determine the Black-Derman-Toy binomial interest rate tree, given data on bonds with various times to maturity - including their yields to maturity, current bond prices, and volatility in year 1 (or time period t = h, where h is one period in the BDT binomial model). This section will begin introduce students to this process for finding $r_0$, $r_u$, and $r_d$.

If $P_h$ is the price of an h-year bond, where h is one time period in the BDT model, then

$P_h = 1/(1 + r_0)$ and thus

$r_0 = 1/P_h - 1$

If $P_{2h}$ is the price of an 2h-year bond, then $R_h = r_d$ and $\sigma_h$ meet the following conditions:

$P_{2h} = 0.5P_h(1/(1 + R_h\exp[2\sigma_h]) + 1/(1 + R_h))$

[To memorize this formula, think of it as $P_{2h} = 0.5P_h(1/(1 + r_u) + 1/(1 + r_d))$]

$r_0 = 0.5\ln[R_h\exp[2\sigma_h]/R_h]$

That is,

$r_0/0.5 = \ln[R_h\exp[2\sigma_h]/R_h]$

$\exp[r_0/0.5] = R_h\exp[2\sigma_h]/R_h$

$\exp[r_0/0.5] = \exp[2\sigma_h]$

$r_0/0.5 = 2\sigma_h$

$\mathbf{r_0 = \sigma_h}$

Now we can make the substitution $r_0 = \sigma_h$ into the expression for $P_{2h}$:

$\mathbf{P_{2h} = 0.5P_h(1/(1 + R_h\exp[2r_0]) + 1/(1 + R_h))}$

**Source:** McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 24, pp. 804-805.

Some of the problems in this section were designed to be similar to problems from past versions of Exam 3F / Exam MFE. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

**Problem DYVBCBTBDTIRM1.** A Black-Derman-Toy interest rate model based on a particular 4-year zero-coupon bond has $P[h, 4, r_u] = 0.7765$ and $P[h, 4, r_d] = 0.8495$. Each time period in the model is equal to one year. Find the volatility in year 1 of a 4-year bond yield. (Note that the 4-year bond yield refers to the yield of a bond that expires 4 years from today.)

**Solution DYVBCBTBDTIRM1.** We are given that T = 4 and $P[h, 4, r_u] = P[1, 4, r_u] = 0.7765$.

Thus, $y[1, 4, r_u] = [1, 4, r_u]^{-1/(4-1)} - 1 = 0.7765^{-1/3} - 1 = 0.0879764962$

$y[1, 4, r_d] = [1, 4, r_d]^{-1/(4-1)} - 1 = 0.8495^{-1/3} - 1 = 0.0558742688$

Yield volatility = $0.5\ln(y[h, T, r_u]/y[h, T, r_d]) = 0.5\ln(y[1, 4, r_u]/y[1, 4, r_d])$ =

$0.5\ln(0.0879764962/0.0558742688)$ = **Yield volatility = 0.2370453955**

**Problem DYVBCBTBDTIRM2.** A Black-Derman-Toy interest rate model based on a particular 9-year zero-coupon bond has $P[h, 9, r_u] = 0.3345$ and $P[h, 9, r_d] = 0.5123$. Each time period in the model is equal to three years. Find the volatility in year 3 of the 9-year bond yield. (Note that the 9-year bond yield refers to the yield of a bond that expires 9 years from today.)

**Solution DYVBCBTBDTIRM2.** We use the formula

Yield volatility = $0.5\ln([P[h, T, r_u]^{-1/(T-h)} - 1]/[P[h, T, r_d]^{-1/(T-h)} - 1])$

Yield volatility = $0.5\ln([P[3, 9, r_u]^{-1/(9-3)} - 1]/[P[3, 9, r_d]^{-1/(9-3)} - 1])$

Yield volatility = $0.5\ln([0.3345^{-1/6} - 1]/[0.5123^{-1/6} - 1])$

Yield volatility = 0.5ln(0.2002378353/0.117924843)

**Yield volatility = 0.2647291692**

**Problem DYVBCBTBDTIRM3.**

**Similar to Question 37 from the Casualty Actuarial Society's Spring 2007 Exam 3:**

The following information is available regarding a particular Black-Derman-Toy interest rate model. Each time period in the model is of length 2 years.

For a six-year zero-coupon bond, the prices at the nodes of the binomial tree are as follows:
$P_0 = 0.6565$; $P_u = 0.7291$; $P_d = 0.8101$; $P_{uu} = 0.8224$; $P_{du} = 0.9$; $P_{dd} = 0.984822179$.

What is the volatility in year 2 of the 6-year bond yield? (Note that the 6-year bond yield refers to the yield of a bond that expires 6 years from today.)

**Solution DYVBCBTBDTIRM3.**

Because we are asked to find the year-2 volatility of the 6-year bond yield (i.e., the yield of the bond whose prices are modeled by this BDT binomial tree), we only need to concern ourselves with $P_u = 0.7291$ and $P_d = 0.8101$. Here, T = 6 and h = 2, so T - h = 4.

We use the formula

Yield volatility = $0.5\ln([P[h, T, r_u]^{-1/(T-h)} - 1]/[P[h, T, r_d]^{-1/(T-h)} - 1])$
Yield volatility = $0.5\ln(P_u^{-1/(T-h)} - 1]/[P_d^{-1/(T-h)} - 1])$
Yield volatility = $0.5\ln(0.7291^{-1/4} - 1]/[0.8101^{-1/4} - 1])$
Yield volatility = $0.5\ln(0.7291^{-1/4} - 1]/[0.8101^{-1/4} - 1])$
Yield volatility = 0.5ln(0.0821892747/0.0540600222)

**Yield volatility = 0.2094649322**

**Problem DYVBCBTBDTIRM4.** A 1-year zero-coupon bond has a current price of 0.8842. A 2-year zero-coupon bond has a current price of 0.7767. In a BDT interest rate model built on the basis of this information, what is $r_0$? One time period in this BDT is equal to one year.

**Solution DYVBCBTBDTIRM4.** We use the formula $r_0 = 1/P_h - 1 = 1/P_1 - 1 = 1/0.8842 - 1 =$

$\mathbf{r_0 = 0.1309658448}$

**Problem DYVBCBTBDTIRM5.** A 1-year zero-coupon bond has a current price of 0.8842. A 2-year zero-coupon bond has a current price of 0.7767. In a BDT interest rate model built on the basis of this information, what is $r_d$? One time period in this BDT is equal to one year.

**Solution DYVBCBTBDTIRM5.** We use the formula $P_{2h} = 0.5P_h(1/(1 + R_h\exp[2r_0]) + 1/(1 + R_h)) =$

$P_{2h} = 0.5P_h(1/(1 + r_d\exp[2r_0]) + 1/(1 + r_d))$. Here, h = 1, $P_h = 0.8842$, and $P_{2h} = 0.7767$. From Solution DYVBCBTBDTIRM4, $r_0 = 0.1309658448$. Thus,

$0.7767 = 0.5*0.8842(1/(1 + r_d\exp[2*0.1309658448]) + 1/(1 + r_d))$

$1.756842343 = (1/(1 + 1.299437774r_d) + 1/(1 + r_d))$

$1.756842343 = [(1 + r_d) + (1 + 1.299437774r_d)]/[(1 + r_d)(1 + 1.299437774r_d)]$

$1.756842343(1 + r_d)(1 + 1.299437774r_d) = 2 + 2.299437774r_d$

$1.756842343(1 + 2.299437774r_d + 1.299437774r_d^2) = 2 + 2.299437774r_d$

$1.756842343 + 4.039749647r_d + 2.282907303r_d^2= 2 + 2.299437774r_d$

$2.282907303r_d^2 + 1.740311873r_d - 0.2431576566 = 0$

By the quadratic formula, the relevant positive value of $r_d$ is $\mathbf{r_d = 0.1206316973}$

# Section 80

# Equity-Linked Insurance Contracts

This section will discuss how to approach problems dealing with equity-linked insurance contracts. There are few existing sample problems available on this topic, so this section will attempt to give some practice questions by which students might be able to learn the basic approach for it.

