The Actuary's Free Study Guide for Exam 3L

Second Edition

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Section 1

Life-Table Probability Functions: Practice Problems and Solutions – Part 1

Before we begin to explore various life-table functions, some actuarial notation is in order.

(x) refers to a **life of age x.**

T(x) refers to the **future lifetime of a life currently at age x.**

X refers to the age of death of a particular life.

F(x) is the **cumulative distribution function of** X**.**

s(x) is the **survival function of X.**

The following are true of F(x) and s(x):

$F(x) = Pr(X \leq x), x \geq 0$.

$s(x) = 1 - F(x) = Pr(X > x), x \geq 0$.

It is always assumed in these models that $F(0) = 0$ and $s(0) = 1$.

The following two expressions are particularly important in studying life tables.

${}_tq_x$ is the **probability that life (x) will die within t years.** It can also be expressed as

${}_tq_x = Pr(T(x) \leq t), t \geq 0$.

${}_tp_x$ is the **probability that life (x) will survive during the next t years and reach the age x + t.** It can also be expressed as

${}_tp_x = 1 - {}_tq_x = Pr(T(x) > t), t \geq 0$.

When a life's current age is zero, **${}_xp_0 = s(x), x \geq 0$.**

When t = 1, we can omit the "t" subscript to the left of the "p" or the "q." Thus,

q_x is the **probability that life (x) will die within 1 year.**

p_x is the **probability that life (x) will survive during the next 1 year and reach the age x + 1.**

Here are some convenient ways of calculating ${}_tp_x$ and ${}_tq_x$.

${}_tp_x = s(x +t)/s(x)$
${}_tq_x = 1 - s(x +t)/s(x)$

The above calculations will be the focus of this section, as it is vital to be able to do them in order to move on to understanding other concepts relevant to Exam 3L.

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1986. First Edition. Society of Actuaries: Itasca, Illinois. pp. 45-48.

Problem S3L1-1. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. Find ${}_3p_5$.

Solution S3L1-1. We use the formula ${}_tp_x = s(x +t)/s(x)$. Thus, ${}_3p_5 = s(5 +3)/s(5) = s(8)/s(5) = e^{-0.34*8}/e^{-0.34*5} = e^{-0.34*3}$ = $\mathbf{{}_3p_5 = e^{-1.02} = 0.3605949402}$

Problem S3L1-2. The life of a giant pin-striped cockroach has the following survival function associated with it: $s(x) = 1 - x/94$, for $0 \le x \le 94$ and 0 otherwise. Find ${}_{55}q_{34}$.

Solution S3L1-2. We use the formula ${}_tq_x = 1 - s(x +t)/s(x)$. Thus, ${}_{55}q_{34} = 1 - s(55 + 34)/s(34) = 1 - s(89)/s(34) = 1 - (1 - 89/94)/(1 - 34/94)$ = $\mathbf{{}_{55}q_{34}}$ **= 11/12 = about 0.916666666667**

Problem S3L1-3. Three-headed donkeys always survive until age 1. Thereafter, the survival function for the life of a three-headed donkey is $s(x) = 1/x$ for all $x > 1$. What is the probability that a three-headed donkey that has survived to age 65 will survive to age 68?

Solution S3L1-3. We wish to find ${}_3p_{65} = s(65 +3)/s(65) = (1/68)/(1/65)$ = $\mathbf{{}_3p_{65}}$ **= 65/68 = about 0.9558823529.**

Problem S3L1-4. Black swans always survive until age 16. After age 16, the lifetime of a black swan can be modeled by the cumulative distribution function $F(x) = 1 - 4x^{-1/2}$, $x > 16$. What is the probability that a black swan that has survived to age 33 will survive to age 34?

Solution S3L1-4. We seek to find $p_{33} = s(33+1)/s(33) = s(34)/s(33)$. We know that $s(x) = 1 - F(x) = 1- (1 - 4x^{-1/2}) = 4x^{-1/2}$. Thus, $s(34)/s(33) = (4*34^{-1/2})/(4*33^{-1/2})$= $\mathbf{p_{33} = \sqrt{(33/34)} =}$ **0.9851843661**.

Problem S3L1-5. The lives of Unicorn-Pegasi can be modeled by the following cumulative distribution function: $F(x) = x/36 - e^{-0.45x}$. If any Unicorn-Pegasus reaches the age of 35, it will get to live forever. Find the probability that a Unicorn-Pegasus currently aged 34 will not get to live forever.

Solution S3L1-5. We wish to find q_{34}, the probability that the Unicorn-Pegasus aged 34 will die within 1 year. We use the formula ${}_tq_x = 1 - s(x +t)/s(x)$. Here, $s(x) = 1 - F(x) = 1 - x/36 + e^{-0.45x}$. $q_{34} = 1 - s(35)/s(34) = 1 - (1 - 35/36 + e^{-0.45*35})/(1 - 34/36 + e^{-0.45*34})$ = **about 0.4999994386.**

Section 2

Life-Table Probability Functions: Practice Problems and Solutions – Part 2

Here, we introduce another life-table probability function.

$_{t|u}q_x$ is the **probability that life (x) will survive for the next t years and die within the subsequent u years.** That is, $_{t|u}q_x$ is the **probability that life (x) will die between ages x + t and x + t + u.**

When u = 1, this function can simply be written as $_{t|}q_x$.

Other expressions for $_{t|u}q_x$ are as follows:

$\mathbf{_{t|u}q_x = Pr[t < T(x) \leq t + u]}$

$\mathbf{_{t|u}q_x = {}_{t+u}q_x - {}_tq_x}$

$\mathbf{_{t|u}q_x = {}_tp_x - {}_{t+u}p_x}$

We can calculate $_{t|u}q_x$ as follows:

$\mathbf{_{t|u}q_x = (s(x + t) - s(x + t + u))/s(x) = {}_tp_x * {}_uq_{x+t}}$

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1986. First Edition. Society of Actuaries: Itasca, Illinois. pp. 47-48.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L2-1. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. Find $_{6|4}q_2$.

Solution S3L2-1. We use the following formula: $_{t|u}q_x = (s(x + t) - s(x + t + u))/s(x)$.Thus,

$_{6|4}q_2 = (s(2 + 6) - s(2 + 6 + 4))/s(2) = (s(8) - s(12))/s(2) = (e^{-0.34*8} - e^{-0.34*12})/(e^{-0.34*2}) =$ $\mathbf{_{6|4}q_2}$ **= about 0.096655409**

Problem S3L2-2. The life of a giant pin-striped cockroach has the following survival function associated with it: s(x) = 1 - x/94, for $0 \leq x \leq 94$ and 0 otherwise. Find $_{5|5}q_{22}$.

Solution S3L2-2. We use the following formula: $_{t|u}q_x = (s(x + t) - s(x + t + u))/s(x)$.Thus,

${}_{5|5}q_{22}$= (s(22 + 5) - s(22 + 5 + 5))/s(22) = (s(27) - s(32))/s(22) = ((1 - 27/94) - (1 - 32/94))/(1 - 22/94) = (5/94)/(72/94) = **${}_{5|5}q_{22}$ = 5/72 = about 0.0694444444444**

Problem S3L2-3. Three-headed donkeys always survive until age 1. Thereafter, the survival function for the life of a three-headed donkey is s(x) = 1/x for all x > 1. What is the probability that a three-headed donkey that has survived to age 23 will senselessly perish (i.e., die) between the ages of 45 and 69?

Solution S3L2-3. We wish to find ${}_{22|24}q_{23}$ = (s(45) - s(69))/s(23) = (1/45 - 1/69)/(1/23) = **${}_{22|24}q_{23}$ = 8/45 = about 0.17777777778**

Problem S3L1-4. You know the following about the lives of jumping slugs. For a jumping slug that has survived to age 39, the probability of surviving to age 45 is 0.835. For a jumping slug that has survived to age 45, the probability of dying within the subsequent 8 years is 0.23. What is the probability that a jumping slug that has survived to age 39 will senselessly perish (i.e., die) between the ages of 45 and 53?

Solution S3L1-4. We use the following formula: ${}_{t|u}q_x = {}_tp_x * {}_uq_{x+t}$. We wish to find ${}_{6|8}q_{39}$ = ${}_6p_{39}$ * ${}_8q_{45}$, where we are given that ${}_6p_{39}$ = 0.835 and ${}_8q_{45}$ = 0.23. Thus, 0.835*0.23 = **${}_{6|8}q_{39}$ = 0.19205**

Problem S3L1-5. There is a 0.05 probability that an ancient Greek god who has survived to age 1245 will die between the ages of 1298 and 1403. Zeus is currently 1245 years old, and his probability of surviving to age 1298 is 0.95. What is the probability that, if Zeus survives to age 1298, he will also survive past age 1403?

Solution S3L1-5. We use the following formula: ${}_{t|u}q_x = {}_tp_x * {}_uq_{x+t}$. We wish to find ${}_{105}p_{1298}$. We are given that ${}_{53|105}q_{1245}$ = 0.05 and ${}_{1298}p_{1245}$ = 0.95. We thus know that 0.05 = 0.95*${}_{105}q_{1298}$, and so ${}_{105}q_{1298}$ = 1/19. 1- ${}_{105}q_{1298}$ = **${}_{105}p_{1298}$ = 18/19 = 0.0526315789**

Section 3

Curate-Future-Lifetime

The **curate-future-lifetime** of life (x) is the number of future years that life (x) completes before death. The curate-future-lifetime of (x) is represented by the random variable K(x).

The probability density function of K(x) can be expressed in the following ways:

Pr[K(x) = k] = Pr[k ≤ T(x) < k + 1]

Pr[K(x) = k] = $_{k}p_x - {}_{k+1}p_x$

Pr[K(x) = k] = $_{k}p_x * q_{x+k}$

Pr[K(x) = k] = $_{k|}q_x$

for all nonnegative integer values of k.

The cumulative distribution function of K(x) can be expressed as follows:
Pr[K(x) ≤ k] = $_{k+1}q_x$ for all nonnegative integer values of k.

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1986. First Edition. Society of Actuaries: Itasca, Illinois. p. 48.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L3-1. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. Find the probability that the curate-future-lifetime of a triceratops life (34) is 7 years.

Solution S3L3-1. We want to find Pr[K(34) = 7].

We use the formula Pr[K(x) = k] = $_{k|}q_x$. In this case, $_{7|}q_{34}$ = (s(34 + 7) - s(34 + 7 + 1))/s(34) =

(s(41) - s(42))/s(34) = $(e^{-0.34*41} - e^{-0.34*42})/(e^{-0.34*34})$ = Pr[K(34) = 7] = **0.0266758231**

Problem S3L3-2. Every year, 0.26 of the lemming population senselessly perishes (i.e., dies), and every lemming has the same likelihood of senselessly perishing. Find the probability that the curate-future-lifetime of a lemming currently aged 3 is 2 years.

Solution S3L3-2. We use the formula Pr[K(x) = k] = $_{k}p_x - {}_{k+1}p_x$. We want to find Pr[K(3) = 2]. Thus, we need to find $_{2}p_3$ and $_{3}p_3$. The probability of a lemming of any age surviving two more years is

$(1 - 0.26)^2 = 0.5476$ so ${}_2p_3 = 0.5476$. The probability of a lemming of any age surviving three more years is $(1 - 0.26)^3 = 0.405224$ so ${}_3p_3 = 0.405224$. Thus, $Pr[K(3) = 2] = {}_2p_3 - {}_3p_3 = 0.5476 - 0.405224 =$ **0.142376**

Problem S3L3-3. The life of a giant pin-striped cockroach has the following survival function associated with it: $s(x) = 1 - x/94$, for $0 \leq x \leq 94$ and 0 otherwise. Find the cumulative distribution function of the curate-future-lifetime of life (36). Use k as your only variable.

Solution S3L3-3. The cumulative distribution function of K(36) can be expressed as follows: $Pr[K(36) \leq k] = {}_{k+1}q_{36}$. We recall from Section 1 that ${}_tq_x = 1 - s(x + t)/s(x)$ and so

${}_{k+1}q_{36} = 1 - s(37 + k)/s(36) = 1 - (1 - (37+k)/94)/(1 - 36/94) = 1 - ((94 - 37 - k)/94)/(58/94)$

$= 1 - (57 - k)/58 = (58 - 57 + k)/58 =$ **$Pr[K(36) \leq k] = (1 + k)/58$**

Problem S3L3-4. For white elephants, the probability that the curate-future-lifetime of life (123) is 4 years is 0.536. The probability that a white elephant that has survived for 127 years will survive for another year is 0.356. Find the probability that a white elephant that has survived for 123 years will survive to age 127.

Solution S3L3-4. We use the formula $Pr[K(x) = k] = {}_kp_x * q_{x+k}$. We are given that $Pr[K(123) = 4]$ is 0.536 and that p_{127} is 0.356. We can quickly find $q_{127} = 1 - 0.356 = 0.644$. Our goal is to find ${}_4p_{123}$. By the formula above, ${}_4p_{123} = Pr[K(123) = 4]/q_{127} = 0.536/0.644 = {}_4p_{123} =$ **0.8322981366**.

Problem S3L3-5. Three-headed donkeys always survive until age 1. Thereafter, the survival function for the life of a three-headed donkey is $s(x) = 1/x$ for all $x > 1$. Eight-tailed oxen also always survive until age 1. Thereafter, the survival function for the life of an eight-tailed ox is $1/x^2$ for all $x > 1$. Find the probability that the curate-future-lifetime of a three-headed donkey life (3) is 5 years *and* that the curate-future-lifetime of an eight-tailed ox life (3) is 5 years.

Solution S3L3-5.

For three-headed donkeys, $Pr[K(3) = 5] = {}_{5|}q_3 = (s(8) - s(9))/s(3) = (1/8 - 1/9)/(1/3) = 1/24$

For eight-tailed oxen, $Pr[K(3) = 5] = {}_{5|}q_3 = (s(8) - s(9))/s(3) = (1/8^2 - 1/9^2)/(1/3^2) = 17/576$

To get the probability that both of these curate-future-lifetimes are 5 years, we multiply the individual probabilities and obtain our desired answer: (1/24)(17/576) = **17/13824 = about 0.0012297454.**

Section 4

Force of Mortality

This section will explore the **force of mortality,** alternatively known as the **failure rate** or the **hazard rate function.** It is abbreviated as μ_x. The force of mortality describes an "instantaneous rate of death" and can be found via the following formulas:

$\boldsymbol{\mu_x = f(x)/(1 - F(x))}$

$\boldsymbol{\mu_x = -s'(x)/s(x)}$

The force of mortality is always greater than or equal to zero, since s(x) is always nonnegative and its derivative, s'(x), is always non-positive, so -s'(x) is always nonnegative.

Meaning of Terms:

X refers to the age of death of a particular life.

F(x) is the **cumulative distribution function of X.**

s(x) is the **survival function of X.**

f(x) is the **probability density function of X.**

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1986. First Edition. Society of Actuaries: Itasca, Illinois. p. 48-49.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L4-1. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. Find μ_x.

Solution S3L4-1. We use the formula $\mu_x = -s'(x)/s(x)$. Since $s(x) = e^{-0.34x}$, it follows that

$s'(x) = -0.34e^{-0.34x}$ and so $\mu_x = (0.34e^{-0.34x})/(e^{-0.34x})$ = $\boldsymbol{\mu_x = 0.34}$

Problem S3L4-2. The life of a giant pin-striped cockroach has the following survival function associated with it: $s(x) = 1 - x/94$, for $0 \le x \le 94$ and 0 otherwise. Find μ_x.

Solution S3L4-2. We use the formula $\mu_x = -s'(x)/s(x)$. Since $s(x) = 1 - x/94$, it follows that

$s'(x) = -1/94$ and so $\mu_x = (1/94)/(1 - x/94) = (1/94)/((94 - x)/94)$ = $\boldsymbol{\mu_x = 1/(94 - x)}$

Problem S3L4-3. Three-headed donkeys always survive until age 1. Thereafter, the survival function for the life of a three-headed donkey is s(x) = 1/x for all x > 1. What is the force of mortality for three-headed donkeys? *Warning:* Make sure you examine all possible ages!

Solution S3L4-3. Prior to age 1, no three-headed donkeys die, so the force of mortality is 0. (We can confirm this by recognizing that s(x) is 1 for $x \leq 1$ and so s'(x) = 0 and thus $\mu_x = 0$).

After age 1, s(x) is 1/x, so $s'(x) = -1/x^2$. We now apply the formula $\mu_x = -s'(x)/s(x) = (1/x^2)/(1/x) = x/x^2 = \mu_x = 1/x$.

Thus, **$\mu_x = 0$, for $0 \leq x \leq 1$ and $\mu_x = 1/x$ for x > 1**.

Problem S3L4-4. Black swans always survive until age 16. After age 16, the lifetime of a black swan can be modeled by the cumulative distribution function $F(x) = 1 - 4x^{-1/2}$, x > 16. What is the force of mortality for black swans? *Warning:* Make sure you examine all possible ages!

Solution S3L4-4. Prior to age 16, no black swans die, so the force of mortality for $x \leq 16$ is 0.

After age 16, $F(x) = 1 - 4x^{-1/2}$, and $f(x) = F'(x) = (-1/2)(-4)x^{-3/2} = f(x) = 2x^{-3/2}$. So we can use the formula $\mu_x = f(x)/(1 - F(x)) = (2x^{-3/2})/(4x^{-1/2}) = 1/(2x)$.

Thus, **$\mu_x = 0$, for $0 \leq x \leq 16$ and $\mu_x = 1/(2x)$ for x > 16**.

Problem S3L4-5. The lives of Unicorn-Pegasi can be modeled by the following cumulative distribution function: $F(x) = x/36 - e^{-0.45x}$. If any Unicorn-Pegasus reaches the age of 35, it will get to live forever. What is the force of mortality for Unicorn-Pegasi? *Warning:* Make sure you examine all possible ages!

Solution S3L4-5. For all x < 35, Unicorn-Pegasi are still mortal. It is given that $F(x) = x/36 - e^{-0.45x}$, and so $f(x) = F'(x) = 1/36 + 0.45e^{-0.45x}$. So we can use the formula

$\mu_x = f(x)/(1 - F(x)) = (1/36 + 0.45e^{-0.45x})/(1 - x/36 + e^{-0.45x}) =$

$(1 + 16.2e^{-0.45x})/(36 - x + 36e^{-0.45x})$.

After reaching age 35, Unicorn-Pegasi no longer die, so their force of mortality becomes 0.

Thus, **$\mu_x = (1 + 16.2e^{-0.45x})/(36 - x + 36e^{-0.45x})$, for $0 \leq x < 35$ and $\mu_x = 0$ for $x \geq 35$**.

Section 5

Identities Involving Life Table Functions and Force of Mortality

Here, we will explore several useful relationships between the force of mortality, discussed in Section 4, and various life table functions:

$_np_x = \exp(-{}_x{}^{x+n}\int \mu_y dy)$

$_np_x = \exp(-{}_0{}^{n}\int \mu_{x+s} ds)$, where s = y - x.

$_xp_0 = s(x) = \exp(-{}_0{}^{x}\int \mu_s ds)$

$F'(x) = f(x) = {}_xp_0 * \mu_x$

$d(1 - {}_tp_x)/dt = -d({}_tp_x)/dt = {}_tp_x\mu_{x+t}$

Meaning of Terms:

X refers to the age of death of a particular life.