When working with any expression for a maximum, it may sometimes be necessary to transform that expression into an expression representing the payoff of an option. In that case, it is useful to keep in mind the identities:

**Formula 80.1:** max(AB, C) = B*max(A, C/B)

**Formula 80.2:** max(A, B) = A + max(0, B - A)

An equity-linked insurance contract will often charge a premium P at the start. In this case, in order for no arbitrage to be possible, the contract's value C(T) at time T = 0 must be equal to P. Remember to set P = C(0) (using whatever values, expressions, or variables for P and C(0) are given) when dealing with this kind of problem.

**Problem ELIC1.** The price of the stock of Treacherous Industries can be expressed as S(T), where T is some time period. The current (T = 0) price of Treacherous Industries stock is $50 per share. A contract expiring in one year has the following payoff:

S(0)*Max[S(1)/S(0), 1.2]. Which of these portfolios will have an equivalent payoff to that of the contract? All options are European, and more than one answer may be correct. Assume that the risk-free interest rate is zero.

(a) A $50-strike put option expiring in one year and $60.
(b) A $50-strike put option expiring in one year and one share of stock.
(c) A $60-strike put option expiring in one year and $60.
(d) A $60-strike call option expiring in one year and $60.
(e) A $60-strike put option expiring in one year and one share of stock.
(f) A $60-strike call option expiring in one year and one share of stock.

**Solution ELIC1.** S(0)*Max[S(1)/S(0), 1.2] = Max[S(1), 1.2S(0)]. We have essentially performed multiplication by S(0) on all terms within the Max expression and compensated for it by dividing by S(0) outside the Max expression.

Max[S(1), 1.2S(0)] = Max[S(1), 1.2*50] = Max[S(1), 60] = S(1) + Max[0, 60 - S(1)]. This is the same as the payoff on one share of stock plus one $60-strike put option. So (e) is a correct answer.

But it is also the case that Max[S(1), 60] = 60 + Max[S(1) - 60, 0]. This is the same as the payoff on $60 plus one $60-strike call option. So (d) is a correct answer.

Thus, **(d) and (e) are correct answers.**

**Problem ELIC2.** A particular equity-linked insurance contract is based on the price of a stock S(T) at time T and has a value of P*(1 - X)*F[S(T)], where P and (1 - X) are constant and F[S(T)] is some function of the stock price S(T). Here, F[S(T)] = $S^2$ + 3S, and S(T) = $50^{(T/3)}$. What is the ratio of the contract's value at T = 3 years to its value at T = 1 year?

**Solution ELIC2.** At T = 3, S(T) = $50^{(3/3)}$ = 50, and F[50] = $50^2$ + 3*50 = F[50] = 2650.

At T = 1, S(T) = $50^{(1/3)}$ = 3.684031499, and

F[3.684031499] = $3.684031499^2$ + 3*3.684031499 = F[3.684031499] = 24.62418258.

Since P and (1 - X) are constant and are present in the expression for the contract's value in both time periods, they simply cancel out in the division and can be ignored. The ratio we seek is thus F[50]/F[3.684031499] = 2650/24.62418258 = **107.6177855**

**Problem ELIC3.** A particular equity-linked insurance contract is based on the price of a stock S(T) at time T and has a value of P*(1 - X)*F[S(T)], where P and (1 - X) are constant and F[S(T)] is some function of the stock price S(T). Here, F[S(T)] = $S^2$ + 3S, and S(T) = $50^{(T/3)}$. Now say that P is the premium that one must pay at time T = 0 to enter into the contract and that no arbitrage is possible. Find the value of X.

**Solution ELIC3.** If no arbitrage is to be possible, the value of the contract at T = 0 must be P. But our expression for the value of the contract also says that its value at T = 0 is P*(1 - X)*F[S(0)]. Thus, we set P = P*(1 - X)*F[S(0)], which means that

1 = (1 - X)*F[S(0)]. Now we find F[S(0)]. S(0) = $50^{(0/3)}$ = 1 and F[1] = $1^2$ + 3*1 = 4. Thus, 1 = (1 - X)*4 and ¼ = 1 - X, so **X = 0.75**

Now we have the practice necessary to attempt an exam-style question.

**Problem ELIC4.**

**Similar to Question 3 from the Society of Actuaries' Sample MFE Questions and Solutions:**

A particular equity-linked insurance contract is based on the price of a stock S(T) at time T and has a value of P*(1 - X)*Max[S(T)/S(0), $(1 + r)^T$]. A premium P is paid for this contract when it is entered into (i.e., now). This contract will reach maturity in 3 years. The minimum guaranteed rate of return r is 0.1. The stock pays no dividend and has a price of $500 per share at time T = 0. Also at time T = 0, the price of one three-year, $665.5-strike European put option on the stock is $55.55. Find the value of X.

**Solution ELIC4.** First we work with the expression Max[S(T)/S(0), $(1 + r)^T$], which, for T = 3 is Max[S(3)/S(0), $(1.1)^3$] = Max[S(3)/S(0), 1.331] = [1/S(0)]Max[S(3), 1.331S(0)] = [1/500]Max[S(3), 665.5], since S(0) = 500. We recall the formula

max(A, B) = A + max(0, B - A).

Thus, Max(S(3), 665.5) = S(3) + Max(0, 665.5 - S(3)), from which it follows that

[1/500]Max[S(3), 665.5] = (1/500)(S(3) + Max(0, 665.5 - S(3))).

The premium P will be equal to the present value of the equity-linked insurance contract, so

P = PV(P*(1 - X)*(1/500)(S(3) + Max(0, 665.5 - S(3))) →

The expression (P/500)*(1 - X)*(S(3) + Max[0, 665.5 - S(3)]) applies to time t = 3. How do we get its present value?

The present value of Max(0, 665.5 - S(3)) is just the price of the put option, 55.55.

The present value of S(3) is the current stock price, S(0). The other terms are not affected by present value considerations.

Thus, PV(P*(1 - X)*(1/500)(S(3) + Max(0, 665.5 - S(3))) = (P/500)*(1 - X)*(S(0) + 55.55), and so

P = (P/500)*(1 - X)*(S(0) + 55.55) = (P/500)*(1 - X)*(500 + 55.55).

Thus, 500 = (1 - X)*555.55, implying that
X = 1 - 500/555.55 = **X = 0.0999909999**.

**Problem ELIC5.**

**Similar to Question 3 from the Society of Actuaries' Sample MFE Questions and Solutions:**

A particular equity-linked insurance contract is based on the price of a stock S(T) at time T and has a value of P*(1 - X)*Max[S(T)/S(0), $(1 + r)^T$]. A premium P is paid for this contract when it is entered into (i.e., now). This contract will reach maturity in 4 years. The minimum guaranteed rate of return r is 0.0064844317. The stock pays no dividend and has a price of $700 per share at time T = 0. Also at time T = 0, the price of one four-year, $900-strike European call option on the stock is $91. Find the value of X.

**Solution ELIC5.**

This problem is somewhat of a twist on Problem ELIC4 and the sample MFE question. Here, instead of being given the value of a put option, we are given the value of a call option, so we will need to transform the Max expression differently.

Max[S(T)/S(0), $(1 + r)^T$] = Max[S(4)/700, $(1.064844317)^4$] = Max[S(4)/700, 9/7] =

(1/*700*)*Max*[S(4), 900] = (1/*700*)(900 + *Max*[S(4) - 900, 0]) (by Formula 80.2, taking advantage of the fact that we are given a call price rather than a put and trying to anticipate the use of that call price.)

We know that the premium is equal to the price of the contract, which is the present value of its value in four years.

Thus, P = PV(P*(1 - X)*(1/*700*)(900 + *Max*[S(4) - 900, 0]))

The present value of 900 in 4 years is 900/$(1.064844317)^4$= 700.

The present value of *Max*[S(4) - 900, 0] is the current price of the call option, or 91. None of the other items in the expression above are affected by present value considerations.

Thus, PV(P*(1 - X)*(1/*700*)(900 + *Max*[S(4) - 900, 0])) = P*(1 - X)*(1/700)(700 + 91).

P*(1 - X)*(1/700)(700 + 91) = P*(1 - X)*(791/700).