F(x) is the **cumulative distribution function of X.**

s(x) is the **survival function of X.**

f(x) is the **probability density function of X.**

$_tp_x$ is the **probability that life (x) will survive during the next t years and reach the age x + t.**

μ_x is the **force of mortality for life (x).**

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1986. First Edition. Society of Actuaries: Itasca, Illinois. p. 49-50.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L5-1. Tyrannosaurs have a force of mortality $\mu_y = 1/y^2$. Find $_{12}p_{34}$ for tyrannosaurs.

Solution S3L5-1. We can let n = 12 and x = 34 and use the formula

$_np_x = \exp(-{}_x{}^{x+n}\int \mu_y dy) = \exp(-{}_{34}{}^{46}\int (1/y^2) dy) = \exp(-0.0076726343) =$ **$_{12}p_{34}$ = 0.9923567253**.

Problem S3L5-2. The force of mortality for octahedral armadillos is $\mu_s = 0$ for $s < 1$ and $1/(95s)$ for $s \geq 1$. What is the survival function for octahedral armadillos?

Solution S3L5-2. Since the force of mortality is 0 for $s < 1$, it follows from this that $s(x) = 1$ for $x < 1$.

For $x \geq 1$, we use the formula $s(x) = \exp(-{}_1{}^x\!\int\mu_s ds) = \exp(-{}_1{}^x\!\int 1/(95s)ds) = \exp(-\ln(s)/95 \,\big|\, {}_1{}^x) =$

$\exp(-\ln(x)/95 + \ln(1)/95) = \exp(-\ln(x)/95 + 0/95) = \exp(-\ln(x)/95) = \exp(\ln(x^{-1/95})) = s(x) = x^{-1/95}$.

Thus, $\mathbf{s(x) = 1}$ **for** $\mathbf{x < 1}$ **and** $\mathbf{s(x) = x^{-1/95}}$ **for** $\mathbf{x \geq 1}$.

Problem S3L5-3. The force of mortality for octahedral armadillos is $\mu_s = 0$ for $s < 1$ and $1/(95s)$ for $s \geq 1$. What is the probability density function for the lives of octahedral armadillos?

Solution S3L5-3. We use the formula $f(x) = {}_xp_0 * \mu_x$. We are given that $\mu_s = 0$ for $s < 1$, so $f(x)$ is also 0 for $s < 1$. From Solution S3L5-2, we know that ${}_xp_1$, the survival function for $x \geq 1$, is $x^{-1/95}$.

Also, we know that for $x \geq 1$, $\mu_x = 1/(95x)$. Thus, we have $f(x) = x^{-1/95} * 1/(95x) = (1/95)x^{-1/96}$.

Therefore, $\mathbf{f(x) = 0}$ **for** $\mathbf{x < 1}$ **and** $\mathbf{f(x) = (1/95)x^{-1/96}}$ **for** $\mathbf{x \geq 1}$.

Problem S3L5-4. Tyrannosaurs have a force of mortality $\mu_y = 1/y^2$. Find $-d({}_{32}p_{56})/dt$ for tyrannosaurs.

Solution S3L5-4. We use the formula $-d({}_tp_x)/dt = {}_tp_x\mu_{x+t}$ for $x = 56$ and $t = 32$. Thus, we need to find $\mu_{88} = 1/88^2$ and ${}_{32}p_{56} = \exp(-{}_{56}{}^{88}\!\int(1/y^2)dy) = \exp(-0.0064935065) = 0.9935275308$. So $-d({}_tp_x)/dt = 0.9935275308/88^2 =$ **about 0.000128296427**

Problem S3L5-5. For orange rabbits, ${}_{18}p_{12} = 0.836$, and μ_y has the form $1/(k - y)$. Find k.

Solution S3L5-5. We use the formula ${}_np_x = \exp(-{}_x{}^{x+n}\!\int\mu_y dy)$, which in this particular case translates to

${}_{18}p_{12} = 0.836 = \exp(-{}_{12}{}^{30}\!\int(1/(k - y)dy)$. Thus,

$0.836 = \exp(\ln(k - y) \,\big|\, {}_{12}{}^{30})$

$0.836 = \exp(\ln(k - 30) - \ln(k - 12))$

$0.836 = \exp(\ln[(k-30)/(k-12)])$

$0.836 = (k-30)/(k-12)$

$0.836(k-12) = (k-30)$

$0.836k - 10.032 = k - 30$

$19.968 = 0.164k$

$\mathbf{k = 121.7560976}$

Section 6

Life-Table Probability Functions: Practice Problems and Solutions – Part 3

If we have a particular group of lives, to which no new members are added, we can let $\mathbf{l_0}$ be the number of lives in the group at time 0. We can also let $\boldsymbol{\ell(x)}$ be the number of survivors in this group at age x. Then the expected number of survivors $E(\ell(x))$ can be expressed as follows:

$\mathbf{E(\ell(x)) = l_x = l_0 * s(x)}$

If we assume that the indicators for the survival of each life are independent, then **$\ell(x)$ has a binomial distribution with parameters $n = l_0$ and $p = s(x)$**. Recall that a binomial probability mass function is of the following form: $\mathbf{p(r) = C(n, r)p^r(1-p)^{n-r}}$

We can also let $_nD_x$ be the **number of deaths between ages x and x + n** and we can define the expected number of such deaths as $\mathbf{E(_nD_x) = {_nd_x}}$. When n = 1, $_nd_x$ is simply written as d_x.

The following formulas hold with respect to l_x and $_nd_x$:

$\mathbf{_nd_x = l_0[s(x) - s(x + n)]}$

$\mathbf{_nd_x = l_x - l_{x+n}}$

$\mathbf{(-1/l_x)(dl_x/dx) = (-1/[s(x)])ds(x)/dx = \mu_x}$

$\mathbf{-dl_x = l_x * \mu_x\, dx}$

Meanings of Terms:

X refers to the age of death of a particular life.

s(x) is the **survival function of X.**

$\boldsymbol{\mu_x}$ is the **force of mortality for life (x).**

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1986. First Edition. Society of Actuaries: Itasca, Illinois. p. 52-53.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L6-1. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. Out of a group of 9035 newborn triceratopses, how many would you expect to remain alive at age 8? Round down to the nearest whole triceratops.

Solution S3L6-1. We want to find l_8, given $l_0 = 9035$ and $s(x) = e^{-0.34x}$. We use the formula $l_x = l_0*s(x)$. Thus, $l_8 = l_0*s(8) = 9035* e^{-0.34*8} = 595.1784062$ = about **595 triceratopses.**

Problem S3L6-2. The lives of Unicorn-Pegasi can be modeled by the following cumulative distribution function: $F(x) = x/36 - e^{-0.45x}$. If any Unicorn-Pegasus reaches the age of 35, it will get to live forever. For a group of newborn 3612 Unicorn-Pegasi, find ${}_2d_7$. Round down to the nearest whole Unicorn-Pegasus.

Solution S3L6-2. We first find $s(x) = 1 - F(x) = 1 - x/36 + e^{-0.45x}$. We are also given $l_0 = 3612$. So we use the formula ${}_nd_x = l_0[s(x) - s(x + n)]$. ${}_2d_7 = 3612[s(7) - s(9)] =$
$3612[(1 - 7/36 + e^{-0.45*7}) - (1 - 9/36 + e^{-0.45*9})] = 292.5189317$ = about **292 Unicorn-Pegasi.**

Problem S3L6-3. Black swans always survive until age 16. After age 16, the lifetime of a black swan can be modeled by the cumulative distribution function $F(x) = 1 - 4x^{-1/2}$, $x > 16$. There is a cohort of 25 newborn black swans. Assuming that the survival of every black swan is independent of the survival of every other black swan, what is the probability that exactly 9 black swans of this group are alive at age 23? (Hint: The answer is extremely small.)

Solution S3L6-3. We want to find $\Pr(\ell(23) = 9)$. We know that $\ell(23)$ follows a binomial distribution with $n = l_0 = 25$ and $p = s(23) = 1 - F(23) = 4*23^{-1/2} = 0.8340576562$.
We use the binomial probability mass function $p(r) = C(n, r)p^r(1-p)^{n-r}$, for $r = 9$.
$p(9) = C(25, 9)*0.8340576562^9*(1 - 0.8340576562)^{16}$
$p(9)$ = about **$1.319298986*10^{-7}$**

Problem S3L6-4. For a particular group of polka-dotted zebras, you are given $l_x = 95/x^4$, for all $x > 1$. Find μ_x for all $x > 1$.

Solution S3L6-4. We use the formula $(-1/l_x)(dl_x/dx) = \mu_x$.
$(dl_x/dx) = -4*95x^{-5} = -380x^{-5}$
Thus, $\mu_x = (-x^4/95)(-380x^{-5}) =$ **$\mu_x = 4/x$, for $x > 1$**.

Problem S3L6-5. The lives of Unicorn-Pegasi can be modeled by the following cumulative distribution function: $F(x) = x/36 - e^{-0.45x}$. If any Unicorn-Pegasus reaches the age of 35, it will get to live forever. For a group of 69 newborn Unicorn-Pegasi, find the probability that at least one of them will get to live forever.

Solution S3L6-5. We want to find $1 - \Pr(\ell(35) = 0)$. We know that $\ell(35)$ follows a binomial distribution with $n = l_0 = 69$ and $p = s(35) = 1 - 35/36 + e^{-0.45*35} = p = 0.0277779223$.

We use the binomial probability mass function $p(r) = C(n, r)p^r(1-p)^{n-r}$, for $r = 0$.
$p(0) = C(69, 0)*p^0(1-p)^{69}$
$p(0) = (1-p)^{69}$
$p(0) = (1-0.0277779223)^{69}$
$p(0) = 0.1431588013$
Our desired answer is $1 - p(0) =$ **0.8568411987**

Section 7

Life-Table Probability Functions: Practice Problems and Solutions – Part 4

A requirement for the force-of-mortality function is useful to remember. For every valid force of mortality function, the following property must hold:

$$\int_0^\infty \mu_x dx = \infty$$

(Technically, we would need to write $\lim_{a \to \infty} \int_0^a \mu_x dx$ instead of $\int_0^\infty \mu_x dx$, but approaching the integration using the rough notation above does not change the substance of what we are trying to determine, while it does save time and space.)

Furthermore, the force of mortality is related to the life table function l_x in the following ways:

$$l_x = l_0 \exp(-\int_0^x \mu_y dy)$$

$$l_{x+n} = l_x \exp(-\int_x^{x+n} \mu_y dy)$$

$$l_x - l_{x+n} = \int_x^{x+n} l_y * \mu_y dy$$

Meaning of terms:

μ_x = **force of mortality** for life (x)

l_x **=** The expected number of survivors at time x of the group of lives in question.

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1986. First Edition. Society of Actuaries: Itasca, Illinois. pp. 51, 53.

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Problem S3L7-1. Which of these functions might serve as a force of mortality? More than one correct answer is possible.

(a) $2x$
(b) $98/x$
(c) $\cos(x)$
(d) $1/x^2$
(e) $-1/x^2$

Solution S3L7-1. We use a rough approach to integration check to see if the requirement $_0{}^{\infty}\int\mu_x dx = \infty$ holds for each of these cases.

For (a): $_0{}^{\infty}\int 2x dx = x^2 \mid_0{}^{\infty} = \infty$, so (a) is a valid force of mortality function.

For (b): $_0{}^{\infty}\int(98/x)dx = 98\ln(x) \mid_0{}^{\infty} = \infty$, so (b) is a valid force of mortality function.

For (c): $_0{}^{\infty}\int(\cos(x))dx = \sin(x) \mid_0{}^{\infty}$. But $\sin(\infty)$ is undefined, so (c) is not a valid force of mortality function.

For (d): $_0{}^{\infty}\int(1/x^2)dx = -1/x \mid_0{}^{\infty} = -1/\infty + 1/0 = -0 + \infty = \infty$, so (d) is a valid force of mortality function.

For (e): $_0{}^{\infty}\int(-1/x^2)dx = 1/x \mid_0{}^{\infty} = 1/\infty - 1/0 = 0 - \infty = -\infty$, so (e) is not a valid force of mortality function.

Thus, **(a), (b), and (d) are functions that might serve as forces of mortality.**

Problem S3L7-2. Tyrannosaurs have a force of mortality $\mu_y = 1/y^2$, starting at age 1. No tyrannosaurs die prior to age 1. From a cohort of 2351 newborn tyrannosaurs, how many would you expect to be alive at age 5? Round down to the nearest whole tyrannosaur.

Solution S3L7-2. We use the formula $l_x = l_0\exp(-_0{}^{x}\int\mu_y dy)$. In this case, the formula can be modified to $l_x = l_0\exp(-_1{}^{x}\int\mu_y dy)$, since no tyrannosaurs die prior to age 1. We are given $l_0 = 2351$. Thus,

$l_5 = 2351\exp(-_1{}^{5}\int(1/y^2)dy)$

$l_5 = 2351\exp((1/y) \mid_1{}^{5})$

$l_5 = 2351\exp((1/y) \mid_1{}^{5})$

$l_5 = 2351\exp(1/5 - 1)$

$l_5 = 2351\exp(-0.8)$

$l_5 = 1056.3723295$ = about **1056 tyrannosaurs**.

Problem S3L7-3. The force of mortality for octahedral armadillos is $\mu_s = 0$ for $s < 1$ and $1/(95s)$ for $s \geq 1$. 810 octahedral armadillos are currently 120 years old. How many octahedral armadillos would you expect to live to be 121 years old? Round down to the nearest whole armadillo.

Solution S3L7-3. We use the formula $l_{x+n} = l_x\exp(-_x{}^{x+n}\int\mu_y dy)$, where $l_{120} = 810$, $x = 120$, and $n = 1$. Thus,

$l_{121} = 810\exp(-{}_{120}{}^{121}\int(1/95y)dy)$

$l_{121} = 810\exp((-1/95)\ln(y) \mid {}_{120}{}^{121})$

$l_{121} = 810\exp((-1/95)(\ln(121) - \ln(120)))$

$l_{121} = 810\exp((-1/95)(\ln(121) - \ln(120)))$

$l_{121} = 810\exp(-0.0000873668191)$

$l_{121} = 810*0.999912648$

l_{121} = 809.9292449 = **about 809 octahedral armadillos** (rounded down).

Problem S3L7-4. A particular group of orange panthers exhibits the following functions:

$l_y = 1359e^{-0.1235y}$ and $\mu_y = 0.1235$. Find out how many of these orange panthers can be expected to senselessly perish (i.e., die) between ages 19 and 35. Round down to the nearest whole panther.

Solution S3L7-4. We want to find $l_{19} - l_{35}$, which we can do by using the formula $l_x - l_{x+n} = {}_x{}^{x+n}\int l_y*\mu_y dy = {}_{19}{}^{35}\int 0.1235*1359e^{-0.1235y}dy = -1359e^{-0.1235y} \mid {}_{19}{}^{35} = 1359e^{-0.1235*19} - 1359e^{-0.1235*35} =$ 112.0316978 = **about 112 orange panthers.**

Problem S3L7-5. For a particular group of polka-dotted zebras, you are given $l_x = 95/x^4$, for all $x > 1$ and $\mu_x = 4/x$, for $x > 1$. How many of this group of zebras would you expect to senselessly perish (i.e., die) between the ages of 2 and 3? Round down to the nearest whole zebra.

Solution S3L7-5. We want to find $l_2 - l_3$, which we can do by using the formula $l_x - l_{x+n} = {}_x{}^{x+n}\int l_y*\mu_y dy = {}_2{}^3\int(95/x^4)*(4/x)dx = {}_2{}^3\int(95*4/x^5)dx = (-95/x^4) \mid {}_2{}^3 = 95/16 - 95/81 = 4.764660494$ = **about 4 polka-dotted zebras.**

Section 8

Deterministic Survivorship Groups

The following are properties of a **deterministic survivorship group**, according to Bowers, Gerber, et. al.

1. Initially, the group is comprised of l_0 lives aged 0.

2. "The members of the group are subject, at each age of their lives, to effective annual rates of mortality (decrement) specified by the values of q_x in the life table."
3. "The group is closed. No future entrants are allowed beyond the initial l_0. The only decreases come as a result of the effective annual rates of mortality (decrement)."

The following formulas illustrate how membership in a deterministic survivorship group changes with the passage of time.

$l_1 = l_0(1- q_0) = l_0 - d_0$

$l_2 = l_1(1- q_1) = l_1 - d_1 = l_0 - (d_0 + d_1)$

...

$\mathbf{l_x = l_{x-1}(1- q_{x-1}) = l_{x-1} - d_{x-1} = l_0 - {}_{y=0}{}^{x-1}\Sigma d_y = l_0(1 - {}_xq_0)}$

The above series of equalities is called the **radix.** We can also write these equalities as follows:

$l_1 = l_0p_0$

$l_2 = l_1p_1 = l_0p_0p_1$

...

$\mathbf{l_x = l_{x-1}p_{x-1} = l_0{*}{}_xp_0}$

We can express q_x, the **effective annual rate of mortality,** as follows:

$\mathbf{q_x = (l_x - l_{x+1})/l_x}$

We can express ${}_nq_x$, the **n-year rate of mortality,** as follows:

$\mathbf{{}_nq_x = (l_x - l_{x+n})/l_x}$

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1986. First Edition. Society of Actuaries: Itasca, Illinois. pp. 60-61.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L8-1. For a particular population of yellow-tailed squawkers, ${}_2q_0$ = 0.012, ${}_{42}q_2$ = 0.24, and l_0 = 13616. Find l_{44}. Round down to the nearest whole yellow-tailed squawker.

Solution S3L8-1. We note that $l_2 = l_0(1 - {}_2q_0)$ and $l_{44} = l_2(1 - {}_{42}q_2)$. Thus, $l_{44} = l_0(1 - {}_2q_0)(1 - {}_{42}q_2)$ = 13616(1 - 0.012)(1 - 0.24) = l_{44} = 10223.98208 = **about 10223 yellow-tailed squawkers.**

Problem S3L8-2. For a particular population of coelacanths, ${}_7p_0$ = 0.8735, ${}_7p_7$ = 0.9623, ${}_7p_{14}$ = 0.5567, and l_0 = 23135. Find l_{21}. Round down to the nearest whole coelacanth.

Solution S3L8-2. We note that $l_7 = l_0 * {}_7p_0$, $l_{14} = l_7 * {}_7p_7$, and $l_{21} = l_{14} * {}_7p_{14}$. Thus, $l_{21} = l_0 * {}_7p_0 * {}_7p_7 * {}_7p_{14}$ = 23135*0.8735*0.9623*0.5567 = 10825.90272 = **about 10825 coelacanths.**

Problem S3L8-3. In a particular deterministic survivorship group of minotaurs, 8742 minotaurs of age 9 are present. A year later, only 7235 of these minotaurs are found to have survived to age 10. What is the effective annual rate of mortality for minotaurs during year 9?

Solution S3L8-3. We use the formula $q_x = (l_x - l_{x+1})/l_x$. We know that l_9 = 8742 and l_{10} = 7235. Thus,

q_9 = (8742 - 7235)/8742 = **q_9 = about 0.1723861817.**

Problem S3L8-4. In a particular deterministic survivorship group of blue squirrels, 10035 squirrels of age 2 are present. Only 8515 of these squirrels survive to age 6. What is the 4-year rate of mortality for blue squirrels from age 2 to age 6?

Solution S3L8-4. We use the formula ${}_nq_x = (l_x - l_{x+n})/l_x$. We know that l_2 = 10035 and l_6 = 8515. Thus, ${}_4q_2$ = (10035 - 8515)/10035 = **${}_4q_2$ = 0.154698555.**

Problem S3L8-5. Sktrfs always survive to age 5. Upon turning 5, sktrfs in a particular deterministic survivorship group face a 4-year rate of mortality of 0.13, followed by an 8-year rate of mortality of 0.262, followed by a 3-year rate of mortality of 0.416. Given that 40006 sktrfs survived to age 20, how many of them must have originally been in the group? Use ordinary rounding conventions to round to the nearest whole sktrf.

Solution S3L8-5. We know that $l_9 = l_5(1 - {}_4q_5)$, $l_{17} = l_9(1 - {}_8q_9)$, and $l_{20} = l_{17}(1 - {}_3q_{17})$. Thus,

$l_{20} = l_5(1 - {}_4q_5)(1 - {}_8q_9)(1 - {}_3q_{17})$ and $l_0 = l_5 = l_{20}/[(1 - {}_4q_5)(1 - {}_8q_9)(1 - {}_3q_{17})]$ =

40006/[(1 - 0.13)(1-0.262)(1-0.416)] = 106693.1823 = **about 106693 sktrfs.**

Section 9

Computing Expected Values for Continuous Random Variables

The following theorem, as given by Bowers, Gerber, et. al., is useful for computing the expected values of continuous random variables.

Theorem 9.1: "If T is a continuous type random variable with cumulative distribution function (c.d.f.) G(t) such that G(0) = 0 and probability density function (p.d.f.) G'(t) = g(t), and z(t) is such that it is a nonnegative, monotonic, differentiable function and E[z(T)] exists, then **$E[z(T)] = {}_0{}^{\infty}\int z(t)g(t)dt = z(0) + {}_0{}^{\infty}\int z'(t)[1 - G(t)]dt$."**

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1986. First Edition. Society of Actuaries: Itasca, Illinois. p. 62.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L9-1. The random variable T has a probability density function of $g(t) = (1/76)e^{-(t/76)}$. Find E[t].

Solution S3L9-1. Here, our z(t) = t and so we can use the formula $E[z(T)] = {}_0{}^{\infty}\int z(t)g(t)dt =$

${}_0{}^{\infty}\int t(1/76)e^{-(t/76)}dt$. We use the tabular method of integration by parts:

Sign-----u-----dv
+...(1/76)t....$e^{-(t/76)}$
-....(1/76)....$-76e^{-(t/76)}$
+........0......$5776e^{-(t/76)}$

Thus, $E[x] = -te^{-(t/76)} - 76e^{-(t/76)} \Big|_0^{\infty} = - 0 - 0 + 0 + 76 =$ **E[x] = 76**.

Problem S3L9-2. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. Find $E[x^3+5]$.

Solution S3L9-2. We use the formula $E[z(T)] = z(0) + {}_0{}^{\infty}\int z'(t)[1 - G(t)]dt$. Here, T = X and z(X) $= x^3+5$, so z(0) = 5. $z'(x) = 3x^2$. G(x) is the cumulative distribution function, which is one minus the survival function. Thus, $[1 - G(x)] = s(x) = e^{-0.34x}$. Hence, $E[x^3+5] = 5 + {}_0{}^{\infty}\int 3x^2e^{-0.34x}dx$.

We use the tabular method of integration by parts:

Sign-----u-----dv
+.......$3x^2$.....$e^{-0.34x}$
-........6x....-$(50/17)e^{-0.34x}$
+.......6......$(2500/289)e^{-0.34x}$
-........0......-$(125000/4913)e^{-0.34x}$

Thus, $E[x^3+5] = 5 + (-3x^2(50/17)e^{-0.34x} - 6x(2500/289)e^{-0.34x} - 6(125000/4913)e^{-0.34x} \Big|_0^\infty) =$

$5 + 6(125000/4913) =$ **about 157.6562182**

Problem S3L9-3. The life of a giant pin-striped cockroach has the following survival function associated with it: $s(x) = 1 - x/94$, for $0 \le x \le 94$ and 0 otherwise. Find $E[x^6 + 2]$.

Solution S3L9-3. We use the formula $E[z(T)] = z(0) + {}_0^\infty\int z'(t)[1 - G(t)]dt$. Here, T = X and

$z(X) = x^6 + 2$, so $z(0) = 2$. $z'(x) = 6x^5$. G(x) is the cumulative distribution function, which is one minus the survival function. Thus, $[1 - G(x)] = s(x) = 1 - x/94$, for $0 \le x \le 94$ and 0 otherwise. Hence,

$E[x^6 + 2] = 2 + {}_0^{94}\int 6x^5(1 - x/94)dx = 2 + {}_0^{94}\int(6x^5 - 3x^6/47)dx = 2 + (x^6 - 3x^7/329) \Big|_0^{94} =$

$2 + (94^6 - 3*94^7/329) =$ **about 98,552,825,867**.

Problem S3L9-4. Three-headed donkeys always survive until age 1. Thereafter, the survival function for the life of a three-headed donkey is $s(x) = 1/x$ for all $x > 1$. What is the expected value of the lifetime of a three-headed donkey?

Solution S3L9-4. We use the formula $E[z(T)] = {}_0^\infty\int z(t)g(t)dt$. Here, X = T and $s(x) = 1/x$, so $G(x) = 1 - 1/x$, and $g(x) = 1/x^2$. Since $z(x) = x$ and every donkey survives to age 1, it follows that $E(x) = {}_1^\infty\int(x/x^2)dx = {}_1^\infty\int(1/x)dx = \ln(x) \Big|_1^\infty = \ln(\infty) - \ln(1) = \infty - 0 = \infty$. So the **expected value of the lifetime of a three-headed donkey is infinite.** Even though some three-headed donkeys will die, this is still the case, because some small number of donkeys will get to live forever - since the survival function asymptotically approaches zero, but never gets there.

Problem S3L9-5. Zigzag-striped plankton never survive past the age of 2. The cumulative distribution function for the lives of zigzag-striped plankton is as follows: $G(t) = t^3/8$, $0 \le t \le 2$, 0 otherwise. What is the expected value of the lifetime of a zigzag-striped plankton?

Solution S3L9-5. We use the formula $E[z(T)] = {}_0^\infty\int z(t)g(t)dt$.Here, our upper bound is 2 rather than infinity, and $z(t) = t$. Moreover, $g(t) = G'(t) = 3t^2/8$. Thus, $E[t] = {}_0^2\int 3t^3/8dt = 3t^4/32 \Big|_0^2 =$ $3*16/32 =$ **E[t] = 3/2.**

Section 10

Complete-Expectation-of-Life

The function $\mathbf{\dot{e}_x}$ is the **complete-expectation-of-life** and is equivalent to E[T(x)], the expected value of the future lifetime of life (x) - a life that has already attained the age of x. The following expressions are equal to $\dot{e}_x$.

$$\mathbf{\dot{e}_x = E[T(x)] = {}_0{}^{\infty}\!\int t * {}_tp_x * \mu_{x+t} * dt = {}_0{}^{\infty}\!\int {}_tp_x dt}$$

It is also useful to be able to find $E[T(x)^2]$ and Var[T(x)].

$$\mathbf{E[T(x)^2] = {}_0{}^{\infty}\!\int t^2 * {}_tp_x * \mu_{x+t} * dt = 2{}_0{}^{\infty}\!\int t * {}_tp_x dt}$$

$$\mathbf{Var[T(x)] = E[T(x)^2] - E[T(x)]^2 = 2{}_0{}^{\infty}\!\int t * {}_tp_x dt - \dot{e}_x{}^2}$$

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1986. First Edition. Society of Actuaries: Itasca, Illinois. pp. 62-63.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L10-1. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. Find $\dot{e}_9$.

Solution S3L10-1. First, we want to find ${}_tp_9 = s(9 + t)/s(9) = e^{-0.34(9+t)}/e^{-0.34(9)} = e^{-0.34t}$. Thus, we can use the formula $\dot{e}_x = {}_0{}^{\infty}\!\int {}_tp_x dt = {}_0{}^{\infty}\!\int {}_tp_9 dt = {}_0{}^{\infty}\!\int e^{-0.34t} dt = (-50/17)e^{-0.34t} \big|_0{}^{\infty}$ = $\mathbf{\dot{e}_9 = 50/17}$ = **about 2.941176471**

Problem S3L10-2. The life of a giant pin-striped cockroach has the following survival function associated with it: $s(x) = 1 - x/94$, for $0 \leq x \leq 94$ and 0 otherwise. Find $E[T(34)^2]$.

Solution S3L10-2. First, we want to find ${}_tp_{34} = s(34 + t)/s(34) = [1 - (34 + t)/94]/[1 - 34/94] =$

$(60 - t)/60$. Thus, we can use the formula $E[T(x)^2] = 2{}_0{}^{\infty}\!\int t * {}_tp_x dt = 2{}_0{}^{60}\!\int t(60 - t)/60 dt = 2{}_0{}^{60}\!\int (t - t^2/60) dt =$

$2(t^2/2 - t^3/180) \big|_0{}^{60} = 2(60^2/2 - 60^3/180) =$ $\mathbf{E[T(34)^2] = 1200.}$ (Note: The upper bound of the integral was changed to 60, because no giant pin-striped cockroach can live more than 60 years past the age of 34.)

Problem S3L10-3. Zigzag-striped plankton never survive past the age of 2. The cumulative distribution function for the lives of zigzag-striped plankton is as follows: $G(t) = t^3/8$, $0 \leq t \leq 2$, 0 otherwise. Find $\dot{e}_1$.

Solution S3L10-3. First, we want to find $_tp_1 = s(1 + t)/s(1)$. Since $G(t) = t^3/8$, $s(t) = 1 - G(t) = 1 - t^3/8$. Thus, $s(1 + t)/s(1) = (1 - (1+t)^3/8)/(1 - 1^3/8) = (1 - (1+t)^3/8)/(7/8) = {}_tp_1 = (8 - (1+t)^3)/7$. Hence, we can use the formula $\mathring{e}_x = {}_0{}^{\infty}\int {}_tp_x dt = {}_0{}^1\int[(8 - (1+t)^3)/7]\, dt = [8t/7 - (1+t)^4/28] \Big|_0^1 = [8/7 - (2)^4/28] - [0 - 1/28] =$

$\mathring{e}_1$ = 17/28 = about 0.6071428571. (Note: The upper bound of the integral was changed to 1, because no zigzag-striped plankton can live more than 1 year past the age of 1.)

Problem S3L10-4. Zigzag-striped plankton never survive past the age of 2. The cumulative distribution function for the lives of zigzag-striped plankton is as follows: $G(t) = t^3/8$, $0 \leq t \leq 2$, 0 otherwise. Find $E[T(1)^2]$.

Solution S3L10-4. We use the formula $E[T(x)^2] = 2{}_0{}^{\infty}\int t^*{}_tp_x dt$. From Solution S3L10-3, we know that $_tp_1 = (8 - (1+t)^3)/7$. Thus, $E[T(1)^2] = 2{}_0{}^1\int[t(8 - (1+t)^3)/7]dt = = 2{}_0{}^1\int[(8t - t(1+t)^3)/7]dt =$

$2{}_0{}^1\int[(8t - t(1+ 3t + 3t^2 + t^3))/7]dt = 2{}_0{}^1\int[(8t - t(1+ 3t + 3t^2 + t^3))/7]dt = 2{}_0{}^1\int[(8t - t - 3t^2 - 3t^3 - t^4)/7]dt =$

$2{}_0{}^1\int[(7t - 3t^2 - 3t^3 - t^4)/7]dt = 2[t^2/2 - t^3/7 - 3t^4/28 - t^5/35] \Big|_0^1 = 2(½ - 1/7 - 3/28 - 1/35) = E[T(1)^2]$ = **31/70 = about 0.4428571429**. (Note: The upper bound of the integral was changed to 1, because no zigzag-striped plankton can live more than 1 year past the age of 1.)

Problem S3L10-5. Zigzag-striped plankton never survive past the age of 2. The cumulative distribution function for the lives of zigzag-striped plankton is as follows: $G(t) = t^3/8$, $0 \leq t \leq 2$, 0 otherwise. Find Var[T(1)].

Solution S3L10-5. We use the formula $Var[T(x)] = E[T(x)^2] - E[T(x)]^2$. From Solutions S3L10-3 and S3L10-4, we know that $E[T(1)^2] = 31/70$ and $E[T(1)] = \mathring{e}_1 = 17/28$. Thus,

$Var[T(1)] = (31/70) - (17/28)^2 =$ **Var[T(1)] = 291/3920 = about 0.0742346939**

Section 11

Median Future Lifetime

m(x) is the **median future lifetime** of life (x). It can be determined by solving either of the following equations:

Pr[T(x) > m(x)] = 1/2

s[x + m(x)]/s(x) = 1/2

Meaning of terms:

T(x) = future lifetime of the life that has already reached the age of x.

s(x) = the survival function for life (x).

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1986. First Edition. Society of Actuaries: Itasca, Illinois. p. 63.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L11-1. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. Find m(24).

Solution S3L11-1. We use the formula s[x + m(x)]/s(x) = ½. Thus,

$e^{-0.34(24+m(24))}/e^{-0.34(24)} = ½$

$e^{-0.34(m(24))} = ½$

-0.34*m(24) = ln(1/2)

m(24) = ln(1/2)/-0.34 = **m(24) = 2.038668178 years**.

Problem S3L11-2. Zigzag-striped plankton never survive past the age of 2. The cumulative distribution function for the lives of zigzag-striped plankton is as follows: $G(t) = t^3/8$, $0 \leq t \leq 2$, 0 otherwise. Find m(1).

Solution S3L11-2. We use the formula s[x + m(x)]/s(x) = ½. We can find $s(x) = 1 - G(t) = 1 - t^3/8$.

Thus, s[1 + m(1)]/s(1) = ½

$(1 - [1 + m(1)]^3/8)/(1 - 1^3/8) = ½$

$(1 - (1+ m(1))^3/8)/(7/8) = ½$

$(8 - (1+ m(1))^3)/7 = ½$

$(8 - (1+ m(1))^3) = 7/2$

$(1+ m(1))^3 = 9/2$

$1 + m(1) = 1.6509636244$

m(1) = about 0.6509636244 years

Problem S3L11-3. Black swans always survive until age 16. After age 16, the lifetime of a black swan can be modeled by the cumulative distribution function $F(x) = 1 - 4x^{-1/2}$, $x > 16$. What is the median age to which newborn black swans live?

Solution S3L11-3. The median age to which newborn black swans live is the same as 16 plus the median age to which 16-year-old black swans live. We know that $s(x) = 1 - F(x) = 4x^{-1/2}$, $x > 16$, and we want to find [16 + m(16)]. We use the formula s[x + m(x)]/s(x) = ½.

$s[16 + m(16)]/s(16) = ½$

$4[16 + m(16)]^{-1/2}/(4(16)^{-1/2}) = ½$

$[16 + m(16)]^{-1/2}/(16)^{-1/2} = ½$

$[16 + m(16)]^{-1/2}/(1/4) = ½$

$[16 + m(16)]^{-1/2} = 1/8$

$([16 + m(16)]^{-1/2})^{-2} = (1/8)^{-2}$

[16 + m(16)] = **64 years.**

Problem S3L11-4. Three-headed donkeys always survive until age 1. Thereafter, the survival function for the life of a three-headed donkey is s(x) = 1/x for all x > 1. What is the median age to which three-headed donkeys that have survived for three years live?

Solution S3L11-4. We want to find [3 + m(3)], since m(3) only tells us the median number of years to which three-year-old three-headed donkeys live *after* their third birthday.

We use the formula s[x + m(x)]/s(x) = ½.

$s[3 + m(3)]/s(3) = ½$

$(1/[3 + m(3)])/(1/3) = ½$

$3/[3 + m(3)] = ½$

We cross-multiply to get 3 + m(3) = **6 years**.

Problem S3L11-5. You know the following about the lives of jumping slugs. For a jumping slug that has survived to age 39, the probability of surviving to age 45 is 0.835. For a jumping slug that has survived to age 45, the probability of dying within the subsequent 8 years is 0.23. A jumping slug that has survived to age 53 has the survival function $s(x) = 1/x^{1/2}$ for all x > 53. Find the median age to which jumping slugs that have survived for 39 years live.

Solution S3L11-5. We seek to find 39 + m(39), i.e., the age at which one-half of the slugs which were alive at age 39 will have died.

By age 45, 1 - 0.835 = 0.165 of the slug population alive at age 39 will have died.

By age 53, another 0.835*0.23 = 0.19205 of the slug population alive at age 39 will have died.

At age 53, only 1 - 0.165 - 0.19205 = 0.64295 of the slug population alive at age 39 is still alive.

We want to know the time at which 0.5 of the slug population alive at age 39 will be alive.

This is 0.5/0.64295 = 0.7776654483 of the slug population alive at age 53.

Thus, we modify our formula thus:

$s[39 + m(39)]/s(53) = 0.7776654483$

We have a tidy survival function, $s(x) = 1/x^{1/2}$, to work with for all x > 53. And we know that

39 + m(39) > 53, because half of the jumping slug population has not yet senselessly perished (i.e., died) at age 53. Thus,

$[39 + m(39)]^{-1/2}/(53)^{-1/2} = 0.7776654483$

$[39 + m(39)]^{-1/2} = 0.1068205645$

39 + m(39) = **87.63755693 years**.

Section 12

Computing Expected Values for Discrete Random Variables

The following theorem, as given by Bowers, Gerber, et. al., is useful for computing the expected values of continuous random variables.

Theorem 12.1: "If K is a discrete random variable with probability only on the non-negative integers, with cumulative distribution function (c. d. f.) G(k) and probability density function (p. d. f.) g(k) = ΔG(k-1), and z(k) is a nonnegative, monotonic function such that E[z(K)] exists, then

$E[z(K)] = {}_{k=0}{}^{\infty}\Sigma z(k)*g(k) = z(0) + {}_{k=0}{}^{\infty}\Sigma[1 - G(k)]\Delta z(k)$.

When K is the curate-future-lifetime of a life (x), then we have **$E[K] = e_x = {}_{k=0}{}^{\infty}\Sigma_{k+1}p_x$**.

Note that E[K] can also be referred to as e_x.

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1986. First Edition. Society of Actuaries: Itasca, Illinois. pp. 64-65.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L12-1. Prxqaks have a 0.36 probability of senselessly perishing (i.e., dying) on their 3rd birthday, 0.41 probability of dying on their 7th birthday, and 0.23 probability of dying on their 13th birthday. They cannot die at any other time. What is the expected value for the age of death of a prxqak?