Hence, P = P*(1 - X)*(*791*/700), and 1 - X = 700/*791*, so **X = 0.115044248**.

# Section 81

# Historical Volatility

Historical volatility calculations are likely to appear on Exam MFE. Here, a systematic method for finding the historical volatility of a given set of stock price data will be given.

Let $S_1$, $S_2$, ... $S_{n+1}$ be the prices of a stock at times 1 through n+1, with each time period being of length h.

Then the non-annualized continuously compounded return $r_j$ for the jth time period (where j can be anything from 1 to n) are as follows: $\mathbf{r_j = ln(S_{j+1}/S_j)}$

Then the variance of all of these non-annualized continuously compounded returns $r_1$ through $r_n$ is $\boldsymbol{\sigma_h^2 = (1/(n-1))_{j=1}{}^n\Sigma(r_j - m)^2}$, where m is the *mean* of all of the returns $r_1$ through $r_n$ and $\sigma_h$ is the *standard deviation of returns for a time period of length h.*

The annualized historical volatility σ of these returns is then $\boldsymbol{\sigma = \sigma_h/\sqrt{(h)}}$

As you might suspect, the calculation of historical volatility is a step-by-step process where it is important to do the steps in order. The first four problems of this section will illustrate the four steps involved in calculating historical volatility, after which you will be ready to attempt an exam-style question.

Some of the problems in this section were designed to be similar to problems from past versions of Exam 3F / Exam MFE. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem HV1.** The monthly prices S of a certain stock are given as follows:

Month 1: S = 654

Month 2: S = 456

Month 3: S = 679

Month 4: S = 524

Month 5: S = 996

Month 6: S = 1000

Find all of the five non-annualized continuously compounded monthly returns on the stock, from $r_1$ through $r_5$.

**Solution HV1.** We use the following formula: $r_j = \ln(S_{j+1}/S_j)$

$r_1 = \ln(S_2/S_1) = \ln(456/654) =$ $\mathbf{r_1 = -0.3606145419}$

$r_2 = \ln(S_3/S_2) = \ln(679/456) =$ $\mathbf{r_2 = 0.398128318}$

$r_3 = \ln(S_4/S_3) = \ln(524/679) =$ $\mathbf{r_3 = -0.2591294432}$

$r_4 = \ln(S_5/S_4) = \ln(996/524) =$ $\mathbf{r_4 = 0.6422555733}$

$r_5 = \ln(S_6/S_5) = \ln(1000/996) =$ $\mathbf{r_5 = 0.0040080214}$

**Problem HV2.** The monthly prices S of a certain stock are given as follows:

Month 1: S = 654

Month 2: S = 456

Month 3: S = 679

Month 4: S = 524

Month 5: S = 996

Month 6: S = 1000

Find the mean m of the non-annualized continuously compounded monthly returns on the stock, from $r_1$ through $r_5$.

**Solution HV2.** We already found $r_1$ through $r_5$ in Solution HV1. Now we take their arithmetic average: (-0.3606145419 + 0.398128318 - 0.2591294432 + 0.6422555733 + 0.0040080214)/5 = **m = 0.0849295855**

**Problem HV3.** The monthly prices S of a certain stock are given as follows:

Month 1: S = 654

Month 2: S = 456

Month 3: S = 679

Month 4: S = 524

Month 5: S = 996

Month 6: S = 1000

Find the variance of the non-annualized continuously compounded monthly returns on the stock, from $r_1$ through $r_5$.

**Solution HV3.** We use the formula $\sigma_h^2 = (1/(n-1))_{j=1}{}^{n}\Sigma(r_j - m)^2$. Here, n = 5, m = 0.0849295855 from Solution HV2, and the $r_j$ are known from Solution HV1. Thus,

$\sigma_h^2 = (1/4)_{j=1}{}^{5}\Sigma(r_j - 0.0849295855)^2 =$

$(1/4)[(r_1 - 0.0849295855)^2 + (r_2 - 0.0849295855)^2 + (r_3 - 0.0849295855)^2 + (r_4 - 0.0849295855)^2 + (r_5 - 0.0849295855)^2] =$

$(1/4)[(-0.3606145419 - 0.0849295855)^2 + (0.398128318 - 0.0849295855)^2 + (-0.2591294432 - 0.0849295855)^2 + (0.6422555733 - 0.0849295855)^2 + (0.0040080214 - 0.0849295855)^2] =$

$\boldsymbol{\sigma_h^2 = 0.1830350467}$

**Problem HV4.** The monthly prices S of a certain stock are given as follows:

Month 1: S = 654

Month 2: S = 456

Month 3: S = 679

Month 4: S = 524

Month 5: S = 996

Month 6: S = 1000

Find the annualized historical volatility of the returns on this stock over the given time period.

**Solution HV4.** We use the formula $\sigma = \sigma_h/\sqrt{(h)}$. From Solution HV3, $\sigma_h^2 = 0.1830350467$, so

$\sigma_h = 0.4278259538$. Here, since each time period is one month, h = 1/12. Thus, the historical volatility $\sigma = 0.4278259538/\sqrt{(1/12)} =$ $\boldsymbol{\sigma = 1.482032578}$

**Problem HV5.**

**Similar to Question 17 from the Society of Actuaries' Sample MFE Questions and Solutions:**

The monthly prices S of a certain stock are given as follows:

Month 1: S = 135

Month 2: S = 90

Month 3: S = 135

Month 4: S = 90

Month 5: S = 60

Month 6: S = 90

Month 7: S = 135

Find the annualized historical volatility of the returns on this stock over the given time period.

**Solution HV5**. First, we attempt to find the monthly continuously compounded returns $r_1$ through $r_6$, using the formula $r_j = \ln(S_{j+1}/S_j)$.

Conveniently enough, 90/135 = 2/3 and 135/90 = 3/2. Similarly, 60/90 = 2/3 and 90/60 = 3/2.

Thus, each of the $r_j$ is either ln(3/2) or ln(2/3) = -ln(3/2). Furthermore, we note that the stock went up in price three times over this time period and went down in price the other three times. Thus, the average return on this stock m is equal to [3ln(3/2) + 3(-ln(3/2))]/6 = 0.

Now we use the formula $\sigma_h^2 = (1/(n-1))_{j=1}{}^{n}\Sigma(r_j - m)^2$ to find the variance of these returns. Here, m = 0, n = 6, and each $(r_j)^2$ is $[-\ln(3/2)]^2$ or $[\ln(3/2)]^2$, both of which are 0.1644019539.

Thus, $\sigma_h^2 = (1/5)_{j=1}{}^{6}\Sigma(0.1644019539) = (1/5)*6*0.1644019539 = \sigma_h^2 = 0.1972823447$, so

$\sigma_h = 0.444164772$ and $\sigma = \sigma_h/\sqrt{(h)}$. Here, h = 1/12, since each time period is a month.

Thus, $\sigma = 0.444164772/\sqrt{(1/12)}$ = **$\sigma$ = 1.538631904**

# Section 82

# Applications of Derivatives, the Garman-Kohlhagen Formula, and Brownian Motion to International Business Contracts

This section will walk you through a problem highly similar to Question 7 from the Society of Actuaries' Sample MFE Questions and Solutions. This problem cannot be categorized under any subject in particular, as it is a quite extensive application of many of the topics covered so far in this study guide, most notably in Section 36 and Section 68.

The basic steps to solving this kind of problem are as follows.

1. Find the option payoff at expiration.

2. Find the value of σ for the Garman-Kohlhagen formula for the currency option price.

3. Find the value of $d_1$ for the Garman-Kohlhagen formula for the currency option price.

4. Find the value of $d_2$ for the Garman-Kohlhagen formula for the currency option price.

5. Find the currency option price using the Garman-Kohlhagen formula. Remember to multiply your result by the number of options under consideration to get the total option cost.

The trick to this kind of problem is recognizing the procedure from the given conditions. To do this, it is necessary to be familiar with some conceptual properties of Brownian motion, including the fact that if ln[X(t)] follows an arithmetic Brownian motion, then X(t) follows a geometric Brownian motion.