Solution S3L12-1. We are trying to find E[k]. We use the formula $E[z(K)] = {}_{k=0}{}^{\infty}\Sigma z(k)*g(k) = {}_{k=0}{}^{\infty}\Sigma k*g(k)$ in this case. g(k) is a discrete p.d.f. with

g(k) = 0.36 for k = 3;

g(k) = 0.41 for k = 7;

g(k) = 0.23 for k = 13.

Thus, $E[k] = {}_{k=0}{}^{\infty}\Sigma k*g(k) = 3*0.36 + 7*0.41 + 13*0.23 =$ **6.94 years.**

Problem S3L12-2. Flying goblins can only die on their 2nd, 3rd, and 6th birthdays. For each of the times t when it can die, the probability that a particular flying goblin will senselessly perish is 1/t. Find $E[t^2]$.

Solution S3L12-2. We use the formula $E[z(K)] = {}_{k=0}{}^{\infty}\Sigma z(k)*g(k)$, where K = T and $z(t) = t^2$ and $g(t) = 1/t$ for t = 2, 3, and 6. Thus, $E[t^2] = 2^2(1/2) + 3^2(1/3) + 6^2(1/6) = 2 + 3 + 6 =$ **$E[t^2] = 11$**

Problem S3L12-3. The cumulative distribution function for the lives of brontosauruses is as follows:

$G(k) = 0$ for $k \leq 3$;

$G(k) = 0.2$ for $3 < k \leq 19$;

$G(k) = 0.3$ for $19 < k \leq 34$;

$G(k) = 0.33$ for $34 < k \leq 36$;

$G(k) = 0.6$ for $36 < k \leq 55$;

$G(k) = 0.9$ for $55 < k \leq 77$;

$G(k) = 1$ for $k > 77$;

Find $E[k^3 + 2]$.

Solution S3L12-3. We use the formula

$E[z(K)] = z(0) + {}_{k=0}{}^{\infty}\Sigma[1 - G(k)]\Delta z(k)$.

Here, $z(0) = 2$, $z(k) = k^3 + 2$, and G(k) is as given above.

Thus, we have

$E[k^3 + 2] = 2 + \Delta z(3) + [1 - 0.2]\Delta z(19) + [1 - 0.3]\Delta z(34) + [1 - 0.33]\Delta z(36) + [1 - 0.6]\Delta z(55) + [1 - 0.9]\Delta z(77)$

$E[k^3 + 2] = 2 + \Delta z(3) + 0.8\Delta z(19) + 0.7\Delta z(34) + 0.67\Delta z(36) + 0.4\Delta z(55) + 0.1\Delta z(77)$

$z(3) = 29$, so $\Delta z(3) = 29 - 2 = 27$

$z(19) = 6861$, so $\Delta z(19) = 6861 - 29 = 6832$

$z(34) = 39306$, so $\Delta z(34) = 39306 - 6861 = 32445$

$z(36) = 46658$, so $\Delta z(36) = 46658 - 39306 = 7352$

$z(55) = 166377$, so $\Delta z(55) = 166377 - 46658 = 119719$

$z(77) = 456535$, so $\Delta z(77) = 456535 - 166377 = 290158$

Thus, $E[k^3 + 2] = 2 + 27 + 0.8*6832 + 0.7*32445 + 0.67*7352 + 0.4*119719 + 0.1*290158$

$E[k^3 + 2] = 110035.31$

Problem S3L12-4. Prxqaks have a 0.36 probability of senselessly perishing (i.e., dying) on their 3rd birthday, 0.41 probability of dying on their 7th birthday, and 0.23 probability of dying on their 13th birthday. They cannot die at any other time. What is e_4 for Prxqaks? (Note: assume that ${}_{k+1}p_x$ takes into account lives that have survived to the (k+1)st birthday but will die on said birthday, immediately after survival statistics are compiled.)

Solution S3L12-4. We use the formula $e_x = {}_{k=0}{}^{\infty}\Sigma_{k+1}p_x$. In this case, the relevant upper bound of the summation is 8, since no prxqaks live beyond 4+(8+1) = 13 years.

Thus, $e_4 = {}_1p_4 + {}_2p_4 + {}_3p_4 + {}_4p_4 + {}_5p_4 + {}_6p_4 + {}_7p_4 + {}_8p_4 + {}_9p_4$

At age 4, 1-0.36 = 0.64 of the original cohort of newborn prxqaks is alive.

No prxqak dies prior to age 7, so

${}_1p_4 = {}_2p_4 = {}_3p_4 = 1$

But at age 7, only 0.23 of the original population or 0.23/0.64 = 0.359375 of the population alive at age 4 remains alive. No prxqak dies thereafter, until age 13, when all remaining prxqaks die. Thus,

${}_4p_4 = {}_5p_4 = {}_6p_4 = {}_7p_4 = {}_8p_4 = {}_9p_4 = 0.359375$.

Hence, $e_4 = 3(1) + 6(0.359375) =$ **$e_4 = 5.15625$ years**

Problem S3L12-5. Flying goblins can only die on their 2nd, 3rd, and 6th birthdays. For each of the times t when it can die, the probability that a particular flying goblin will senselessly perish is 1/t. Find e_1. (Note: assume that ${}_{k+1}p_x$ takes into account lives that have survived to the (k+1)st birthday but will die on said birthday, immediately after survival statistics are compiled.)

Solution S3L12-5. We use the formula $e_x = {}_{k=0}{}^{\infty}\Sigma_{k+1}p_x$. In this case, the relevant upper bound of the summation is 4, since no prxqaks live beyond 1+(4+1) = 6 years.

Thus, $e_1 = {}_1p_1 + {}_2p_1 + {}_3p_1 + {}_4p_1 + {}_5p_1$

At age 2, ½ of the goblins will die, so ${}_1p_1 = 1$ and ${}_2p_1 = 1/2$

At age 2, another 1/3 of the goblins will die. No further goblins die until age 6, when the remaining goblins die. So ${}_3p_1 = {}_4p_1 = {}_5p_1 = ½ - 1/3 = 1/6$.

Thus, $e_1 = 1 + ½ + 3(1/6) =$ **$e_1 = 2$ years**.

Section 13

Life-Table Probability Functions: Practice Problems and Solutions – Part 5

The function $\mathbf{L_x}$ is "the total expected number of years lived between ages x and x+1 by survivors of the initial group of l_0 lives" (Bowers, Gerber, et. al.). We can find it as follows.
$\mathbf{L_x={}_0{}^1\int l_{x+t}dt}$

The function $\mathbf{m_x}$ is the **central-death-rate at age x.** We can find it as follows.

$\mathbf{m_x= (l_x - l_{x+1})/L_x}$

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1986. First Edition. Society of Actuaries: Itasca, Illinois. p. 65.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L13-1. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. Of a group of 13503 newborn triceratopses, how many years will members of this group in aggregate live between the ages of 5 and 6?

Solution S3L13-1. We use the formula $L_x = {}_0{}^1\int l_{x+t}dt$.Here,$l_0 = 13503$. We recall from Section 6 that $l_x = l_0*s(x)$, so $l_{x+t} = l_0*s(x+t) = 13503e^{-0.34(x+t)}$ and therefore $l_0*s(5+t) = 13503e^{-0.34(5+t)}$.

We want to find $L_5 = {}_0{}^1\int 13503e^{-0.34(5+t)}dt = 13503e^{-0.34(5)}{}_0{}^1\int e^{-0.34t}dt = 13503e^{-0.34(5)}(-50/17)e^{-0.34t}\big|_0{}^1 = 13503e^{-0.34(5)}(50/17)(1 - e^{-0.34}) = L_5 =$ **2091.170419 years**.

Problem S3L13-2. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. For a group of 13503 newborn triceratopses, find the central-death-rate at age 5.

Solution S3L13-2. We want to find m_5, which we will do using the formula$m_x = (l_x - l_{x+1})/L_x$.

We know from Solution S3L13-1 that $L_5 = 2091.170419$ years. Moreover, we recall from Section 6 that $l_x = l_0*s(x)$, so $l_5 = 13503e^{-0.34(5)}$ and $l_6 = 13503e^{-0.34(6)}$. Thus,

$m_5 = (l_5 - l_6)/L_5 = (13503e^{-0.34(5)} -13503e^{-0.34(6)})/2091.170419 =$ $\mathbf{m_5 = 0.34}$.

Problem S3L13-3. For a particular group of polka-dotted zebras, you are given $l_x = 95/x^4$, for all $x > 1$. Find L_4.

Solution S3L13-3. We use the formula $L_x = {}_0{}^1\int l_{x+t}dt$. Here, x = 4, so $l_{x+t} = 95/(4+t)^4$ and

$L_4 = {}_0{}^1\int[95/(4+t)^4]dt = -95/[3(4+t)^3] \Big|_0{}^1 = 95/(3(4)^3) - 95/(3(5)^3)$ = **L_4 = 1159/4800 = about 0.2414583333.**

Problem S3L13-4. For a particular group of polka-dotted zebras, you are given $l_x = 95/x^4$, for all x > 1. Find m_4.

Solution S3L13-4. We use the formula$m_x = (l_x - l_{x+1})/L_x$.

We know from Solution S3L13-3 that L_4 = 0.2414583333years.

Moreover, $l_4 = 95/4^4$ and $l_5 = 95/5^4$.

Thus, $m_4 = (95/4^4 - 95/5^4)/0.2414583333$ = **m_4 = about 0.9073770493.**

Problem S3L13-5. Black swans always survive until age 16. After age 16, the lifetime of a black swan can be modeled by the cumulative distribution function $F(x) = 1 - 4x^{-1/2}$, x > 16. There is a cohort of 3511 newborn black swans. How many years will members of this group in aggregate live between the ages of 31 and 32?

Solution S3L13-5. We use the formula $L_x = {}_0{}^1\int l_{x+t}dt$. We recall from Section 6 that $l_x = l_0*s(x)$. Here, l_0 = 3511 and $s(x) = 1 - F(x) = 4x^{-1/2}$, x > 16. Thus, for x > 16, $l_{x+t} = 3511*4(x+t)^{-1/2}$. We seek to find L_{31}, so here x = 31. Thus, $L_{31} = {}_0{}^1\int 3511*4(31+t)^{-1/2}dt = 3511*8(31+t)^{1/2} \Big|_0{}^1 = 3511*8[(32)^{1/2} - (31)^{1/2}]$ = **L_{31} = 2502.356737 years**.

Section 14

Life-Table Probability Functions: Practice Problems and Solutions – Part 6

The life table function $\mathbf{T_x}$represents "**the total number of years lived beyond age x** by the survivorship group with l_0 initial members" (Bowers, Gerber, et. al.). This function can be found as follows:

$$\mathbf{T_x = {}_0^{\infty}\int l_{x+t} dt}$$

Moreover, the following relationship holds:

$$\mathbf{T_x/l_x = \dot{e}_x}$$

The life table function **a(x)** represents "**the average number of years lived between ages x and x + 1 by those of the survivorship group who die between those ages."** (Bowers, Gerber, et. al.).

The direct determination of a(x) is rather time-consuming:

$$\mathbf{a(x) = [{}_0^1\int t*l_{x+t}*\mu_{x+t}dt]/[{}_0^1\int l_{x+t}*\mu_{x+t}dt]}$$

An approximation of a(x) which assumes that deaths are uniformly distributed in the year of age is

$$\mathbf{a(x) = {}_0^1\int tdt = ½}$$

A faster way to determine a(x) is through the following identity:

$$\mathbf{L_x = a(x)l_x + [1 - a(x)]l_{x+1}}$$

Thus, $\mathbf{a(x) = (L_x - l_{x+1})/(l_x - l_{x+1})}$

Meaning of variables:
$\mathbf{L_x}$ is "the total expected number of years lived between ages x and x+1 by survivors of the initial group of l_0 lives" (Bowers, Gerber, et. al.).

$\mathbf{\dot{e}_x}$ is the **complete-expectation-of-life** for life (x).

$\mathbf{l_x}$ is the expected number of survivors at age x of an original group of l_0 members.

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1986. First Edition. Society of Actuaries: Itasca, Illinois. pp. 66-67.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L14-1. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. Of a group of 7765 newborn triceratopses, how many total years will be lived beyond age 30 by the group's members?

Solution S3L14-1. We want to find $T_{30} = {}_0{}^\infty\int l_{30+t}dt$. We know that $l_0 = 7765$, so $l_x = l_0*s(x) = 7765e^{-0.34x}$. Thus, $T_{30} = {}_0{}^\infty\int 7765e^{-0.34(30+t)}dt = 7765e^{-0.34(30)}\ {}_0{}^\infty\int e^{-0.34t}dt =$

$7765e^{-0.34(30)}(-50/17)e^{-0.34t}\,|\,_0{}^\infty = 7765e^{-0.34(30)}(50/17) =$ **0.8489044841 years**

Problem S3L14-2. For a certain group of burgundy crickets,

$l_x = 354(1 - 0.0625x^2)$ for $0 \le x \le 4$ and 0 otherwise. Find $\dot{e}_x$ for this group of burgundy crickets.

Solution S3L14-2.

We first find T_x by using the formula $T_x = {}_0{}^\infty\int l_{x+t}dt$. But the upper bound of our integral is not infinity, but 4-x, since at t = 4-x, x+t = x + (4 - x) = 4, and we know that $l_4 = 0$, so all members of the group of burgundy crickets will have died by age 4. Therefore,

${}_0{}^{4-x}\int 354(1 - 0.0625(x+t)^2)dt = {}_0{}^{4-x}\int(354 - 22.125(x+t)^2)dt = (354t - 7.375(x+t)^3)\,|\,_0{}^{4-x} =$

$1416 - 354x - 7.375(4)^3 + 7.375x^3 = T_x = 944 - 354x + 7.375x^3 =$

We use the formula $T_x/l_x = \dot{e}_x$ to find that

$\dot{e}_x = (944 - 354x + 7.375x^3)/[354(1 - 0.0625x^2)]$.

We can simplify this expression as follows:

$\dot{e}_x = [354(8/3 - x + (1/48)x^3)]/[354(1 - 0.0625x^2)]$.

$\boldsymbol{\dot{e}_x = (8/3 - x + (1/48)x^3)/(1 - 0.0625x^2)}$.

Problem S3L14-3. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. For a group of 7765 newborn triceratopses, find a(5).

Solution S3L14-3. We know that $l_0 = 7765$, so $l_x = l_0*s(x) = 7765e^{-0.34x}$.

Then we find L_5 using the formula $L_x = {}_0{}^1\int l_{x+t}dt$.

$L_5 = {}_0{}^1\int 7765e^{-0.34(5+t)}dt = 7765e^{-0.34(5)}\,{}_0{}^1\int e^{-0.34t}dt = 7765e^{-0.34(5)}(-50/17)e^{-0.34t}\,|\,_0{}^1 =$

$7765e^{-0.34(5)}(50/17)(1 - e^{-0.34}) = L_5 = 1202.543013$ years.

We can also find $l_5 = 7765e^{-0.34*5} = 1418.537564$ and $l_6 = 7765e^{-0.34*6} = 1009.67294$.

Now we can use the formula $a(x) = (L_x - l_{x+1})/(l_x - l_{x+1}) = (L_5 - l_6)/(l_5 - l_6) =$

$(1202.543013 - 1009.67294)/(1418.537564 - 1009.67294) =$ **$a(5) = 0.4717211057$**

Problem S3L14-4. For a certain group of burgundy crickets,

$l_x = 354(1 - 0.0625x^2)$ for $0 \leq x \leq 4$ and 0 otherwise. Find a(2) for this group of burgundy crickets.

Solution S3L14-4. We first find L_2 using the formula $L_x = {}_0{}^1\int l_{x+t}dt$.

$L_2 = {}_0{}^1\int l_{2+t}dt = {}_0{}^1\int 354(1 - 0.0625(2+t)^2)dt = {}_0{}^1\int(354 - 22.125(2+t)^2)dt =$

$(354t - 7.375(2+t)^3) \Big|_0^1 = 354 - 7.375*27 + 7.375*8 = 354 - 7.375*19 = L_2 = 213.875$.

We can also find $l_2 = 354(1 - 0.0625*2^2) = 265.5$ and $l_3 = 354(1 - 0.0625*3^2) = 154.875$.

Now we can use the formula $a(x) = (L_x - l_{x+1})/(l_x - l_{x+1}) = (L_2 - l_3)/(l_2 - l_3) =$

$(213.875 - 154.875)/(265.5 - 154.875) =$ **$a(2) = 8/15 =$ about 0.5333333333333**

Problem S3L14-5. In a certain group of ichthyosaurs, 6086 ichthyosaurs are alive at age 7, $L_7 = 5440$, and $a(7) = 0.687$. Find out how many of these ichthyosaurs will senselessly perish (i.e., die) before reaching their 8th birthday. Round up to the nearest whole ichthyosaur.

Solution S3L14-5. We use the formula $a(x) = (L_x - l_{x+1})/(l_x - l_{x+1})$. We want to find $l_7 - l_8$.

We first find l_8, for which we need to rearrange the formula thus:
$a(x)(l_x - l_{x+1}) = (L_x - l_{x+1})$

$a(x)l_x - a(x)l_{x+1} = L_x - l_{x+1}$

$a(x)l_x - L_x = a(x)l_{x+1} - l_{x+1}$

$a(x)l_x - L_x = (a(x) - 1)l_{x+1}$

$(a(x)l_x - L_x)/(a(x) - 1) = l_{x+1}$

Thus, $l_8 = (a(7)l_7 - L_7)/(a(7) - 1) = (0.687*6086 - 5440)/(0.687 - 1) = l_8 = 4022.102236$

Thus, $l_7 - l_8 = 6086 - 4022.102236 = 2063.897764 =$ **2064 ichthyosaurs**.

Section 15

The Uniform-Distribution-of-Deaths Assumption for Fractional Ages

To work with life table functions for fractional ages, it is often useful to adopt the assumption of the uniform distribution of deaths. The assumption results in the following formula:

s(x+t) = (1-t)*s(x) +t*s(x+1), where $0 \le t \le 1$.

Under a uniform distribution of deaths for fractional ages, the following equalities hold:

$\mathbf{{}_tq_x = t*q_x}$

$\mathbf{{}_tp_x = 1 - t*q_x}$

$\mathbf{{}_yq_{x+t} = yq_x/(1 - t*q_x)}$, where $0 \le t \le 1$, where $0 \le y \le 1$, $y + t \le 1$.

$\mathbf{\mu_{x+t} = q_x/(1 - t*q_x)}$

$\mathbf{{}_tp_x\mu_{x+t} = q_x}$

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1986. First Edition. Society of Actuaries: Itasca, Illinois. pp. 67-68.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L15-1. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. Assuming a uniform distribution of deaths for fractional ages, find s(17.6).

Solution S3L15-1. We use the formula s(x+t) = (1-t)*s(x) +t*s(x+1), where x = 17 and t = 0.6. Then s(17.6) = 0.4*s(17) + 0.6*s(18) = $0.4e^{-0.34*17} + 0.6e^{-0.34*18}$ = **s(17.6) = 0.0025545597.**

Problem S3L15-2. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. Assuming a uniform distribution of deaths for fractional ages, find ${}_{0.2}q_5$.

Solution S3L15-2. First, we find $q_5 = 1 - s(6)/s(5) = 1 - (e^{-0.34*6})/(e^{-0.34*5}) = 1 - e^{-0.34} =$ 0.2882296772. Now we can use the formula ${}_tq_x = t*q_x$. So ${}_{0.2}q_5 = 0.2q_5 = 0.2*0.2882296772 =$

$\mathbf{{}_{0.2}q_5 = 0.0576459354.}$

Problem S3L15-3. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. Assuming a uniform distribution of deaths for fractional ages, find ${}_{0.4}q_{3.3}$.

Solution S3L15-3. We use the formula ${}_yq_{x+t} = yq_x/(1 - t*q_x)$. Here, x = 3, t = 0.3 and y = 0.4. Moreover, $q_3 = 1 - s(4)/s(3) = 1 - (e^{-0.34*4})/(e^{-0.34*3}) = 1 - e^{-0.34} = 0.2882296772$. Thus,

${}_{0.4}q_{3.3} = 0.4*0.2882296772/(1 - 0.3*0.2882296772) =$ $\mathbf{{}_{0.4}q_{3.3} = 0.1262046484}$.

Problem S3L15-4. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. Assuming a uniform distribution of deaths for fractional ages, find $\mu_{1.9}$.

Solution S3L15-4. We use the formula $\mu_{x+t} = q_x/(1 - t*q_x)$. Here, x = 1, t = 0.9.

Moreover, $q_1 = 1 - s(2)/s(1) = 1 - (e^{-0.34*2})/(e^{-0.34*1}) = 1 - e^{-0.34} = 0.2882296772$.

Thus, $\mu_{1.9} = 0.2882296772/(1 - 0.9*0.2882296772) =$ $\boldsymbol{\mu_{1.9} = 0.389187535}$.

Problem S3L15-5. The lives of a group of π-legged grasshoppers do not follow a neat survival function. However, you are given that deaths for fractional ages are uniformly distributed and that ${}_{0.3}p_2 = 0.656$ and $\mu_{2.3} = 0.448$. Find q_2 for this group of π-legged grasshoppers.

Solution S3L15-5. We use the formula ${}_tp_x\mu_{x+t} = q_x$. Thus, $q_2 = {}_{0.3}p_2*\mu_{2.3} = 0.656*0.448 =$ $\mathbf{q_2 = 0.293888}$

Section 16

The Uniform-Distribution-of-Deaths Assumption for Fractional Ages: Practice Problems and Solutions – Part 2

We define the following random variables:
T = time until death
K = curate-future-lifetime
S = the fractional part of a year lived in the year of death

The following hold irrespective of the assumptions we make for fractional ages.
T = K + S
$\Pr[k < T \leq k + s] = {}_{k|s}q_x = {}_kp_x * {}_sq_{x+k}$

Then, under the assumption of the uniform distribution of deaths for fractional ages, the following relationships hold:

$\Pr[k < T \leq k + s] = s * {}_{k|}q_x = s * {}_kp_x * q_{x+k}$
$\mathring{e}_x = e_x + ½$
Var[T] = Var[K] + 1/12

Meanings of variables:

$\mathring{e}_x$= complete-expectation-of-life for a life currently at age x.
e_x= expected value of the curate-future-lifetime for a life currently at age x.

The variables K and S are independent under the uniform distribution of deaths assumption.

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1986. First Edition. Society of Actuaries: Itasca, Illinois. pp. 70-71.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L16-1. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. For a 4-year-old triceratops, find $\Pr[12 < T(4) \leq 12.67]$ without assuming a uniform distribution for fractional ages or making any other special assumptions.

Solution S3L16-1. We use the formula $\Pr[k < T \leq k + s] = {}_{k|s}q_x$.
Thus, $\Pr[12 < T(4) \leq 12.67] = {}_{12|0.67}q_4$ = (s(4 + 12) - s(4 + 12 + 0.67))/s(4) = (s(16) - s(16.67))/s(4) = $(e^{-0.34*16} - e^{-0.34*16.67})/(e^{-0.34*4})$ = **$\Pr[12 < T(4) \leq 12.67]$ = about 0.0034443297.**

Problem S3L16-2. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. For a 4-year-old triceratops, find $\Pr[12 < T(4) \leq 12.67]$ by assuming a uniform distribution of deaths for fractional ages.

Solution S3L16-2. We use the formula $\Pr[k < T \leq k + s] = s*{}_{k|}q_x$. Thus, $\Pr[12 < T(4) \leq 12.67]$ =
$0.67\,{}_{12|}q_4 = 0.67(e^{-0.34*16} - e^{-0.34*17})/(e^{-0.34*4})$ = **$\Pr[12 < T(4) \leq 12.67]$ = about 0.0032650664.**

Problem S3L16-3. For a certain population of newborn rabid rabbits, the expected value of the curate-future-lifetime is 5.56 years. Using the assumption of the uniform distribution of deaths for fractional ages, find the complete-expectation-of-life for newborn rabid rabbits.

Solution S3L16-3. The expected value of the curate-future-lifetime is defined as $E[K] = e_x$. The complete-expectation-of-life is $\dot{e}_x$. We want to find $\dot{e}_0$.We thus use the formula $\dot{e}_x = e_x + ½$.

$\dot{e}_0 = e_0 + ½ = 5.56 + 0.5 =$ **6.16 years.**

Problem S3L16-4. The variance of the total future lifetime of 7-year-old giant dragonflies is 4. Find the variance of the curate-future-lifetime of 7-year-old giant dragonflies.

Solution S3L16-4. We use the formula Var[T] = Var[K] + 1/12, where we want to find Var[K] = Var[T] - 1/12 = 4 - 1/12 = **47/12 = about 3.916666666667**

Problem S3L16-5. Augustus the Actuary makes a bet with Gilgamesh the Gambler that it is possible to reasonably analyze the populations of √(3)-winged hippopotami by assuming a uniform distribution of deaths for fractional ages. Augustus needs to use the assumption to figure out the probability that a newborn √(3)-winged hippopotamus will die sometime between age 5 and age 5.33. Augustus will win the bet if his result using the assumption is within 0.003 of the true result. It is known that the survival function for √(3)-winged hippopotami is $s(x) = \exp[-0.00003(5^x - 1)]$. Who will win the bet? Justify your answer.

Solution S3L16-5. We need to find $\Pr[5 < T(0) \leq 5.33]$ in two ways.

First, without any assumptions,

$\Pr[5 < T(0) \leq 5.33] = {}_{5|0.33}q_0 = (s(5) - s(5.33))/s(0) = (s(5) - s(5.33)) =$
$\exp[-0.00003(5^5 - 1)] - \exp[-0.00003(5^{5.33} - 1)] = 0.0996302456$. We will call this value A.

Now, assuming a uniform distribution of deaths for fractional ages,

$\Pr[5 < T(0) \leq 5.33] = 0.33*{}_{5|}q_0 = 0.33(s(5) - s(6))/s(0) = 0.33(s(5) - s(6)) =$
$0.33(\exp[-0.00003(5^5 - 1)] - \exp[-0.00003(5^6 - 1)]) = 0.0939625149$. We will call this value B.

Now we need to find A - B = 0.0996302456 - 0.0939625149 = **0.0056677309 > 0.003**, so **Gilgamesh wins the bet.**

Section 17

Exam-Style Questions on Life-Table Functions – Part 1

The problems in this section were designed to be similar to problems from past versions of the Casualty Actuarial Society's Exam 3L and the Society of Actuaries' Exam MLC. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L17-1.

Similar to Question 13 from the Casualty Actuarial Society's Spring 2008 Exam 3L.

The future lifetimes of bald chimpanzees can be modeled by the survival function $s(x) = ((94 - x)/94)^{3.1}$. Find $\dot{e}_{44}$, the expected future lifetime of a 44-year-old bald chimpanzee.

Solution S3L17-1. We use the formula $\dot{e}_x = {}_0^{\infty}\int {}_tp_x dt$ from Section 10. We need to find ${}_tp_{44}$ = $s(44+t)/s(44) = ((94 - 44 - t)/94)^{3.1}/((94 - 44)/94)^{3.1} = ((50 - t)/94)^{3.1}/((50)/94)^{3.1} = ((50 - t)/50)^{3.1}$.

In determining the upper bound for our integral, we need to recognize that bald chimpanzees do not live indefinitely. In fact, no bald chimpanzee lives past the age of 94, since s(94) = 0. In determining the future lifetime of a chimpanzee *already* aged 44, we need to take into account the fact that the *most* this chimpanzee will live is 94 - 44 = 50 years. Thus, the upper bound of our integral is 50.

Thus, $\dot{e}_{44} = {}_0^{50}\int((50 - t)/50)^{3.1}dt = (50 - t)^{4.1}/[4.1(50)^{3.1}] \,|\, {}_0^{50} = (50)^{4.1}/[4.1(50)^{3.1}] = 50/4.1 =$ $\boldsymbol{\dot{e}_{44}}$ **= about 12.19512195 years**.

Problem S3L17-2.

Similar to Question 14 from the Casualty Actuarial Society's Spring 2008 Exam 3L.

This question uses the **Illustrative Life Table, which can be found here**.

In the Illustrative Life Table, the following adjustments are made, whereby every adjusted value of q_x is replaced by the value of q_x*.

$q_{24}* = 3q_{24}$

$q_{25}* = (1/2)q_{25}$

$q_{26}* = q_{24}$

$q_{27}* = 8q_5$

$q_x* = (1/2)q_x$ for $28 \leq x \leq 30$.

Find $_{2|5}q_{24}$ using the modified values (q_x*).

Solution S3L17-2. From the Illustrative Life Table, we know the following:

$q_5 = 0.00098$

$q_{24} = 0.00118$

$q_{25} = 0.00122$

$q_{28} = 0.00139$

$q_{29} = 0.00146$

$q_{30} = 0.00153$

Thus,

$q_{24}* = 3q_{24} = 0.00354$

$q_{25}* = (1/2)q_{25} = 0.00061$

$q_{26}* = q_{24} = 0.00118$

$q_{27}* = 8q_5 = 0.00784$

$q_{28}* = (1/2)q_{28} = 0.000695$

$q_{29}* = (1/2)q_{29} = 0.00073$

$q_{30}* = (1/2)\ q_{30} = 0.000765$

We need to figure out $_{2|5}q_{24} = [s(26)-s(31)]/s(24) = [s(26|24) - s(31|24)]/s(24|24)$.

(That is, we can modify our survival function to $s(x|24)$. This is the survival function for all lives that have survived to age 24. So $s(24|24) = 1$.)

Moreover, s(26|24) and s(31|24) are simple to find, since we already know the probabilities q_x*.

s(26|24) = (1 - q_{24}*)(1 - q_{25}*) = (1 - 0.00354)(1 - 0.00061) = s(26|24) = 0.9958521594

s(31|24) = (1 - q_{24}*)(1 - q_{25}*)(1 - q_{26}*)(1 - q_{27}*)(1 - q_{28}*)(1 - q_{29}*)(1 - q_{30}*) =

0.9958521594(1 - 0.00118)(1 - 0.00784)(1 - 0.000695)(1 - 0.00073)(1 - 0.000765) =

s(31|24) = 0.9847190973.

Thus, [s(26|24) - s(31|24)]/s(24|24) = (0.9958521594 - 0.9847190973)/1 = $_{2|5}q_{24}$ **=about 0.0111330621**

Problem S3L17-3.

Similar to Question 15 from the Casualty Actuarial Society's Spring 2008 Exam 3L.

Fuzzy cucumber plants have the following survival function:

$s(x) = \exp[-99x^{130}]$. Find the force of mortality μ_x for fuzzy cucumber plants.

Solution S3L17-3. We use the formula $\mu_x = -s'(x)/s(x)$.

$-s'(x) = -[-99x^{130}]'\exp[-99x^{130}] = 99*130x^{129}\exp[-99x^{130}] = 12870x^{129}\exp[-99x^{130}]$

Thus, $\mu_x = -s'(x)/s(x) = (12870x^{129}\exp[-99x^{130}])/\exp[-99x^{130}]$ = $\mathbf{\mu_x = 12870x^{129}}$

Problem S3L17-4.

Similar to Question 16 from the Casualty Actuarial Society's Spring 2008 Exam 3L.

Ordinary talking phytoplankton have the following life table associated with them:

x.........l_x
53.......10000
54.......9954
55.......8312

Use the uniform distribution of deaths assumption for fractional ages to find $_{0.3|0.9}q_{53.6}$ for ordinary talking phytoplankton. Do not round your answers at any step. Fractional answers are possible.

Solution S3L17-4. We use the formula $_{t|u}q_x = (s(x + t) - s(x + t + u))/s(x)$ from Section 2.

So $_{0.3|0.9}q_{53.6}$ = (s(53.9) - s(54.8))/s(53.6)

By age 53.6 years, we assume that 0.6 of the phytoplankton to die in the 54^{th} year will have died. Thus, 0.6(10000 - 9954) = 27.6 phytoplankton have died and thus 10000 - 27.6 = 9972.4 are alive at age 53.6. So s(53.6) = 9972.4.

By age 53.9 years, we assume that 0.9 of the phytoplankton to die in the 54^{th} year will have died.

Thus, 0.9(10000 - 9954) = 41.4 phytoplankton have died and thus 10000 - 41.4 = 9958.6 are alive at age 53.9. So s(53.9) = 9958.6.

By age 54.8 years, we assume that 0.8 of the phytoplankton to die in the 55^{th} year will have died.

Thus, 0.8(9954 - 8312) = 1313.6 phytoplankton have died and thus 9954 - 1313.6 = 8640.4 are still alive at age 54.8. So s(54.8) = 8640.4.

Hence, ${}_{0.3|0.9}q_{53.6}$ = (s(53.9) - s(54.8))/s(53.6) = (9958.6 - 8640.4)/9972.4 = **about 0.1321848301**.

Problem S3L17-5.

Similar to Question 5 from the Casualty Actuarial Society's Spring 2007 Exam 3.

This question uses the **Illustrative Life Table, which can be found here**.

In the Illustrative Life Table, find $10000_{4|1}q_{35}$.

Solution S3L17-5. We use the formula ${}_{t|u}q_x$ = (s(x + t) - s(x + t + u))/s(x) from Section 2.

So ${}_{4|1}q_{35}$ = (s(39) - s(40))/s(35) = (s(39 | 35) - s(40 | 35))/s(35 | 35)

s(35 | 35) = 1

s(39 | 35) = (1 - q_{35})(1 - q_{36})(1 - q_{37})(1 - q_{38}) =

(1 - 0.00201)(1 - 0.00214)(1 - 0.00228)(1 - 0.00243) = 0.9911693541

s(40 | 35) = (1 - q_{35})(1 - q_{36})(1 - q_{37})(1 - q_{38})(1 - q_{39}) = 0.9911693541*(1 - 0.0026) = 0.9885923048.

Thus, ${}_{4|1}q_{35}$ = 0.9911693541 - 0.9885923048 = **about 0.0025770403.**

Section 18

Exam-Style Questions on Life-Table Functions – Part 2

The problems in this section were designed to be similar to problems from past versions of the Casualty Actuarial Society's Exam 3L and the Society of Actuaries' Exam MLC. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L18-1.

Similar to Question 6 from the Casualty Actuarial Society's Spring 2007 Exam 3.

The survival function for elephantine fleas is $s(x) = (1444 - x^2)/1444$ for $0 \leq x \leq 38$. Find $\dot{e}_{21}$, the expected future lifetime of a 21-year-old elephantine flea.

Solution S3L18-1. We use the formula $\dot{e}_x = {}_0^{\infty}\int {}_tp_x dt$ from Section 10. We need to find ${}_tp_{21}$ = $s(21+t)/s(21) = [(1444 - (21+t)^2)/1444]/[(1444 - (21)^2)/1444] = (1444 - (21+t)^2)/(1444 - (21)^2)$ =

$(1444 - (21+t)^2)/1003 = 1444/1003 - (21+t)^2/1003$.

In determining the upper bound for our integral, we need to recognize that elephantine fleas do not live indefinitely. In fact, no elephantine flea lives past the age of 38, since s(38) = 0. In determining the future lifetime of a flea *already* aged 21, we need to take into account the fact that the *most* this flea will live is 38 - 21 = 17 years. Thus, the upper bound of our integral is 17.

Thus, $\dot{e}_{21} = {}_0^{17}\int(1444/1003 - (21+t)^2/1003)dt = (1444t/1003 - (21+t)^3/3009) \Big|_0^{17}$ =

$(1444*17/1003 - (38)^3/3009) + (21)^3/3009$ = **$\dot{e}_{21}$ =about 9.316384181 years.**

Problem S3L18-2.

Similar to Question 31 from the Casualty Actuarial Society's Fall 2007 Exam 3.

You know the following data about orange bluebirds:

${}_xp_0 = 0.8345$

$s(x + h) = 0.6664$

${}_{h|z}q_x = 0.113$

Find $s(x + h + z)$.

Solution S3L18-2. We know that ${}_xp_0 = s(x)/s(0) = s(x) = 0.8345$.

Furthermore, we know from Section 2 that ${}_{h|z}q_x = [s(x + h) - s(x + h + z)]/s(x)$.

Thus, $0.113 = [0.6664 - s(x + h + z)]/0.8345$

$0.0942985 = 0.6664 - s(x + h + z)$

$s(x + h + z) = 0.5721015$.

Problem S3L18-3.

Similar to Question 32 from the Casualty Actuarial Society's Fall 2007 Exam 3.

The lives of a group of conical armadillos are described by the following life table.

x.........l_x

76.......10000

77.......7893

78.......7765

Use the uniform distribution of deaths assumption for fractional ages to find ${}_{0.8}p_{76.4}$ for conical armadillos. Do not round your answers at any step. Fractional answers are possible.

Solution S3L18-3. ${}_{0.8}p_{76.4} = s(77.2)/s(76.4)$.

By age 76.4 years, we assume that 0.4 of the armadillos to die in the 77^{th} year will have died. Thus, 0.4(10000 - 7893) = 842.8 armadillos have died and thus 10000 - 842.8= 9157.2 are alive at age 76.4. So s(76.4) = 9157.2.

By age 77.2 years, we assume that 0.2 of the armadillos to die in the 78^{th} year will have died. Thus, 0.2(7893 - 7765) = 25.6 armadillos have died and thus 7893 - 25.6 = 7867.4 are alive at age 77.2. So s(77.2) = 7867.4.

Hence, 7867.4/9157.2 = **${}_{0.8}p_{76.4}$ = about 0.8591490849.**

Problem S3L18-4. Similar to Question 11 from the Casualty Actuarial Society's Fall 2006 Exam 3.

This question uses the **Illustrative Life Table, which can be found here**.

In the Illustrative Life Table, find the average number of *complete* years lived between the ages of 34 and 37.

Solution S3L18-4. We seek to find a truncated version of $E[K] = e_x = {}_{k=0}{}^{\infty}\Sigma_{k+1}p_x$. In our case, this will be ${}_{k=0}{}^{2}\Sigma_{k+1}p_{34} = {}_1p_{34} + {}_2p_{34} + {}_3p_{34}$. The easiest way to use the life table here entails the recognition that

${}_tp_x = l_{x+t}/l_x$ and that in each of our three computations for the values of the "p" functions, there will be a common denominator $l_x = l_{34} = 9438571$.