Some of the problems in this section were designed to be similar to problems from past versions of Exam 3F / Exam MFE. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

**Problem ADGKFBMIBC1.** One Yap piece of stone (YPS) currently trades for 10 Golden Hexagons (GH). Company Ξ is an international company that holds Golden Hexagons. Company Λ is a Yapese company. Company Ξ makes a deal to sell flying superwidgets to Company Λ, and the deal will be settled in YPS. To prevent possible losses on its position, Company Ξ has purchased at-the-money European put options on YPS, denominated in Golden Hexagons. The deal will be settled 4 months from now at a price of 300 YPS, which Company Λ will pay to Company Ξ. You know the following:

The annual continuously compounded GH-denominated risk-free interest rate is 5.6%. The annual continuously compounded YPS-denominated risk-free interest rate is 2%. The natural logarithm of the YPS per GH exchange rate is an arithmetic Brownian motion with daily volatility 0.44%.

Assume that 1 year is 365 days and 4 months is the same as 1/3 years.

Let X(t) be the exchange rate in terms of GH per YPS at time t.

Find the expression for the payoff at expiration of one put option purchased by Company Ξ, as described above.

**Solution ADGKFBMIBC1.** The generic expression for the payoff on a put option is Max[K - S, 0], where K is the strike price and S is the underlying asset price.

This put will expire in 4 months - that is, at time t = 1/3. Since this is a dollar-denominated put on YPS, its underlying asset price is the exchange rate X(t) in terms of GH per YPS at time t. Thus, at expiration, the underlying asset price will be X(1/3).

The put was at the money - i.e., at its strike price - when the deal was entered into, i.e., when the exchange rate was 10 GH per YPS. Thus, the strike price for this put is 10 GH/YPS and so the payoff at expiration of the put is **Max[10 - X(1/3), 0]**

**Problem ADGKFBMIBC2.** One Yap piece of stone (YPS) currently trades for 10 Golden Hexagons (GH). Company Ξ is an international company that holds Golden Hexagons. Company Λ is a Yapese company. Company Ξ makes a deal to sell flying superwidgets to Company Λ, and the deal will be settled in YPS. To prevent possible losses on its position, Company Ξ has purchased at-the-money European put options on YPS, denominated in Golden Hexagons. The deal will be settled 4 months from now at a price of 300 YPS, which Company Λ will pay to Company Ξ. You know the following:

The annual continuously compounded GH-denominated risk-free interest rate is 5.6%. The annual continuously compounded YPS-denominated risk-free interest rate is 2%. The natural logarithm of the YPS per GH exchange rate is an arithmetic Brownian motion with daily volatility 0.44%.

Assume that 1 year is 365 days and 4 months is the same as 1/3 years.

Let X(t) be the exchange rate in terms of GH per YPS at time t.

Use this information to find σ, the annual exchange rate volatility relevant for the Garman-Kohlhagen formula for the price of one put option purchased by Company Ξ.

**Solution ADGKFBMIBC2.** We are given that the natural logarithm of the YPS per GH exchange rate is an arithmetic Brownian motion with daily volatility 0.44%. The YPS per GH exchange rate is $1/X(t)$. We know that $\ln[1/X(t)] = \alpha dt + \beta dZ(t)$, by the definition of an arithmetic Brownian motion. It is also the case that $\ln[X(t)] = -\ln[1/X(t)]$. But volatility is always nonnegative, so if the daily volatility factor $\beta > 0$, then $-\ln[1/X(t)]$ is not $-\alpha dt - \beta dZ(t)$, but rather $-\alpha dt + \beta dZ(t)$. Thus, $\ln[X(t)] = -\alpha dt + \beta dZ(t)$, and $\ln[X(t)]$ follows an arithmetic Brownian motion with daily volatility 0.44%.

We recall from Section 68 that if $\ln[X(t)]$ follows an arithmetic Brownian motion, then $X(t)$ follows a *geometric* Brownian motion. Also, the Black-Scholes model (and the Garman-Kohlhagen model, which is just the Black-Scholes model applied to currency options) is based on an assumption that the underlying asset price follows a geometric Brownian motion. Therefore, to find the annual exchange rate volatility $\sigma$, all we need to do is use the equation $\sigma = \sigma_h/\sqrt{(h)}$, where $h = 1/365$ and $\sigma_h = 0.0044$. Thus, $\sigma = 0.0044/\sqrt{(1/365)}$ = $\boldsymbol{\sigma = 0.084061882}$.

**Problem ADGKFBMIBC3.** One Yap piece of stone (YPS) currently trades for 10 Golden Hexagons (GH). Company Ξ is an international company that holds Golden Hexagons. Company Λ is a Yapese company. Company Ξ makes a deal to sell flying superwidgets to Company Λ, and the deal will be settled in YPS. To prevent possible losses on its position, Company Ξ has purchased at-the-money European put options on YPS, denominated in Golden Hexagons. The deal will be settled 4 months from now at a price of 300 YPS, which Company Λ will pay to Company Ξ. You know the following:

The annual continuously compounded GH-denominated risk-free interest rate is 5.6%. The annual continuously compounded YPS-denominated risk-free interest rate is 2%. The natural logarithm of the YPS per GH exchange rate is an arithmetic Brownian motion with daily volatility 0.44%.

Assume that 1 year is 365 days and 4 months is the same as 1/3 years.

Let $X(t)$ be the exchange rate in terms of GH per YPS at time t.

Use this information to find the value of $d_1$ relevant for the Garman-Kohlhagen formula for the price of this put option.

**Solution ADGKFBMIBC3.** We recall from Section 36 that in the Garman-Kohlhagen formula, $d_1 = [\ln(x/K) + (r - f + 0.5\sigma^2)T]/[\sigma\sqrt{(T)}]$. From Solution ADGKFBMIBC2, we know that $\sigma$ = 0.084061882. It is also given that $T = 1/3$. The "domestic" (GH-denominated) interest rate r is 0.056. The "foreign" (YPS-denominated) interest rate f is 0.02. Furthermore, $K = 10$, and x, the current exchange rate, is also 10, since the put option is at-the-money. Therefore,

$d_1 = [\ln(10/10) + (0.056 - 0.02 + 0.5*0.084061882^2)(1/3)]/[0.084061882\sqrt{(1/3)}] =$

$d_1 = [(0.056 - 0.02 + 0.5*0.084061882^2)(1/3)]/[0.084061882\sqrt{(1/3)}] =$

$\boldsymbol{d_1 = 0.2715202554}$

**Problem ADGKFBMIBC4.** One Yap piece of stone (YPS) currently trades for 10 Golden Hexagons (GH). Company Ξ is an international company that holds Golden Hexagons. Company Λ is a Yapese company. Company Ξ makes a deal to sell flying superwidgets to Company Λ, and the deal will be settled in YPS. To prevent possible losses on its position, Company Ξ has purchased at-the-money European put options on YPS, denominated in Golden Hexagons. The deal will be settled 4 months from now at a price of 300 YPS, which Company Λ will pay to Company Ξ. You know the following:

The annual continuously compounded GH-denominated risk-free interest rate is 5.6%. The annual continuously compounded YPS-denominated risk-free interest rate is 2%. The natural logarithm of the YPS per GH exchange rate is an arithmetic Brownian motion with daily volatility 0.44%.

Assume that 1 year is 365 days and 4 months is the same as 1/3 years.

Let X(t) be the exchange rate in terms of GH per YPS at time t.

Use this information to find the value of $d_2$ relevant for the Garman-Kohlhagen formula for the price of this put option.

**Solution ADGKFBMIBC4.** In the Garman-Kohlhagen formula, $d_2 = d_1 - \sigma\sqrt{(T)}$. We know that $T = 1/3$, $\sigma = 0.084061882$ (from Solution ADGKFBMIBC2), and $d_1 = 0.2715202554$ (from Solution ADGKFBMIBC3).

Thus, $d_2 = 0.2715202554 - 0.084061882\sqrt{(1/3)}$ = $\mathbf{d_2 = 0.2229871052}$

**Problem ADGKFBMIBC5.** One Yap piece of stone (YPS) currently trades for 10 Golden Hexagons (GH). Company Ξ is an international company that holds Golden Hexagons. Company Λ is a Yapese company. Company Ξ makes a deal to sell flying superwidgets to Company Λ, and the deal will be settled in YPS. To prevent possible losses on its position, Company Ξ has purchased at-the-money European put options on YPS, denominated in Golden Hexagons. The deal will be settled 4 months from now at a price of 300 YPS, which Company Λ will pay to Company Ξ. You know the following:

The annual continuously compounded GH-denominated risk-free interest rate is 5.6%. The annual continuously compounded YPS-denominated risk-free interest rate is 2%. The natural logarithm of the YPS per GH exchange rate is an arithmetic Brownian motion with daily volatility 0.44%.

Assume that 1 year is 365 days and 4 months is the same as 1/3 years.

Let X(t) be the exchange rate in terms of GH per YPS at time t.

Use this information to find the total cost of the put options to Company Ξ.

**Solution ADGKFBMIBC5.** To find the price of one put option, we can use the Garman-Kohlhagen formula for currency put options: $P(x, K, \sigma, r, T, f) = Ke^{-rT}N(-d_2) - xe^{-fT}N(-d_1)$

From Solutions ADGKFBMIBC3-4, we know that

$d_1 = 0.2715202554$ and $d_2 = 0.2229871052$. Furthermore, we know that $K = x = 10$.

And that $r = 0.056$ and $f = 0.02$.

We enter "=NormSDist(-0.2715202554)" in MS Excel to find $N(-d_1) = 0.392995462$

We enter "=NormSDist(-0.2229871052)" in MS Excel to find $N(-d_2) = 0.411772771$

Thus, for one put option *on one YPS*, $P(x, K, \sigma, r, T, f) = Ke^{-rT}N(-d_2) - xe^{-fT}N(-d_1) =$

$10[e^{-0.056/3}0.411772771 - e^{-0.02/3}0.392995462] = P = 0.1377343549$ GH

But the deal will be settled at a price of 300 YPS, so Company Ξ must have purchased 300 put options in order to safeguard itself against YPS devaluation. Therefore, its total option cost is 300*0.1377343549 = **41.32030647 GH**.

# Section 83

# Valuing Claims on Derivatives Whose Price is the Underlying Asset Price Taken to Some Power

If S is the price of some underlying asset and $S^a$ is the price of some derivative based on that claim, then it is possible to use Ito's Lemma to value a claim on $S^a$, where a is some number. Hereafter, we will call $S^a$ the ***power derivative***of S. (This term is my own invention, and I will use it for the sake of word economy and the ability to concisely express ideas regarding these kinds of derivatives.) Valuation of power derivatives is a new addition to the MFE syllabus (as of Spring 2008) and thus is likely to appear in at least one question.

Let us say we have an asset whose price S follows this geometric Brownian motion.

$dS(t)/S(t) = (\alpha - \delta)dt + \sigma dZ(t)$

Here, $\alpha$ is the expected return on the asset, $\delta$ is the asset's dividend yield, and $\sigma$ is the asset price volatility.

Then the price of the power derivative, $S^a$, follows this geometric Brownian motion.

$d(S^a)/S^a = (a(\alpha - \delta) + (1/2)a(a-1)\sigma^2)dt + a\sigma dZ(t)$

It is important to note that *the power derivative will often have a different expected return and a different dividend yield than the underlying asset*! We can call the expected return on the power derivative $\gamma$ and the dividend yield on the power derivative $\delta^*$. We can find these values via the following formulas.

$\gamma = a(\alpha - r) + r$, where r is the annual continuously compounded risk-free interest rate.

$\delta^* = r - a(r - \delta) - 0.5a(a-1)\sigma^2$

We can also neatly express both the forward price $F_{0,T}[S(T)^a]$ and the prepaid forward price $F^P_{0,T}[S(T)^a]$ of a power derivative:

$F_{0,T}[S(T)^a] = S(0)^a\exp[(a(r - \delta) + 0.5a(a-1)\sigma^2)T]$

$F^P_{0,T}[S(T)^a] = e^{-rT}S(0)^a\exp[(a(r - \delta) + 0.5a(a-1)\sigma^2)T]$

Note that the only difference between the forward price and the prepaid forward price is that the prepaid forward price has an additional factor of $e^{-rT}$.

In this section, we will work through practice problems to help you memorize these formulas, and then we will undertake an exam-style question.

**Source:** Actuarial Brew. "Chapter 20 Review Note. Brownian Motion and Ito's Lemma. Section 20.7, Valuing a Claim on $S^a$."

Some of the problems in this section were designed to be similar to problems from past versions of Exam 3F / Exam MFE. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem VCDWPUAPTSP1.** The price S of the stock of Obsequious Co. follows this Brownian motion: $dS(t)/S(t) = 0.34dt + 0.23dZ(t)$

The stock's annual continuously compounded dividend yield is 0.05, and the annual continuously compounded risk-free interest rate is 0.13.

Calculate the annual expected return on $S^5$, a power derivative based on this stock.

**Solution VCDWPUAPTSP1.** We use the formula $\gamma = a(\alpha - r) + r$. We are given a = 5 and r = 0.13, but what is α? We recall that, in a geometric Brownian motion like the one given, $dS(t)/S(t) = (\alpha - \delta)dt + \sigma dZ(t)$. Therefore, $(\alpha - \delta) = 0.34$. Since it is given that $\delta = 0.05$, we find that $\alpha = 0.34 + 0.05 = 0.39$. Therefore,

$\gamma = 5(0.39 - 0.13) + 0.13 =$ **$\gamma = 1.43$** (Yes, that is an expected return of 143%.)

**Problem VCDWPUAPTSP2.** The price S of the stock of Obsequious Co. follows this Brownian motion: $dS(t)/S(t) = 0.34dt + 0.23dZ(t)$

The stock's annual continuously compounded dividend yield is 0.05, and the annual continuously compounded risk-free interest rate is 0.13.

Calculate the annual continuously compounded dividend yield of $S^5$, a power derivative based on this stock.

**Solution VCDWPUAPTSP2.** We use the formula $\delta^* = r - a(r - \delta) - 0.5a(a-1)\sigma^2$, where a = 5, r = 0.13, $\delta = 0.05$, and $\sigma = 0.23$, from the given Brownian motion.

Thus, $\delta^* = 0.13 - 5(0.13 - 0.05) - 0.5*5*4*0.23^2 =$ **$\delta^* = -0.799$**

Yes, this is a dividend yield of -79.9%. Negative dividend yields on power derivatives will occur frequently when a > 1. The general idea for why this happens is that the expected capital gains on the power derivative are so large (as in Solution VCDWPUAPTSP1, where the expected *annual* return is 143%) that the owner of the derivative has to pay out an effective dividend for the privilege of enjoying such high returns in the future. The party that sold him the derivative would have rather held on to it if it did not receive some form of compensation in terms of dividend payouts.

**Problem VCDWPUAPTSP3.** The price S of the stock of Obsequious Co. follows this Brownian motion: $dS(t)/S(t) = 0.34dt + 0.23dZ(t)$. The stock's annual continuously compounded dividend yield is 0.05, and the annual continuously compounded risk-free interest rate is 0.13.

Calculate the forward price of a 4-year forward contract on $S^5$, a power derivative based on this stock. The current stock price is $3 per share.

**Solution VCDWPUAPTSP3.**

We use the formula $F_{0,T}[S(T)^a] = S(0)^a\exp[(a(r - \delta) + 0.5a(a-1)\sigma^2)T]$, where $a = 5$, $r = 0.13$, $\delta = 0.05$, $T = 4$, $S(0) = 3$, and $\sigma = 0.23$.

Thus, $F_{0,4}[S(T)^5] = 3^5\exp[(5(0.13 - 0.05) + 0.5*5*4*0.23^2)4] =$
$F_{0,4}[S(T)^5] = 243*\exp(3.716) =$ $\mathbf{F_{0,4}[S(T)^5] = \$9987.218888}$

**Problem VCDWPUAPTSP4.** The price S of the stock of Lucrative Co. follows this Brownian motion: $dS(t)/S(t) = 0.24dt + 0.431dZ(t)$

The stock's annual continuously compounded dividend yield is 0.08, and the annual continuously compounded risk-free interest rate is 0.22.