Thus, ${}_1p_{34} + {}_2p_{34} + {}_3p_{34} = (l_{35} + l_{36} + l_{37})/l_{34} = (9420657 + 9401688 + 9381566)/9438571 =$ **about 2.988154755.**

Problem S3L18-5. Similar to Question 11 from the Casualty Actuarial Society's Fall 2006 Exam 3.

Vercingetorix and Bob are two speckled slugs. Vercingetorix is currently 4 years old, and Bob is currently 6 years old. We know the following about speckled slugs:

$s(4) = 0.89$
$s(6) = 0.77$
${}_{4|3}q_0 = 0.28$
${}_{6|3}q_0 = 0.45$

Find the probability that at least one of the two speckled slugs will be alive three years from now.

Solution S3L18-5. We note that ${}_{4|3}q_0 = (s(4) - s(7))/s(0) = (s(4) - s(7))$
Thus, $0.28 = 0.89 - s(7)$ and so $s(7) = 0.61$.
The probability that Vercingetorix will be alive 3 years from now is $s(7)/s(4) = 0.61/0.89 = 61/89$.

Furthermore, ${}_{6|3}q_0 = (s(6) - s(9))/s(0) = (s(6) - s(9))$
Thus, $0.45 = 0.77 - s(9)$ and so $s(9) = 0.32$.
The probability that Bob will be alive 3 years from now is $s(9)/s(6) = 0.32/0.77 = 32/77$.

The probability that *neither* slug will be alive 3 years from now is $(1 - 61/89)(1 - 32/77)$. Thus, the probability that *at least one* will be alive 3 years from now is $1 - (1 - 61/89)(1 - 32/77) =$ **799/979 = about 0.8161389173.**

Section 19

Temporary Complete Life Expectancy

Another important life table function is the **n-year temporary complete life expectancy** of life (x), which describes the average number of years lived between the ages of x and x + n by a group of l_x members, all aged x. This function is denoted by $\mathring{e}_{x:n\neg}$ and can be found as follows.

$$\mathbf{{}_nL_x/l_x = \mathring{e}_{x:n\neg} = {}_0{}^n\!\int {}_tp_x dt}$$

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1997. Second Edition. Society of Actuaries: Itasca, Illinois. p. 71.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L19-1. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. Find $\mathring{e}_{9:4\neg}$ for triceratopses.

Solution S3L19-1. We use the formula $\mathring{e}_{x:n\neg} = {}_0{}^n\!\int {}_tp_x dt$. Here, ${}_tp_9 = s(9 + t)/s(9) = e^{-0.34(9+t)}/e^{-0.34(9)} = e^{-0.34t}$

Thus, $\mathring{e}_{9:4\neg} = {}_0{}^4\!\int e^{-0.34t}dt = (-50/17)e^{-0.34t} \big|_0{}^4 = 50/17(1 - e^{-0.34*4})$ = **about2.186291832 years.**

Problem S3L19-2. The life of a giant pin-striped cockroach has the following survival function associated with it: $s(x) = 1 - x/94$, for $0 \leq x \leq 94$ and 0 otherwise. Find $\mathring{e}_{17:24\neg}$ for pin-striped cockroaches.

Solution S3L19-2. We use the formula $\mathring{e}_{x:n\neg} = {}_0{}^n\!\int {}_tp_x dt$. Here, ${}_tp_{17} = s(17 + t)/s(17) =$

$(1 - (17+t)/94)/(1 - 17/94) = (77 - t)/77$. Thus, $\mathring{e}_{17:24\neg} = {}_0{}^{24}\!\int[(77 - t)/77]dt = -(77 - t)^2/154 \big|_0{}^{24}$

$= (77^2 - 53^2)/154 =$ **about20.25974026 years**.

Problem S3L19-3. Zigzag-striped plankton never survive past the age of 2. The cumulative distribution function for the lives of zigzag-striped plankton is as follows: $G(t) = t^3/8$, $0 \leq t \leq 2$, 0 otherwise. Find $\mathring{e}_{0.3:0.6\neg}$ for zigzag-striped plankton without using any assumptions for the distribution of deaths for fractional ages.

Solution S3L19-3. We use the formula $\mathring{e}_{x:n\neg} = {}_0{}^n\!\int {}_tp_x dt$.

First, we want to find ${}_tp_{0.3} = s(0.3 + t)/s(0.3)$. Since $G(t) = t^3/8$, $s(t) = 1 - G(t) = 1 - t^3/8$.

Thus, $s(0.3 + t)/s(0.3) = (1 - (0.3+t)^3/8)/(1 - 0.3^3/8) = (8 - (0.3+t)^3)/(63784/8000) =$

${}_tp_{0.3}$ = (64000 - 8000(0.3+t)^3)/63784.

Thus, $\dot{e}_{0.3:0.6\neg}$ = ${}_0{}^{0.6}\int$[(64000 - 8000(0.3+t)^3)/63784]dt =

[64000t/63784 - 8000(0.3+t)^4/255136] $|_0{}^{0.6}$ = **about 0.5817132823 years**.

Problem S3L19-4. The survival function for elephantine fleas is s(x) = (1444 - x^2)/1444 for 0 ≤ x ≤ 38. Find $\dot{e}_{21:5\neg}$ for elephantine fleas.

Solution S3L19-4. We use the formula $\dot{e}_{x:n\neg}$ = ${}_0{}^n\int {}_tp_x$dt. First, we want to find ${}_tp_{21}$ = s(21 + t)/s(21) =

[(1444 - (21+t)^2)/1444]/[(1444 - (21)^2)/1444] = (1444 - (21+t)^2)/1003. Thus,

$\dot{e}_{21:5\neg}$= ${}_0{}^5\int$[(1444 - (21+t)^2)/1003]dt = [1444t/1003 - (21+t)^3/3009] $|_0{}^5$ = **about 4.435028249 years.**

Problem S3L19-5. The future lifetimes of bald chimpanzees can be modeled by the survival function

s(x) = ((94 - x)/94)$^{3.1}$. Find $\dot{e}_{11:19\neg}$ for bald chimpanzees.

Solution S3L19-5. We use the formula $\dot{e}_{x:n\neg}$ = ${}_0{}^n\int {}_tp_x$dt. First, we want to find ${}_tp_{11}$ = s(11 + t)/s(11) =

((94 - (11+t))/94)$^{3.1}$/((94 - 11)/94)$^{3.1}$) = (94 - (11+t))$^{3.1}$/(83)$^{3.1}$ = (1/83)$^{3.1}$(94 - (11+t))$^{3.1}$.

Thus, $\dot{e}_{11:19\neg}$ = ${}_0{}^{19}\int$ [(1/83)$^{3.1}$(94 - (11+t))$^{3.1}$]dt = -(1/83)$^{3.1}$(94 - (11+t))$^{4.1}$/4.1 $|_0{}^{19}$ = **about 13.27102631 years.**

Section 20

Recursion Formulas

Two basic kinds of recursion formulas will be used on Exam 3L.

The **backward recursion formula:**

$u(x) = c(x) + d(x)*u(x + 1)$

The **forward recursion formula:**

$u(x + 1) = -c(x)/d(x) + u(x)/d(x)$

What are the functions u, c, and d?

u(x) is the function/life table characteristic being evaluated.

c(x) and **d(x)** are functions relating u(x) to u(x + 1). They are typically given, along with a starting value of u(x) or u(x +1). Or u(x) and u(x + 1) are given, and one is asked to find c(x) and d(x).

Here, we will do some fairly simple practice problems with recursion formulas, whose main objective will be to get students to memorize the formulas.

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1997. Second Edition. Society of Actuaries: Itasca, Illinois. p. 73.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L20-1. It is known that $u(x + 1) = 3x^3$, $d(x) = 3/x$, $c(x) = 5x^2$. Find u(x) using a recursion formula.

Solution S3L20-1. We use a backward recursion formula:

$u(x) = c(x) + d(x)*u(x + 1) = 5x^2 + (3/x)3x^3 = 5x^2 + 9x^2 =$ $\mathbf{u(x) = 14x^2}$.

Problem S3L20-2. It is known that $c(x) = 15x^2$, and $d(x) = 3$. You know that u(16) = 35. Find u(17) using a recursion formula.

Solution S3L20-2. We use a forward recursion formula:

$u(x + 1) = -c(x)/d(x) + u(x)/d(x) = -15x^2/3 + u(x)/3$. We know that x = 16 and u(x) = 35. Thus, $u(17) = -15*17^2/3 + 35/3 =$ **u(17) = -1433.3333333333**.

Problem S3L20-3. You are trying to establish a recursion formula between $_{t}p_{x}$ and $_{t+1}p_{x}$. Find c(x) and d(x) for this formula, satisfying the requirement that at least one of these functions must have a "p" in it.

Solution S3L20-3. We know that $_{t}p_{x} = s(x + t)/s(x)$ and $_{t+1}p_{x} = s(x + t + 1)/s(x)$
Now we can relate $_{t}p_{x}$ to $_{t+1}p_{x}$ via a backward recursion formula:
$_{t}p_{x} = {}_{t+1}p_{x}[s(x + t)/s(x + t + 1)]$
We know that $s(x + t + 1)/s(x + t) = p_{x+t}$,
so $[s(x + t)/s(x + t + 1)] = 1/p_{x+t}$
Thus, $_{t}p_{x} = [1/p_{x+t}]_{t+1}p_{x}$
$_{t}p_{x} = 0 + [1/p_{x+t}]_{t+1}p_{x}$.
This follows the format of the backward recursion formula $u(x) = c(x) + d(x)*u(x + 1)$.

Thus, **$c(x) = 0$ and $d(x) = [1/p_{x+t}]$.**

Problem S3L20-4. You are trying to establish a recursion formula between a(x) and a(x+1) for a population of aquatic aardvarks. It is known that $(L_x - l_{x+1}) = (7/8)(L_{x+1} - l_{x+2})$ for all x and $(l_x - l_{x+1}) = (3/4)(l_{x+1} - l_{x+2})$ for all x. Find c(x) and d(x) for this formula. Your answers should be expressed in terms of the functions "L," "l," and a(x). (If you need to, feel free to refresh yourself on the formula for L_x in Section 13 and the formula for a(x) from Section 14.)

Solution S3L20-4.

We know from Section 14 that $a(x) = (L_x - l_{x+1})/(l_x - l_{x+1})$.
Likewise, $a(x+1) = (L_{x+1} - l_{x+2})/(l_{x+1} - l_{x+2})$
Now we can relate a(x) to a(x+1):
$a(x) = a(x+1)[(l_{x+1} - l_{x+2})/(L_{x+1} - l_{x+2})][(L_x - l_{x+1})/(l_x - l_{x+1})]$
$a(x) = a(x+1)[(l_{x+1} - l_{x+2})/(l_x - l_{x+1})][(L_x - l_{x+1})/(L_{x+1} - l_{x+2})]$
We know that $(L_x - l_{x+1}) = (7/8)(L_{x+1} - l_{x+2})$ and $(l_x - l_{x+1}) = (3/4)(l_{x+1} - l_{x+2})$.
So $[(l_{x+1} - l_{x+2})/(l_x - l_{x+1})] = 4/3$ and $[(L_x - l_{x+1})/(L_{x+1} - l_{x+2})] = 7/8$
Thus, $a(x) = a(x+1)(4/3)(7/8)$
$a(x) = (7/6)a(x+1)$

Hence, using the format of the backward recursion formula, **$c(x) = 0$ and $d(x) = 7/6$**.

Problem S3L20-5. You are trying to establish a recursion formula between x^3 and $2(x+1)^3$. Find c(x) in this formula, given that $u(x) = x^3$ and $u(x+1) = 2(x+1)^3$, and $d(x) = 2$.

Solution S3L20-5. We use the format of a backward recursion formula:
$u(x) = c(x) + d(x)*u(x + 1)$
$x^3 = c(x) + d(x)*2(x+1)^3$
$x^3 = c(x) + d(x)*2(x+1)^3$
$x^3 = c(x) + d(x)(2x^3 + 6x^2 + 6x + 2)$
$x^3 = c(x) + 2(2x^3 + 6x^2 + 6x + 2)$
$c(x) = x^3 - (4x^3 + 12x^2 + 12x + 4)$
$c(x) = - 3x^3 - 12x^2 - 12x - 4$

Section 21

Term Life Insurance

The following is defined to be the **present-value function.**

$\mathbf{z_t = Z = b_t v_t}$

$\mathbf{z_t = Z}$ is the present value, at policy issue, of the benefit payment.

$\mathbf{b_t}$ is the **benefit function.**

$\mathbf{v_t}$ is the **discount function.** **v** is the one-year discount factor by which a sum of money payable one year from now is multiplied to get its present value today. If the annual effective interest rate is r, then v = 1/(1+r).

n-year term life insurance makes a payment if and only if the insured person dies within n years of the policy's issue. A policy for which a sum of 1 is paid at the death of the insured person has the following functions associated with it, where t is the time from the present moment until death.

$\mathbf{b_t = 1 \text{ if } t \leq n;}$

$\mathbf{b_t = 0 \text{ if } t > n.}$

$\mathbf{v_t = v^t \text{ for } t \geq 0;}$

$\mathbf{Z = v^T \text{ if } T \leq n;}$

$\mathbf{Z = 0 \text{ if } T > n.}$

The expectation of Z, E[Z], is the **actuarial present value** of the insurance. E[Z] is also denoted as

$\bar{A}^1_{x:n\neg}$ for life (x) and can be found using the following formulas:

$\mathbf{\bar{A}^1_{x:n\neg} = {}_0^{\infty}\int z_t * f_T(t)dt}$

$\mathbf{\bar{A}^1_{x:n\neg} = {}_0^{n}\int v^t * {}_tp_x * \mu_x(t)dt}$

In the symbol $\bar{A}^1_{x:n\neg}$, the superscript "1" means that 1 unit of money is paid for a life currently aged x if that life dies prior to reaching the age of x + n. It is possible to have some other amount than 1, say, k, paid instead, in which case the symbol for the actuarial present value of the insurance would become $\bar{A}^k_{x:n\neg}$.

Meaning of variables:

$\mathbf{f_T(t)}$ **=** the probability density function of T, the time to death of life (x).

$\mathbf{\mu_x(t)}$ **=** the force of mortality for life (x) at time t.

$_t\mathbf{p_x}$ = the probability that life (x) will survive to time x+t.

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1997. Second Edition. Society of Actuaries: Itasca, Illinois. pp. 94-95.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L21-1. Imhotep the Immortal lives forever and has decided to commit life insurance fraud. He plans to take out a 56-year term life insurance policy for $100,000 and fake his own death 55 years from now, thereafter pretending to be his own non-existent son (Imhotep II) and claiming the benefits from his policy. Imhotep can foresee that the annual effective rate of interest will be 0.03 for the next 35 years and 0.05 for every year thereafter. What is the present value of his policy to him, given that he is sure of his ability to collect on it? Round your answer to the nearest cent.

Solution S3L21-1. Imhotep will collect $100,000 after 55 years. For the first 35 of those years, the annual discount factor will be 1/1.03. For the remaining 20 years, the annual discount factor will be 1/1.05. Thus, the present value of Imhotep's policy is $100000(1/1.03)^{35}(1/1.05)^{20}$**= about $13,394.03**

Problem S3L21-2. Mandy the Mortal takes out a 56-year term life insurance policy for $100,000. She has a 0.02 probability of dying 5 years from now, a 0.1 probability of dying 23 years from now, a 0.6 probability of dying 35 years from now, a 0.2 probability of dying 53 years from now, and a 0.08 probability of dying 999 years from now. Mandy can foresee that the annual effective rate of interest will be 0.03 for the next 35 years and 0.05 for every year thereafter. Find the actuarial present value of Mandy's policy to her. Round your answer to the nearest cent.

Solution S3L21-2. For the first 35 years from now, the annual discount factor will be 1/1.03. For all subsequent years, the annual discount factor will be 1/1.05. Mandy will only collect if she dies within the next 56 years. That is, if she dies 999 years from now, she will not collect. Thus, the actuarial present value of Mandy's policy is $100000*(0.02/1.03^5 + 0.1/1.03^{23} + 0.6/1.03^{35} + 0.2/(1.03^{35}*1.05^{18}))$ = **about $31068.52.**

Problem S3L21-3. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual effective interest rate in Triceratopsland is 0.07. Jerry the Triceratops is currently 3 years old has a 6-year term life insurance policy, which will pay him 1 Triceratops Currency Unit (TCU) upon death. Find the actuarial present value of this policy. Set up the integral and then use any calculator to evaluate it.

Solution S3L21-3. We use the formula $\bar{A}^1_{x:n\neg} = {}_0{}^n\int v^t * {}_tp_x * \mu_x(t)dt$.
We know that x = 3 and n = 6.
We find ${}_tp_3 = s(x + t)/s(x) = s(3 + t)/s(3) = e^{-0.34(3+t)}/e^{-0.34(3)} = e^{-0.34t}$
We find $\mu_3(t) = -s'(x)/s(x) = 0.34e^{-0.34t}/e^{-0.34t} = 0.34$
We find $v^t = (1/1.07)^t$
Thus, $\bar{A}^1_{3:6\neg} = {}_0{}^6\int(1/1.07)^t * e^{-0.34t} * 0.34dt$.
$\mathbf{\bar{A}^1_{3:6\neg} = {}_0{}^6\int(0.34e^{-0.34t}/1.07^t)dt}$ **= about 0.7617676461 TCU.**

Problem S3L21-4. Xerxes the Spiky Tarantula is 4 years old and has a 4-year term life insurance policy, which will pay 5 Golden Hexagons (GH) upon death. The probability density function for the future lifetime of 4-year-old spiky tarantulas is $f_T(t) = 0.4462871026*0.64^t$. The annual effective interest rate is 0.04. Find the actuarial present value of Xerxes's policy. Set up the integral and evaluate by hand, using a calculator only for the numerical computations.

Solution S3L21-4. We use the formula $\bar{A}^k_{x:n\neg} = {}_0{}^\infty\int z_t * f_T(t)dt$.
Here, $k = b_t = 5$, and $v_t = (1/1.04)^t$.
So $z_t = b_tv_t = 5/1.04^t$ for $t \leq 4$ and 0 otherwise.
It is given that $f_T(t) = 0.4462871026*0.64^t$.
Thus, $\bar{A}^5_{4:4\neg} = {}_0{}^4\int(5/1.04^t)*0.4462871026*0.64^t dt$
$\bar{A}^5_{4:4\neg} = {}_0{}^4\int 2.231435513*0.6153846154^t dt$
$\bar{A}^5_{4:4\neg} = [2.231435513/\ln(0.6153846154)]0.6153846154^t \Big|_0^4$
$\bar{A}^5_{4:4\neg} = [2.231435513/\ln(0.6153846154)]0.6153846154^4 - [2.231435513/\ln(0.6153846154)]$
$\mathbf{\bar{A}^5_{4:4\neg}}$ **= about 3.936950241GH.**

Problem S3L21-5. The life of a giant pin-striped cockroach has the following survival function associated with it: s(x) = 1 - x/94, for $0 \leq x \leq 94$ and 0 otherwise. Orgox the Giant Pin-Striped Cockroach is currently 23 years old and has a 40-year term life insurance policy which will pay 10 Golden Hexagons (GH) upon death. The annual effective interest rate is 0.11. Find the actuarial present value of Orgox's policy.