Calculate the price of a 10-year prepaid forward contract on $S^{-3}$, a power derivative based on this stock. The current stock price is $3.45 per share.

**Solution VCDWPUAPTSP4.** We use the formula
$F^P_{0,T}[S(T)^a] = e^{-rT}S(0)^a\exp[(a(r - \delta) + 0.5a(a-1)\sigma^2)T]$, where
$a = -3$, $r = 0.22$, $\delta = 0.08$, $T = 10$, $S(0) = 3.45$, and $\sigma = 0.431$.
Thus, $F^P_{0,10}[S(T)^{-3}] = e^{-0.22*10}*3.45^{-3}\exp[(-3(0.22 - 0.08) + 0.5(-3)(-4)*0.431^2)10] =$
$e^{-2.2}*3.45^{-3}\exp[6.94566] = F^P_{0,10}[S(T)^{-3}] = 115.0837357*3.45^{-3} =$
$\mathbf{F^P_{0,10}[S(T)^{-3}] = \$2.802571271}$

**Problem VCDWPUAPTSP5.**

**Similar to Question 16 from the Society of Actuaries' Sample MFE Questions and Solutions:**

The price of Imperious LLC stock obeys the Black-Scholes framework. The stock pays no dividends. The stock price volatility is 0.46, and the annual continuously-compounded risk-free interest rate is 0.06.

Let S(t) be the price of Imperious LLC stock at some time $t \geq 0$.

For time T such that T > t, the power derivative $S(T)^x$, where x is some power, has a prepaid forward price such that the following equality holds:
$F^P_{0,T}[S(T)^x] = S(t)^x$. There are two solutions to this equation, one of which is x = 1.What is the other solution? Fractional or irrational-number values of x are entirely possible.

**Solution VCDWPUAPTSP5.**

We use the formula

$F^P_{0,T}[S(T)^a] = e^{-rT}S(0)^a\exp[(a(r - \delta) + 0.5a(a-1)\sigma^2)T]$. Here, T = T, a = x, r = 0.06, σ = 0.46, and δ = 0, since the stock pays no dividends.

Thus, $F^P_{0,T}[S(T)^x] = e^{-0.06T}S(0)^x\exp[(x*0.06 + 0.5x(x-1)*0.46^2)T] =$

$e^{-0.06T}S(0)^x\exp[(x*0.06 + 0.1058x(x-1))T]$

We know that the following equation holds:
$e^{-0.06T}S(0)^x\exp[(x*0.06 + 0.1058x(x-1))T] = S(t)^x$.

Fortunately, we know that 1 is a solution for x. Thus, it is true that

$e^{-0.06T}S(0)^1\exp[(1*0.06 + 0.1058*1(1-1))T] = S(t)^1$ and

$e^{-0.06T}S(0)e^{0.06T} = S(t)$, so S(0) = S(t) and t = 0.

Thus, the following general equation holds:

$e^{-0.06T}S(0)^x\exp[(x*0.06 + 0.1058x(x-1))T] = S(0)^x$, in which case we can cancel $S(0)^x$ on both sides:

$e^{-0.06T}\exp[(x*0.06 + 0.1058x(x-1))T] = 1$ and

$\exp[(x*0.06 + 0.1058x(x-1))T] = \exp[0.06T]$, so

$(x*0.06 + 0.1058x(x-1))T = 0.06T$ and $(x*0.06 + 0.1058x(x-1)) = 0.06$.

We expand this expression and get it into the form of a quadratic equation:

$0.1058x^2 - 0.0458x - 0.06 = 0$

Solving this equation, we get x = 1 or x = -300/529. Thus, our desired answer is

**x = -300/529= -0.5671077505**

# Section 84

# Assorted Exam-Style Questions and Solutions for Exam 3F / Exam MFE

To provide additional practice, I offer five exam-style questions relating to material covered throughout the previous sections. All material from the study guide is fair game in this section.

The problems in this section were designed to be similar to problems from past versions of Exam 3F / Exam MFE. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

**Problem AESQ1.**

**Similar to Question 18 from the Society of Actuaries' Sample MFE Questions and Solutions:**

Okonkwo is a market-maker who has sold 7000 European 2-year gap options, each on one share of the stock of Yams, Inc., which does not pay any dividends. The stock price volatility is 0.55. Each gap option has a payment trigger price of \$555 and a strike price of \$666. The stock currently trades for \$545. The annual continuously-compounded risk-free interest rate is 0. Okonkwo has decided to delta-hedge his position. How many shares are initially in the delta-hedge?

**Solution AESQ1.** A modified Black-Scholes formula can be used to price a gap call:
$C_{gap}(S, K_1, K_2, \sigma, r, T, \delta) = Se^{-\delta T}N(d_1) - K_1e^{-rT}N(d_2)$, where
$d_1 = (\ln[(Se^{-\delta T})/(K_2e^{-rT})] + 0.5\sigma^2T)/(\sigma\surd(T))$ and $d_2 = d_1 - \sigma\surd(T)$

Here, S is the stock price, $K_1$ is the strike price, and $K_2$is the trigger price.

We also know an expression for delta for a regular call: $\Delta_{\text{regular call}} = e^{-\delta T}N(d_1)$

If this gap option were a regular call option, then the strike price would be equal to 100, or the trigger price. So $C(S, K_1, K_2, \sigma, r, T, \delta) = Se^{-\delta T}N(d_1) - K_2e^{-rT}N(d_2)$.

The delta of this option can be expressed as $\partial C/\partial S = e^{-\delta T}N(d_1)$.

But $C_{gap}(S, K_1, K_2, \sigma, r, T, \delta) = Se^{-\delta T}N(d_1) - K_1e^{-rT}N(d_2)$, and so
$C_{gap} = C - (K_1 - K_2)e^{-rT}N(d_2)$.

Here, we are given that r and δ are both zero, so

$C_{gap} = C - (K_1 - K_2)N(d_2)$ and
$\Delta_{gap\ call} = \partial C/\partial S - \partial[(K_1 - K_2)N(d_2)]/\partial S =$
$\Delta_{gap\ call} = \Delta_{regular\ call} - \partial[(K_1 - K_2)N(d_2)]/\partial S =$
$\Delta_{gap\ call} = \Delta_{regular\ call} - (K_1 - K_2)N'(d_2)\partial[d_2]/\partial S$.

We recall that $d_2 = d_1 - \sigma\sqrt{(T)} = (\ln[(Se^{-\delta T})/(K_2e^{-rT})] + 0.5\sigma^2T)/(\sigma\sqrt{(T)}) - \sigma\sqrt{(T)}$, which in this case is $(\ln[S/K_2] + 0.5\sigma^2T)/(\sigma\sqrt{(T)}) - \sigma\sqrt{(T)})$. The partial derivative of this expression with respect to S is $1/(S\sigma\sqrt{(T)})$, so $\Delta_{gap\ call} = \Delta_{regular\ call} - (K_1 - K_2)N'(d_2)/(S\sigma\sqrt{(T)})$

Now, what is the derivative of $N(d_2)$? We recall that $N'(x)$ is the probability density function for a standard normal random variable. That is, $N'(x) = [1/\sqrt{(2\pi)}]\exp[-x^2/2]$

So $N'(d_2) = [1/\sqrt{(2\pi)}]\exp[-d_2^2/2]$

Thus, we have

$\Delta_{gap\ call} = \Delta_{regular\ call} - (K_1 - K_2)[1/\sqrt{(2\pi)}]\exp[-d_2^2/2]/(S\sigma\sqrt{(T)})$

$\Delta_{gap\ call} = N(d_1) - (K_1 - K_2)[1/\sqrt{(2\pi)}]\exp[-d_2^2/2]/(S\sigma\sqrt{(T)})$

Now we need to find $d_1$:

$d_1 = (\ln[(Se^{-\delta T})/(K_2e^{-rT})] + 0.5\sigma^2T)/(\sigma\sqrt{(T)}) = (\ln[(S)/(K_2)] + 0.5\sigma^2T)/(\sigma\sqrt{(T)})$.

Here, S = 545, $K_1$ = 666, $K_2$= 555, σ = 0.55, and T = 2. Thus,

$d_1 = (\ln[(545)/(555)] + 0.5*0.55^2*2)/(0.55\sqrt{(2)}) = d_1 = 0.3655326549$

Now we need to find $d_2$: $d_2 = d_1 - \sigma\sqrt{(T)} = 0.3655326549 - 0.55\sqrt{(2)} =$

$d_2 = -0.4122848044$

We enter "=NormSDist(0.3655326549)" in MS Excel to get $N(d_1) = 0.642643082$

Thus, $\Delta_{gap\ call} = 0.642643082 -$

$(666- 555)[1/\sqrt{(2\pi)}]\exp[-(-0.4122848044)^2/2]/(545*0.55\sqrt{(2)})$,

so $\Delta_{gap\ call} = 0.5466923182$

Since Okonkwo has sold 7000 of these gap calls, his unhedged position would have a total delta of 7000*-0.5466923182 = -3826.846227. Since each share of stock has a delta of 1, Okonkwo would need to buy **3826.846227shares** to delta-hedge this position.

**Problem AESQ2.**

**Similar to Question 19 from the Society of Actuaries' Sample MFE Questions and Solutions:**

A forward start option is an option that, at expiration, will give the owner a European call option equal with a strike price that is the same as the stock price at that time. Consider a forward start option that, in 2 years, will give its owner a 3-year European call option under these conditions. The European call option is based on a stock that pays no dividends and has a volatility of 0.4. The 2-year forward price for delivery of one share of such a stock is $235. The annual continuously compounded risk-free interest rate is 0.06. Find the price today of one forward start option under these conditions, assuming that the Black-Scholes framework holds.

**Solution AESQ2.**

The price of the forward start option should be the price of the call option in two years' time, discounted by two years. First, we find the price of the given European call option in 2 years using the Black-Scholes formula:

$C(S, K, \sigma, r, T, \partial) = Se^{-\partial T}N(d_1) - Ke^{-rT}N(d_2)$
where $d_1 = [\ln(S/K) + (r - \partial + 0.5\sigma^2)T]/[\sigma\sqrt{(T)}]$ and $d_2 = d_1 - \sigma\sqrt{(T)}$

Here, S = 235, the 2-year forward price, and K is likewise 235. $\partial = 0$, r = 0.06, σ = 0.4, and T = 3. Thus, we have
$d_1 = [\ln(235/235) + (0.06 - 0 + 0.5*0.4^2)3]/[0.4\sqrt{(3)}] = d_1 = 0.6062177826$
$d_2 = d_1 - \sigma\sqrt{(T)} = 0.6062177826 - 0.4\sqrt{(3)} = d_2 = -0.0866025404$
We enter "=NormSDist(0.6062177826)" in MS Excel to get $N(d_1) = 0.727814982$
We enter "=NormSDist(-0.0866025404)" in MS Excel to get $N(d_2) = 0.465493658$

Hence, $C = SN(d_1) - Ke^{-rT}N(d_2) = 235*0.727814982 - 235e^{-0.06*3}*0.465493658 = C =$ 79.66546903. This is what we can expect the call option price to be in two years. We need to discount this by a factor of $e^{-0.06*2}$ to get the price of the forward start option:
$79.66546903e^{-0.06*2}$ = **70.65693258**

**Problem AESQ3.**

**Similar to Question 20 from the Society of Actuaries' Sample MFE Questions and Solutions:**

Artaxerxes and Mardonius are two investors with portfolios composed of certain call and put options on the stock of Persepolis Construction, Ltd. The stock price is 565, the call price is 53, and the put price is 34.

Artaxerxes' portfolio contains 4 calls and 5 puts. The elasticity of his portfolio is 4.3.

Mardonius' portfolio contains 8 calls and 3 written puts. The delta of his portfolio is 9.9.

What is the put option elasticity?

**Solution AESQ3.**

From Section 44, we have $\Omega_{portfolio} = {}_{i=1}{}^{n}\Sigma\omega_i\Omega_i$ where $\Omega_i$ is the elasticity of the ith call option and $\omega_i$ is the percentage of the portfolio comprised of the ith call option.

Artaxerxes' portfolio has total value of 4C + 5P = 4*53 + 5*34 = 382, of which the $\omega_{put}$ is 5*34/382 = $\omega_{put}$ = 0.445026178. Thus, $\omega_{call} = 1 - \omega_{put} = 0.554973822$.

Thus, we can set up equation (i): $4.3 = 0.445026178\Omega_{put} + 0.554973822\Omega_{call}$.

The delta of a portfolio is just the sum of the individual option deltas in that portfolio. From Mardonius' portfolio, we can also set up equation (ii): $9.9 = 8\Delta_{call} - 3\Delta_{put}$.

From Section 42, we have the formula $\Omega = S\Delta/X$ for any option, where X is the option price. Thus, $\Delta = \Omega X/S$ and so we can modify equation (ii):

$9.9 = 8(C/S)\Omega_{call} - 3(P/S)\Omega_{put}$

$9.9 = 8(53/565)\Omega_{call} - 3(34/565)\Omega_{put}$

(ii)*: $9.9 = -0.1805309735\Omega_{put} + 0.7504424779\Omega_{call}$.

Now we have a system of equations:
(i): $4.3 = 0.445026178\Omega_{put} + 0.554973822\Omega_{call}$.

(ii)*: $9.9 = -0.1805309735\Omega_{put} + 0.7504424779\Omega_{call}$.

On the actual exam, you will need to solve for this through algebraic manipulation, which is messy but not conceptually difficult. The best way to solve this system outside an exam setting is by constructing a 2 by 3 augmented matrix and having a calculator put it into reduced row echelon form. On a TI-83 calculator, you can let matrix

[0.445026178 0.554973822 | 4.3]

[-0.1805309735 0.7504424779 | 9.9]

Then rref

[1 0 | -5.22239819]

[0 1 | 11.93588634]

Thus, $\mathbf{\Omega_{put} = -5.22239819}$

**Problem AESQ4.**

**Similar to Question 35 from the Casualty Actuarial Society's Spring 2007 Exam 3:**

A certain stock price is a function of time and of the Brownian process Z(t). Ito's Lemma is used to characterize the behavior of the stock as a function of Z(t). A lognormal stock price can be expressed as $S(t) = S(0)\exp[(2\alpha - 6\delta - 8\sigma^2)t + 4\sigma Z(t)]$. Which of the following is an expression for dS(t)?

(a) $(2\alpha - 6\delta - 8\sigma^2)S(t)dt + \sigma S(t)dZ(t) + 16\sigma^2S(t)dt$
(b) $(2\alpha - 6\delta - 8\sigma^2)S(t)dt + 4\sigma S(t)dZ(t) + 16\sigma^2S(t)dt$
(c) $(2\alpha - 6\delta - 8\sigma^2)S(t)dt + 4\sigma S(t)dZ(t) + 8\sigma^2S(t)dt$
(d) $(2\alpha - 6\delta)S(t)dt + \sigma S(t)dZ(t) + 8\sigma^2S(t)dt$
(e) $(2\alpha - 6\delta)S(t)dt + 4\sigma S(t)dZ(t) + 8\sigma^2S(t)dt$

**Solution AESQ4.** A lognormal stock price means that the change in the stock price follows a geometric Brownian motion $dS(t) = aS(t)dt + bS(t)dZ(t)$. In this Brownian motion, the drift factor a is $(2\alpha - 6\delta)$ and the volatility factor b is $4\sigma$.

Thus, we have $dS(t) = (2\alpha - 6\delta)S(t)dt + 4\sigma S(t)dZ(t)$.

We can subtract and then add $8\sigma^2S(t)dt$ to this expression:
$dS(t) = (2\alpha - 6\delta)S(t)dt + 4\sigma S(t)dZ(t) - 8\sigma^2S(t)dt + 8\sigma^2S(t)dt$, which is the same as

**(c):** $(2\alpha - 6\delta - 8\sigma^2)S(t)dt + 4\sigma S(t)dZ(t) + 8\sigma^2S(t)dt$.

So **(c)** is the correct answer.