Solution S3L21-5. We use the formula $\bar{A}^k_{x:n\neg} = k* {}_0{}^n\int v^t * {}_tp_x * \mu_x(t)dt$.
We know that x = 23 and n = 40.
We find ${}_tp_{23} = s(x + t)/s(x) = s(23 + t)/s(23) = (1 - (23+t)/94)/(1 - 23/94) = (71 - t)/71$
We find $\mu_{23}(t) = -s'(x)/s(x) = (1/94)/(1 - x/94) = (-1/94)/[(94 - x)/94] = 1/(94 - x) = 1/(94 - (23-t))$ $= 1/(71 - t)$. Conveniently enough, ${}_tp_{23} * \mu_{23}(t) = ((71 - t)/71)(1/(71 - t)) = 1/71$.
We find $v^t = (1/1.11)^t$
Thus, $\bar{A}^{10}_{23:40\neg} = 10*{}_0{}^{40}\int(0.9009009009)^t(1/71)dt =$
$(10/71)*(1/\ln(0.9009009009))(0.9009009009)^t \Big|_0^{40}$
$= -1.349607606(0.9009009009)^t \Big|_0^{40} = 1.349607606(1 - 0.9009009009^{40}) =$ **about 1.328844689 GH**.

Section 22

Moments of the Present Value of a Term Life Insurance Policy

In Section 21, the formula for $\bar{A}^1_{x:n\neg}$, the actuarial present value of an n-year term life insurance policy paying a benefit of 1 upon the death of life (x), was introduced:
$E[Z] = \bar{A}^1_{x:n\neg} = {}_0^{\infty}\int z_t * f_T(t)dt = {}_0^{n}\int v^t * {}_tp_x * \mu_x(t)dt$

Here, we introduce formulas for the **j-th moment of the distribution of Z,** denoted by $\mathbf{E[Z^j]}$. In one of these formulas, **δ,** the **force of interest**, will be used. We recall from Course 2/FM that v = $e^{-\delta}$.

$\mathbf{E[Z^j] = {}_0^{n}\int (v^t)^j * {}_tp_x * \mu_x(t)dt}$

$\mathbf{E[Z^j] = {}_0^{n}\int e^{-\delta jt} * {}_tp_x * \mu_x(t)dt}$

We note that $E[Z^j]$ when the force of interest is δ is equal to E[Z] when the force of interest is jδ. More generally, for any deterministic force of interest, expressed as δ_t, the following property, known as the **rule of moments,** holds.

$\mathbf{E[Z^j] \ @ \ \delta_t = E[Z] \ @ \ j\delta_t}$

The rule of moments enables us to calculate the variance of Z:
$\mathbf{Var(Z) = {}^2\bar{A}^1_{x:n\neg} - (\bar{A}^1_{x:n\neg})^2}$

Meaning of variables:

$\bar{\mathbf{A}}^{\mathbf{1}}_{\mathbf{x:n\neg}}$= the actuarial present value of n-year term life insurance which pays a benefit of 1 upon the death of life (x). The force of interest is assumed to be δ.

$\mathbf{{}^2\bar{A}^1_{x:n\neg} = E[Z^2]}$ = the actuarial present value of n-year term life insurance which pays a benefit of 1 upon the death of life (x). **Important:** The force of interest for ${}^2\bar{A}^1_{x:n\neg}$ is assumed to be ***2δ***.

$\mathbf{f_T(t)}$ **=** the probability density function of T, the time to death of life (x).

$\boldsymbol{\mu_x(t)}$ **=** the force of mortality for life (x) at time t.

$\mathbf{{}_tp_x}$ = the probability that life (x) will survive to time x+t.

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1997. Second Edition. Society of Actuaries: Itasca, Illinois. pp. 95-96.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L22-1. An n-year term life insurance policy has an actuarial present value of ψ for a constant force of interest 0.07. At what force of interest will the 3rd moment of the distribution of the present-value random variable for this policy be equal to ψ?

Solution S3L22-1. By the rule of moments, $E[Z^j]$ @ δ_t = $E[Z]$ @ $j\delta_t$.

Here, j = 3. Moreover, $E[Z] = \psi$.

Thus, $E[Z^3]$ @ δ_t = ψ @ $3*\delta_t$.

We are given that $3*\delta_t = 0.07$. Thus, $\delta_t = 0.07/3$ = **about 0.02333333333**.

Problem S3L22-2. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is currently 0.06. Theodosius the Triceratops is currently 2 years old and takes out a 3-year term life insurance policy paying a benefit of 1. Find the actuarial present value of this policy.

Solution S3L22-2. We use the formula $\bar{A}^1_{x:n\neg} = {}_0{}^n\int v^t * {}_tp_x * \mu_x(t)dt$. We are asked to find $\bar{A}^1_{2:3\neg}$.

We find ${}_tp_2 = s(x + t)/s(x) = s(2 + t)/s(2) = e^{-0.34(2+t)}/e^{-0.34(2)} = e^{-0.34t}$

We find $\mu_2(t) = -s'(x)/s(x) = 0.34e^{-0.34x}/e^{-0.34x} = 0.34$

We find $v^t = e^{-0.06t}$, since $\delta = 0.06$

Thus, $\bar{A}^1_{2:3\neg} = {}_0{}^3\int e^{-0.06t} * e^{-0.34t} * 0.34dt = {}_0{}^3\int 0.34 * e^{-0.40t}dt = (-34/40)e^{-0.40t} \Big|_0{}^3 = (34/40)(1 - e^{-0.40*3})$

$\bar{A}^1_{2:3\neg}$ = about 0.5939849199.

Problem S3L22-3. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is currently 0.06. Theodosius the Triceratops is currently 2 years old and takes out a 3-year term life insurance policy paying a benefit of 1. Find the second moment of the distribution of the present-value random variable for this policy.

Solution S3L22-3. We want to find $E[Z^2]$, which we can do using the formula
$E[Z^j] = {}_0{}^n\int e^{-\delta jt} * {}_tp_x * \mu_x(t)dt$, using j = 2.
As in Solution S3L22-1, we have ${}_tp_2 = e^{-0.34t}$, $\mu_2(t) = 0.34$, and n = 3.
However, v^t is changed from $e^{-0.06t}$ to $e^{-0.06*2t} = e^{-0.12t}$.

Thus, $E[Z^2] = {}_0{}^3\int e^{-0.12t} * e^{-0.34t} * 0.34dt = {}_0{}^3\int 0.34 * e^{-0.46t}dt = (-34/46)e^{-0.46t} \Big|_0{}^3 = (34/46)(1 - e^{-0.46*3})$

${}^2\bar{A}^1_{x:n\neg} = E[Z^2]$ = about 0.5531810695.

Problem S3L22-4. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is currently 0.06. Theodosius the Triceratops is currently 2 years old and takes out a 3-year term life insurance policy paying a benefit of 1. Find the variance of the distribution of the present-value random variable for this policy.

Solution S3L22-4. We use the formula $Var(Z) = {}^2\bar{A}^1_{x:n\neg} - (\bar{A}^1_{x:n\neg})^2$.

From Solution S3L22-1, we know that $\bar{A}^1_{2:3\neg} = 0.5939849199$.

From Solution S3L22-2, we know that ${}^2\bar{A}^1_{x:n\neg} = 0.5531810695$.

Thus, $Var(Z) = 0.5531810695 - 0.5939849199^2 =$ **Var(Z) = about 0.2003629844.**

Problem S3L22-5. The life of a giant pin-striped cockroach has the following survival function associated with it: $s(x) = 1 - x/94$, for $0 \leq x \leq 94$ and 0 otherwise. Sparkles the Giant Pin-Striped Cockroach is currently 11 years old and has a 50-year term life insurance policy which will pay 10 Golden Hexagons (GH) upon death. The annual force of interest is 0.3. Find the fifth moment of the distribution of the present-value random variable for this policy.

Solution S3L22-5.
The random variable for the present value of the insurance policy is 10Z.
We seek to find $E[(10Z)^5]$. We use the formula $E[Z^j] = {}_0{}^n\int e^{-\delta jt} * {}_tp_x * \mu_x(t)dt$.

We know that $j = 5$, $\delta = 0.3$, $x = 11$, and $n = 50$.

We find ${}_tp_{11} = s(x + t)/s(x) = s(11 + t)/s(11) = (1 - (11+t)/94)/(1 - 11/94) = (83 - t)/83$

We find $\mu_{11}(t) = -s'(x)/s(x) = (1/94)/(1 - x/94) = (-1/94)/[(94 - x)/94] = 1/(94 - x) = 1/(94 - (11-t))$ $= 1/(83 - t)$.

Conveniently enough, ${}_tp_{11} * \mu_{11}(t) = ((83 - t)/83)(1/(83 - t)) = (1/83)$ We find $e^{-\delta jt} = e^{-5*0.3t} = e^{-1.5t}$

Thus, $E[(10Z)^5] = 10^5E[Z^5] = 10^5 * {}_0{}^{50}\int e^{-1.5t}(1/83)dt$
$10^5E[Z^5] = 10^5*(-2/249)e^{-1.5t} \Big|_0^{50} = 10^5*(2/249)(1 - e^{-75}) = 10^5*(2/249)$ (for all practical purposes).

$10^5E[Z^5]$ = 200000/249 = about 803.2128514.

Section 23

Whole Life Insurance

As in Section 21, the following is defined to be the **present-value function.**

$\mathbf{z_t = Z = b_t v_t}$

$\mathbf{z_t = Z}$ is the present value, at policy issue, of the benefit payment.

$\mathbf{b_t}$ is the **benefit function.**

$\mathbf{v_t}$ is the **discount function. v** is the one-year discount factor by which a sum of money payable one year from now is multiplied to get its present value today. If the annual effective interest rate is r, then v = 1/(1+r).

Whole life insurance makes a payment at the time of death of the insured person, no matter when that time might be. A policy for which a sum of 1 is paid at the death of the insured person has the following functions associated with it, where t is the time from the present moment until death.

$\mathbf{b_t = 1 \text{ for } t \geq 0;}$

$\mathbf{v_t = v^t \text{ for } t \geq 0;}$

$\mathbf{Z = v^T \text{ for } t \geq 0.}$

The **actuarial present value** for life (x) of a whole life insurance policy for which the benefit is 1 is denoted as $\mathbf{E[Z] = \bar{A}_x = {}_0{}^{\infty}\!\int v^t * {}_tp_x * \mu_x(t)dt.}$

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1997. Second Edition. Society of Actuaries: Itasca, Illinois. pp.96-97.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L23-1. Maximus the Mortal takes out a whole life insurance policy for $100,000. He has a 0.3 probability of dying 15 years from now, a 0.3 probability of dying 43 years from now, a 0.2 probability of dying 65 years from now, and a 0.2 probability of dying 120 years from now. Maximus can foresee that the annual effective rate of interest will be 0.04 for the next 40 years and 0.01 for every year thereafter. Find the actuarial present value of Maximus's policy to him. Round your answer to the nearest cent.

Solution S3L23-1. For the first 40 years from now, the annual discount factor will be 1/1.04. For all subsequent years, the annual discount factor will be 1/1.01. Maximus's estate will collect

$100,000 at the time of his death irrespective of when he dies. Thus, the actuarial present value of Maximus's policy is

$100000*(0.3/1.04^{15} + 0.3/(1.04^{40}*1.01^{3}) + 0.2/(1.04^{40}*1.01^{25}) + 0.2/(1.04^{40}*1.01^{80})) =$
about $27850.44.

Problem S3L23-2. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is 0.07. François the Triceratops is currently 3 years old has a whole life insurance policy, which will pay him 1 Triceratops Currency Unit (TCU) upon death. Find the actuarial present value of this policy.

Solution S3L23-2. We use the formula $\bar{A}_x = {}_0^{\infty}\int v^t * {}_tp_x * \mu_x(t)dt$.

We know that x = 3 and δ = 0.07.

We find ${}_tp_3 = s(x + t)/s(x) = s(3 + t)/s(3) = e^{-0.34(3+t)}/e^{-0.34(3)} = e^{-0.34t}$

We find $\mu_3(t) = -s'(x)/s(x) = 0.34e^{-0.34t}/e^{-0.34t} = 0.34$

We find $v^t = e^{-0.07t}$

Thus, $\bar{A}_3 = {}_0^{\infty}\int e^{-0.07t} * e^{-0.34t} * 0.34dt$.

$\bar{A}_3 = {}_0^{\infty}\int 0.34e^{-0.41t}dt = (-34/41)e^{-0.41t} \Big|_0^{\infty} =$ **34/41 = about 0.8292682927 TCU**.

Problem S3L23-3. Lysander the Spiky Tarantula is 4 years old and has a whole life insurance policy, which will pay 5 Golden Hexagons (GH) upon death. The probability density function for the future lifetime of 4-year-old spiky tarantulas is $f_T(t) = 0.4462871026*0.64^t$. The annual effective interest rate is 0.04. Find the actuarial present value of Lysander's policy.

Solution S3L23-3. We use the formula $\bar{A}_x = {}_0^{\infty}\int v^t * {}_tp_x * \mu_x(t)dt = {}_0^{\infty}\int v^t * f_T(t)dt$.

We want to find $5\bar{A}_4$. We know that $f_T(t) = 0.4462871026*0.64^t$ and $v^t = (1/1.04)^t$.

Thus,

$5\bar{A}_4 = {}_0^{\infty}\int (5/1.04^t)*0.4462871026*0.64^t dt$

$5\bar{A}_4 = {}_0^{\infty}\int 2.231435513*0.6153846154^t dt$

$5\bar{A}_4 = [2.231435513/\ln(0.6153846154)]0.6153846154^t \Big|_0^{\infty}$

$5\bar{A}_4 = -[2.231435513/\ln(0.6153846154)]$

$5\bar{A}_4$ =about 4.596085667 GH.

Problem S3L23-4. The life of a giant pin-striped cockroach has the following survival function associated with it: s(x) = 1 - x/94, for $0 \le x \le 94$ and 0 otherwise. Hcaorkcoc the Giant Pin-Striped Cockroach is currently 56 years old and has a whole life insurance policy which will pay 10 Golden Hexagons (GH) upon death. The annual force of interest is 0.02. Find the actuarial present value of Hcaorkcoc's policy.

Solution S3L23-4. We use the formula $\bar{A}_x = {}_0{}^{\infty}\int v^t * {}_tp_x * \mu_x(t)dt$.

We want to find $10\bar{A}_{56}$. Since no giant pin-striped cockroach lives past the age of 94, our integral's upper bound will be 94-56 = 38, because Hcaorkcoc will not live for more than 38 additional years.

We find ${}_tp_{56}$ = s(x + t)/s(x) = s(56 + t)/s(56) = (1 - (56+t)/94)/(1 - 56/94) = (38 - t)/38

We find $\mu_{56}(t)$ = -s'(x)/s(x) = (1/94)/(1 - x/94) = (-1/94)/[(94 - x)/94] = 1/(94 - x) =

1/(94 - (56+t)) = 1/(38 - t). Conveniently enough, ${}_tp_{56} * \mu_{56}(t)$ = ((38 - t)/38)(1/(38 - t)) = 1/38.
We find $v^t = e^{-0.02t}$.

Thus, $10\bar{A}_{56} = 10 * {}_0{}^{38}\int e^{-0.02t} * (1/38)dt$

$10\bar{A}_{56} = 10*(-50/38)e^{-0.02t} \Big|_0^{38} = 10(50/38)(1 - e^{-0.76})$ = **$10\bar{A}_{56}$ = about 7.004389118 GH**.

Problem S3L23-5. For burgundy crickets, the survival function is $s(x) = (1 - 0.0625x^2)$ for $0 \le x \le 4$ and 0 otherwise. Among burgundy crickets, interest is determined in an unusual manner, and the discount factor v^t is equal to (1/3)/(1/3 + (1/6)t). Hatshepsut the Burgundy Cricket is 2 years old and has a whole life insurance policy paying 1 Golden Hexagon (GH) upon death. Find the actuarial present value of this policy.

Solution S3L23-5.
We use the formula $\bar{A}_x = {}_0{}^{\infty}\int v^t * {}_tp_x * \mu_x(t)dt$.
We want to find $\bar{A}_2$. Since no giant pin-striped cockroach lives past the age of 4, our integral's upper bound will be 4-2 = 2, because Hatshepsut will not live for more than 2 additional years.
${}_tp_2 = s(x + t)/s(x) = s(2 + t)/s(2) = (1 - 0.0625(2+t)^2)/(1 - 0.0625(2)^2)$ =
$(1 - 0.0625(2+t)^2)/(0.75) = (0.75 - 0.25t - 0.0625t^2)/(0.75) = {}_tp_2 = 1 - (1/3)t - (1/12)t^2$.
$\mu_2(t) = -s'(x)/s(x) = 0.125x/(1 - 0.0625x^2) = 0.125*(2+t)/(1 - 0.0625*(2+t)^2)$ =
$(0.25 + 0.125t)/(0.75 - 0.25t - 0.0625t^2) = \mu_2(t) = (1/3 + (1/6)t)/(1 - (1/3)t - (1/12)t^2)$.
It is given that v^t =(1/3)/(1/3 + (1/6)t).
Thus, $\bar{A}_2 = {}_0{}^2\int v^t * {}_tp_2 * \mu_2(t)dt$ =
${}_0{}^2\int[(1/3)/(1/3 + (1/6)t)](1 - (1/3)t - (1/12)t^2)((1/3 + (1/6)t)/(1 - (1/3)t - (1/12)t^2))dt$ =
${}_0{}^2\int(1/3)dt = \bar{A}_2$ = 2/3 = about 0.6666666666

Section 24

Makeham's Law of Mortality

Makeham's law of mortality corresponds to the following survival distribution.

$s(x) = \exp[-Ax - m(c^x-1)]$

The force of mortality under Makeham's Law is as follows.

$\mu_x = A + Bc^x$

Under Makeham's Law, A, B, and c are given constants, and m is defined as **$m = B/\ln(c)$**

The restrictions for using Makeham's Law are as follows:

$B > 0$, $A \geq -B$, $c > 1$, $x \geq 0$.

A special case of Makeham's Law is **Gompertz's Law,** where A = 0.

Thus, **under Gompertz's law, $s(x) = \exp[-m(c^x-1)]$ and $\mu_x = Bc^x$.**

If c = 1 under Gompertz's and Makeham's laws, we get a simple exponential distribution.

A under Makeham's law could be interpreted as the **accident hazard,** while Bc^x could be interpreted as the **hazard of aging.**

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1997. Second Edition. Society of Actuaries: Itasca, Illinois. p. 78.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L24-1. The lives of three-horned rhinoceroses follow Makeham's law with an accident hazard of 0.31, B = 0.43, and c = 2. Find μ_1 for three-horned rhinoceroses.

Solution S3L24-1. We use the formula $\mu_x = A + Bc^x$.

We are given that x = 1, A = 0.31, B = 0.43, and c = 2. Thus, **μ_1 =** $0.31 + 0.43*2^1$ = **μ_1 = 1.17.**

Problem S3L24-2. The lives of three-horned rhinoceroses follow Makeham's law with an accident hazard of 0.31, B = 0.43, and c = 2. Find s(3) for three-horned rhinoceroses.