**Note:** In Problem 35 of CAS's Spring 2007 exam, the given lognormal stock price should have been expressed as $S(t) = S(0)\exp[(\alpha - \delta - 0.5\sigma^2)t + \sigma Z(t)]$, and the answer to that question is (B): $(\alpha - \delta - 0.5\sigma^2)S(t)dt + \sigma S(t)dZ(t) + 0.5\sigma^2S(t)dt$, obtained using the same reasoning as above.

**Problem AESQ5.**

**Similar to Question 36 from the Casualty Actuarial Society's Spring 2007 Exam 3:**

A certain Vasicek bond price model has the following parameters: a = 0.35, b = 0.44, r = 0.09, and $\sigma = 0.12$. Find E[dr]/dt, the annual expected change in the interest rate.

**Solution AESQ5.** For a Vasicek model, $dr = a(b - r)dt + \sigma dZ(t)$.

Here, $dr = 0.35(0.44 - r)dt + 0.12dZ(t)$.

$E[dr] = E[0.35(0.44 - r)dt + 0.12dZ(t)] =$

$E[0.35(0.44 - r)dt] + E[0.12dZ(t)]$

But Z follows a standard normal distribution, which has a mean of zero, so

E[0.12dZ(t)] = 0 and E[dr] = E[0.35(0.44 - r)dt] = 0.35(0.44 - r)dt for some given value of r, which in our case is 0.09. Thus, E[dr] = 0.35(0.44 - 0.09)dt = E[dr] = 0.1225dt.

Since we are looking for the *annual* expected change in the interest rate, we need to divide E[dr] by dt, the time increment over which we expect E[dr] to occur. Thus, the annual expected change in the interest rate is 0.1225dt/dt = **0.1225**.

**Note:** In Problem 36 of CAS's Spring 2007 exam, the given "γ" should be "r", and the question should have asked to express the expected *annual* change in the interest rate.

The answer to that question is (E): At least 0.005, obtained using the same reasoning as above.

**Source for corrections to CAS's Spring 2007 questions:** Actuarial Brew's CAS/CIA Exam 3 2007 Solutions.

# Section 85

# Yield to Maturity of an Infinitely Lived Bond in the Vasicek Model

An important formula to memorize for the Vasicek model is the formula for $\bar{r}$ (r-bar), the yield to maturity of an infinitely-lived bond. This formula was already given in Section 70, but more repeated exposure to it in actual applications is needed to facilitate adequate memorization. The following five problems use the exact same formula and procedure. They are intended to immerse you in the usage of the formula. You are advised to try to solve the problems without looking at the formula or at the solutions. By the time you solve all five of the problems, you will have the formula memorized - adding another tool to your arsenal for this exam.

The formula for the yield to maturity of an infinitely lived bond in the Vasicek model is

$\bar{r} = b + \sigma\varphi/a - 0.5\sigma^2/a^2$.

**Meanings of variables:**

$\varphi$ = Sharpe ratio (assumed to be constant).
$\bar{r}$ = yield to maturity on an infinitely lived bond.
$\sigma$ = volatility factor.
a = drift factor.
b = the mean around which mean reversion occurs.
r = the short-term interest rate.

**Sources:** McDonald, R.L., *Derivatives Markets* (Second Edition), Addison Wesley, 2006, Ch. 24, pp. 786-787.

Here are some ideas to help with memorizing the formula.

Note that b is the only term that stands alone. The only term that ever appears in a denominator is a. The numerators of the non-b terms contain at least one $\sigma$.

**Original Practice Problems and Solutions from the Actuary's Free Study Guide:**

**Problem YMILBVM1.** A particular Vasicek model has the following parameters:
a = 0.56; b = 0.347; $\sigma$ = 0.66; $\varphi$ = 0.99. Find the yield to maturity on an infinitely lived bond in this model.

**Solution YMILBVM1.** We use the formula $\bar{r} = b + \sigma\varphi/a - 0.5\sigma^2/a^2$.

Thus, $\bar{r} = 0.347 + 0.66*0.99/0.56 - 0.5*0.66^2/0.56^2$ = **0.8129704082**

**Problem YMILBVM2.** A particular Vasicek model has the following parameters: a = 0.356; b = 0.46; σ = 0.11; φ = 0.35. Find the yield to maturity on an infinitely lived bond in this model.

**Solution YMILBVM2.** We use the formula $\bar{r} = b + \sigma\varphi/a - 0.5\sigma^2/a^2$.

Thus, $\bar{r} = 0.46 + 0.11*0.35/0.356 - 0.5*0.11^2/0.356^2 =$ **0.5204090393**

**Problem YMILBVM3.** A particular Vasicek model has the following parameters: a = 0.555; b = 0.52; σ = 0.71; φ = 0.88. Find the yield to maturity on an infinitely lived bond in this model.

**Solution YMILBVM3.** We use the formula $\bar{r} = b + \sigma\varphi/a - 0.5\sigma^2/a^2$.

Thus, $\bar{r} = 0.52 + 0.71*0.88/0.555 - 0.5*0.71^2/0.555^2 =$ **0.8274880286**

**Problem YMILBVM4.** A particular Vasicek model has the following parameters: a = 0.516; b = 0.5151; σ = 0.63; φ = 0.68. Find the yield to maturity on an infinitely lived bond in this model.

**Solution YMILBVM4.** We use the formula $\bar{r} = b + \sigma\varphi/a - 0.5\sigma^2/a^2$.

Thus, $\bar{r} = 0.5151 + 0.63*0.68/0.516 - 0.5*0.63^2/0.516^2 =$ **0.5999972418**

**Problem YMILBVM5.** A particular Vasicek model has the following parameters: a = 0.776; b = 0.8681; σ = 0.5; φ = 0.94. Find the yield to maturity on an infinitely lived bond in this model.

**Solution YMILBVM5.** We use the formula $\bar{r} = b + \sigma\varphi/a - 0.5\sigma^2/a^2$.

Thus, $\bar{r} = 0.8681 + 0.5*0.94/0.776 - 0.5*0.5^2/0.776^2 =$ **1.266189595**

# About Mr. Stolyarov

**Gennady Stolyarov II (G. Stolyarov II) is an actuary, science-fiction novelist, independent philosophical essayist, poet, amateur mathematician, composer, and Editor-in-Chief of The Rational Argumentator, a magazine championing the principles of reason, rights, and progress.**

**In December 2013, Mr. Stolyarov published *Death is Wrong*, an ambitious children's book on life extension illustrated by his wife Wendy. *Death is Wrong* can be found on Amazon in paperback and Kindle formats.**

**Mr. Stolyarov has contributed articles to the Institute for Ethics and Emerging Technologies (IEET), The Wave Chronicle, Le Quebecois Libre, Brighter Brains Institute, Immortal Life, Enter Stage Right, Rebirth of Reason, The Liberal Institute, and the Ludwig von Mises Institute. Mr. Stolyarov also published his articles on Associated Content (subsequently the Yahoo! Contributor Network) from 2007 until its closure in 2014, in an effort to assist the spread of rational ideas. He held the highest Clout Level (10) possible on the Yahoo! Contributor Network and was one of its Page View Millionaires, with over 3.1 million views.**

**Mr. Stolyarov holds the professional insurance designations of Associate of the Society of Actuaries (ASA), Associate of the Casualty Actuarial Society (ACAS), Member of the American Academy of Actuaries (MAAA), Chartered Property Casualty Underwriter (CPCU), Associate in Reinsurance (ARe), Associate in Regulation and Compliance (ARC), Associate in Personal Insurance (API), Associate in Insurance Services (AIS), Accredited Insurance Examiner (AIE), and Associate in Insurance Accounting and Finance (AIAF).**

**Mr. Stolyarov has written a science fiction novel, *Eden against the Colossus*, a philosophical treatise, *A Rational Cosmology*, a play, *Implied Consent*, and a free self-help treatise, *The Best Self-Help is Free*. You can watch his YouTube Videos. Mr. Stolyarov can be contacted at gennadystolyarovii@gmail.com.**