Solution S3L24-2. We use the formula $s(x) = \exp[-Ax - m(c^x-1)]$.

We are given that x = 3, A = 0.31, B = 0.43, and c = 2. Thus, m = B/ln(c) = 0.43/ln(2) = 0.6203588676.

Hence, s(3) = exp[-0.31*3 - 0.6203588676(2^3-1)] = **s(3) = about 0.005130705663.**

Problem S3L24-3. The lives of orange orangutans follow Gompertz's law with s(40) = 0.65 and c = 1.03. Find μ_{40} for orange orangutans.

Solution S3L24-3. We first find B using the formula

s(x) = exp[-m(c^x-1)] = exp[-B(c^x-1)/ln(c)]

We are given that s(40) = 0.65, x = 40, and c = 1.03. Thus,

0.65 = exp[-B(1.03^{40}-1)/ln(1.03)]

0.65 = exp[-76.52670678B]

ln(0.65) = -76.52670678B

B = ln(0.65)/-76.52670678

B = about 0.005629184.

μ_{40} = Bc^x = 0.005629184*1.03^{40} = **about 0.0183626111.**

Problem S3L24-4. The lives of pterodactyls follow Makeham's law with accident hazard of 0.23 and hazard of aging of 0.02 for 20-year-old pterodactyls and 0.06 for 60-year-old pterodactyls. Find s(20) for pterodactyls.

Solution S3L24-4. We are given that A = 0.23, Bc^{20} = 0.02, and Bc^{60} = 0.06

Thus, c^{40} = (Bc^{60})/(Bc^{20})= 0.06/0.02 = 3. Thus, c = $3^{1/40}$ = 1.027845956.

Hence, B = 0.02c^{-20} = 0.02*1.027845956^{-20} = B = 0.0115470054.

We now use the formula

s(x) = exp[-Ax - m(c^x-1)] = exp[-Ax - B(c^x-1)/ln(c)]

s(20) = exp[-0.23*20 - 0.0115470054(1.027845956^{20}-1)/ln(1.027845956)]

s(20) = exp[-4.907769891]

s(20) = about 0.0073889481.

Problem S3L24-5. The lives of jabberwocks follow Makeham's law with B = 0.1, c = 1.0003, and s(35) = 0.02. Find μ_5 for jabberwocks.

Solution S3L24-5. We use the formula $\mu_x = A + Bc^x$ and note that A is at present an unknown quantity. We try to find A by using the formula

$s(x) = \exp[-Ax - m(c^x-1)] = \exp[-Ax - B(c^x-1)/\ln(c)]$

We are given s(35), so we work with the conditions for x = 35:

$0.02 = \exp[-35A - 0.1(1.0003^{35}-1)/\ln(1.0003)]$

$0.02 = \exp[-35A - 3.518436707]$

$\ln(0.02) = -35A - 3.518436707$

$-35A = \ln(0.02) + 3.518436707$

$A = [\ln(0.02) + 3.518436707]/-35$

A = 0.0112453228.

So $\mu_5 = A + Bc^5 = 0.0112453228 + 0.1*1.0003^5 =$ **μ_5 = about 0.1113954128.**

Section 25

Weibull's Law of Mortality

Weibull's law of mortality corresponds to the following survival distribution.

$\mathbf{s(x) = exp[-ux^{n+1}]}$

The force of mortality under Weibull's law is as follows.

$\boldsymbol{\mu_x = kx^n}$

Under Weibull's law, k and n are given constants, and u is defined as **u = k/(n + 1).**

The restrictions for using Weibull's law are as follows:

$\mathbf{k > 0, n > 0, x \geq 0.}$

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1997. Second Edition. Society of Actuaries: Itasca, Illinois. p. 78.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L25-1. The lives of rabbit-eared basilisks follow Weibull's law with k = 0.2 and n = 0.4. Find μ_{18} for rabbit-eared basilisks.

Solution S3L25-1. We use the formula $\mu_x = kx^n$. Thus, $\mu_{18} = 0.2*18^{0.4}$ = $\boldsymbol{\mu_{18}}$ **= 0.6355343046**.

Problem S3L25-2. The lives of rabbit-eared basilisks follow Weibull's law with k = 0.2 and n = 0.4. Find s(22) for rabbit-eared basilisks. The answer will be very small!

Solution S3L25-2. We use the formula $s(x) = exp[-ux^{n+1}]$. We find u = k/(n+1) = 0.2/1.4 = 1/7.

Thus, $s(22) = exp[-(1/7)22^{1.4}]$ = **s(22) = about 0.00001996248653**.

Problem S3L25-3. The lives of wild rainbow-spotted anchovies follow Weibull's law with k = 0.11. You know that $\mu_3 = 0.25$ for wild rainbow-spotted anchovies. Find s(10) for wild rainbow-spotted anchovies.

Solution S3L25-3. First, we need to find n by using the formula $\mu_x = kx^n$ with x = 3.

Hence, $\mu_3 = 0.11*3^n$. Thus, $0.25 = 0.11*3^n$ and $3^n = 25/11$, so n*ln(3) = ln(25/11) and n = ln(25/11)/ln(3) = n = 0.7472887028.

Now we find u = k/(n+1) = 0.11/1.7472887028 = u = 0.0629546793.

Now we use the formula $s(x) = \exp[-ux^{n+1}]$ for x = 10.

$s(10) = \exp[-0.0629546793*10^{1.7472887028}]$ = **s(10) = about 0.03072459**

Problem S3L25-4. The lives of four-legged fish follow Weibull's law with n = 1 and s(20) = 0.14. Find k for four-legged fish.

Solution S3L25-4. We use the formula $s(x) = \exp[-ux^{n+1}]$ to recognize that

$s(20) = 0.14 = \exp[-u*20^2]$

Thus, $\ln(0.14) = -u*20^2$

Thus, u = ln(0.14)/-400 = 0.0049152821

But we know that u = k/(n+1), so k = u*(n+1) = 0.0049152821*2 = **k = about 0.0098305643**.

Problem S3L25-5. The lives of rabbit-eared basilisks follow Weibull's law with k = 0.2 and n = 0.4. Find ${}_3p_{34}$ for rabbit-eared basilisks.

Solution S3L25-5. We know that ${}_3p_{34} = s(37)/s(34)$ and that $s(x) = \exp[-ux^{n+1}]$. We find u = k/(n+1) = 0.2/1.4 = u = 1/7. Thus, ${}_3p_{34} = \exp[-(1/7)37^{1.4}]/\exp[-(1/7)34^{1.4}] = \exp[-(1/7)37^{1.4} +(1/7)34^{1.4}]$ =

exp[-2.501563687] = **${}_3p_{34}$ = 0.0819567436**.

Section 26

Probability Density Functions and Cumulative Distribution Functions of Present-Value Random Variables

Z is the random variable representing the present value of a life insurance policy with a benefit of one unit. How might the probability density function (p.d.f.) of Z and the cumulative distribution function (c.d.f.) of Z be expressed?

We assume we know the p.d.f. and c.d.f. of T, the future-lifetime random variable for life (x). These will be denoted as $f_T(t)$ and $F_T(t)$.

Then

$F_Z(z) = 0$ for $z \leq 0$;

$F_Z(z) = 1 - F_T(\ln(z)/\ln(v))$ for $0 < z < 1$;

$F_Z(z) = 1$ for $1 \leq z$.

Moreover,

$f_Z(z) = f_T(\ln(z)/\ln(v))(1/(\delta z))$ for $0 < z < 1$ and 0 elsewhere.

Meaning of variables:

δ = annual force of interest = ln(1+r), where r is the annual effective interest rate.

v = the discount factor for one year ((1/(1+r)), where r is the annual effective interest rate.

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1997. Second Edition. Society of Actuaries: Itasca, Illinois. pp. 97-98.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L26-1. The future lifetime of 2-year-old centaurs can be modeled by the following

p. d. f.: $f_T(t) = 0.05e^{-0.05t}$. The annual force of interest is 0.07. Find $f_Z(z)$ for a whole life insurance policy entered into by 2-year-old centaurs that pays a one-unit benefit upon death.

Solution S3L26-1. We use the formula $f_Z(z) = f_T(\ln(z)/\ln(v))(1/(\delta z))$ for $0 < z < 1$. Here, $\delta = 0.07$ and $v = e^{-0.07}$. So $\ln(v) = -0.07$. Thus, $f_Z(z) = 0.05e^{-0.05(\ln(z)/\ln(v))}(1/(0.07z))$

$f_Z(z) = (5/7z)e^{(5/7)\ln(z)}$

$f_Z(z) = (5/7z)e^{\ln(z\wedge(5/7))}$

$f_Z(z) = (5/7z)z^{5/7}$

$f_Z(z) = (5/7)z^{-2/7}$ for $0 < z < 1$ and 0 elsewhere.

Problem S3L26-2. The future lifetime of 2-year-old centaurs can be modeled by the following

p. d. f.: $f_T(t) = 0.05e^{-0.05t}$. The annual force of interest is 0.07. Find $F_Z(z)$ for a whole life insurance policy entered into by 2-year-old centaurs that pays a one-unit benefit upon death.

Solution S3L26-2. We use the formula $F_Z(z) = 1 - F_T(\ln(z)/\ln(v))$ for $0 < z < 1$.

$f_T(t) = 0.05e^{-0.05t}$ implies that $s_T(t) = e^{-0.05t}$ and $F_T(t) = 1 - s_T(t) = 1 - e^{-0.05t}$. (The future lifetime of centaurs follows an exponential distribution.)

Then $F_Z(z) = 1 - (1 - e^{-0.05(\ln(z)/\ln(v))})$

$F_Z(z) = e^{-0.05(\ln(z)/\ln(v))}$

We know from Solution S3L26-1 that $\ln(v) = -0.07$, so

$F_Z(z) = e^{-0.05(\ln(z)/-0.07)}$

$F_Z(z) = e^{(5/7)\ln(z)}$

$F_Z(z) = e^{\ln(z\wedge(5/7))}$ for $0 < z < 1$.

Then

$F_Z(z) = 0$ for $z \leq 0$;

$F_Z(z) = z^{5/7}$ for $0 < z < 1$;

$F_Z(z) = 1$ for $1 \leq z$.

Problem S3L26-3. The future lifetime of 11-year-old microdragons can be modeled by the following p. d. f.: $f_T(t) = 1/66$ for $0 \leq t \leq 66$ and 0 otherwise. The annual force of interest is 0.1. Find $f_Z(z)$ for a whole life insurance policy entered into by 11-year-old microdragons that pays a one-unit benefit upon death.

Solution S3L26-3. We use the formula $f_Z(z) = f_T(\ln(z)/\ln(v))(1/(\delta z))$ for $0 < z < 1$. Here, $\delta = 0.1$

and $f_T(\ln(z)/\ln(v)) = 1/66$, since the value of $f_T(\square)$ does not depend on the value of $\square$.

Thus, $f_Z(z) = (1/66)(1/0.1z) = f_Z(z) = 10/(66z)$, but for which bounds?

We know that $\Pr\{T > 66\} = 0$, so $\Pr\{0 < Z < v^{66}\} = 0$ (There is no way that the present value of one unit of benefit will be greater than v^{66}.)

So our bounds are $v^{66} < z < 1$. We find $v^{66} = e^{-0.1*66}$ = about 0.001360368

Thus, **$f_Z(z) = 10/(66z)$ for $0.001360368 < z < 1$ and 0 elsewhere.**

Problem S3L26-4. The future lifetime of 11-year-old microdragons can be modeled by the following p. d. f.: $f_T(t) = 1/66$ for $0 \le t \le 66$ and 0 otherwise. The annual force of interest is 0.1. Find $F_Z(z)$ for a whole life insurance policy entered into by 11-year-old microdragons that pays a one-unit benefit upon death.

Solution S3L26-4. We know from Solution S3L26-3 that $f_Z(z) = 10/(66z)$ for $0.001360368 < z < 1$ and 0 elsewhere.We also know that $F_Z(z) = \int f_Z(z)dz = (10/66)\ln(z)$ for $0.001360368 < z < 1$.

Then

$F_Z(z) = 0$ for $z \le 0.001360368$;

$F_Z(z) = (10/66)\ln(z)$ for $0.001360368 < z < 1$;

$F_Z(z) = 1$ for $1 \le z$.

Problem S3L26-5. The cumulative distribution function for the future lifetime of 4-year-old wolf-rabbits is $F_T(t) =1 - \exp(-0.2x^3)$. The annual effective interest rate is 0.04. Find $F_Z(z)$ for a life insurance policy for 4-year-old wolf-rabbits that pays a one-unit benefit upon death.

Solution S3L26-5. We use the formula $F_Z(z) = 1 - F_T(\ln(z)/\ln(v))$ for $0 < z < 1$.
$1 - F_T(\ln(z)/\ln(v)) = 1 - [1 - \exp(-0.2(\ln(z)/\ln(v))^3)]$
Since r = 0.04, it follows that v = 1/1.04 = 0.9615384615 and thus ln(v) = -0.0392207132.
Thus, $F_Z(z) = \exp(-0.2(\ln(z)/-0.0392207132)^3)$
$F_Z(z) = \exp(3315(\ln(z))^3)$ for $0 < z < 1$.
Then
$F_Z(z) = 0$ for $z \le 0$;
$F_Z(z) = \exp(3315(\ln(z))^3)$for $0 < z < 1$;
$F_Z(z) = 1$ for $1 \le z$.

Section 27

Pure Endowments

As in Section 21, the following is defined to be the **present-value function.**

$\mathbf{z_t = Z = b_t v_t}$

$\mathbf{z_t = Z}$ is the present value, at policy issue, of the benefit payment.

$\mathbf{b_t}$ is the **benefit function.**

$\mathbf{v_t}$ is the **discount function. v** is the one-year discount factor by which a sum of money payable one year from now is multiplied to get its present value today. If the annual effective interest rate is r, then v = 1/(1+r).

An **n-year pure endowment** makes a payment at the conclusion of n years if and only if the insured person is alive n years after the policy has been issued. An endowment that pays one unit in benefits has the following functions associated with it:

$\mathbf{b_t = 0 \text{ if } t \leq n;}$

$\mathbf{b_t = 1 \text{ if } t > n.}$

$\mathbf{v_t = v^n \text{ for } t \geq 0;}$

$\mathbf{Z = 0 \text{ if } T \leq n;}$

$\mathbf{Z = v^n \text{ if } T > n.}$

The actuarial present value of an n-year pure endowment entered into by life (x) and paying a unit benefit is denoted as $\mathbf{A^1_{x:n\neg}}$ and has the following formulas associated with it.

$\mathbf{E[Z] = A^1_{x:n\neg} = v^n * {}_np_x}$

$\mathbf{Var(Z) = v^{2n} * {}_np_x * {}_nq_x}$

$\mathbf{Var(Z) = {}^2A^1_{x:n\neg} - (A^1_{x:n\neg})^2}$

Meaning of Variables:

$\mathbf{{}^2A^1_{x:n\neg} = E[Z^2]}$ = the actuarial present value of n-year pure endowment which pays a benefit of 1 upon the death of life (x). **Important:** The force of interest for ${}^2\bar{A}^1_{x:n\neg}$ is assumed to be $\boldsymbol{2\delta}$.

$_n\mathbf{p}_x$ = probability that life (x) will survive to age x+n.

$_n\mathbf{q}_x$ = probability that life (x) will not survive to age x+n.

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1997. Second Edition. Society of Actuaries: Itasca, Illinois. p. 101.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L27-1. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual effective interest rate in Triceratopsland is 0.07. Jugurtha the Triceratops is currently 8 years old has a 3-year pure endowment, which will pay him 1 Triceratops Currency Unit (TCU) if he survives to age 11. Find the actuarial present value of this policy.

Solution S3L27-1. We use the formula $A^1_{x:n\neg} = v^n * {}_np_x$. We know that x = 8, n = 3, and v = (1/1.07). Thus, $v^3 = (1/1.07)^3$.

We find ${}_3p_8 = s(11)/s(8) = e^{-0.34*3}$. Hence, $A^1_{8:3\neg} = (1/1.07)^3e^{-0.34*3}$ =

$\mathbf{A^1_{8:3\neg}}$ **= about 0.2943528841**.

Problem S3L27-2. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual effective interest rate in Triceratopsland is 0.07. Jugurtha the Triceratops is currently 8 years old has a 3-year pure endowment, which will pay him 1 Triceratops Currency Unit (TCU) if he survives to age 11. Find the variance of the present-value random variable for this policy.

Solution S3L27-2. We use the formula $Var(Z) = v^{2n} * {}_np_x * {}_nq_x$.

We know from Solution S3L27-1 that ${}_3p_8 = e^{-0.34*3}$. ${}_nq_x = 1 - {}_np_x$. Thus, ${}_3q_8 = 1 - e^{-0.34*3}$.

$v^{2n} = v^6 = (1/1.07)^6$.

Hence, $Var(Z) = (1/1.07)^6 * e^{-0.34*3} * (1 - e^{-0.34*3})$ = **Var(Z) = about 0.153636014**.

Problem S3L27-3. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual effective interest rate in Triceratopsland is 0.07. Jugurtha the Triceratops is currently 8 years old has a 3-year pure endowment, which will pay him 1 Triceratops Currency Unit (TCU) if he survives to age 11. Find the second moment of the present-value random variable for this policy.

Solution S3L27-3. We use the formula $Var(Z) = {}^2A^1_{x:n\neg} - (A^1_{x:n\neg})^2$, rearranging it thus:

${}^2A^1_{x:n\neg} = Var(Z) + (A^1_{x:n\neg})^2$. We want to find ${}^2A^1_{8:3\neg}$. From Solutions S3L27-1 and S3L27-2, we know that Var(Z) = 0.153636014 and $A^1_{8:3\neg} = 0.2943528841$. Thus,

${}^2A^1_{8:3\neg} = 0.153636014 + 0.2943528841^2 = $ **${}^2A^1_{8:3\neg}$ = about 0.2402796343**.

Problem S3L27-4. The life of a giant pin-striped cockroach has the following survival function associated with it: $s(x) = 1 - x/94$, for $0 \leq x \leq 94$ and 0 otherwise. Odoacer the Giant Pin-Striped Cockroach is currently 44 years old and has a 13-year endowment which will pay 1 Golden Hexagon (GH) if he reaches age 57. The annual force of interest is 0.02. Find the actuarial present value of Odoacer's policy.

Solution S3L27-4. We use the formula $A^1_{x:n\neg} = v^n{*}_np_x$ to find $A^1_{44:13\neg}$. We know that x = 44, n = 13, and

$v = e^{-0.02}$. Thus, $v^{13} = e^{-0.02*13} = e^{-0.26}$.

We find ${}_np_x = {}_{13}p_{44} = s(57)/s(44) = (37/94)/(50/94) = 37/50$.

Thus, $A^1_{44:13\neg} = e^{-0.26}(37/50) = $ **$A^1_{44:13\neg}$ = about 0.5705781735**.

Problem S3L27-5. The life of a giant pin-striped cockroach has the following survival function associated with it: $s(x) = 1 - x/94$, for $0 \leq x \leq 94$ and 0 otherwise. Odoacer the Giant Pin-Striped Cockroach is currently 44 years old and has a 13-year endowment which will pay 1 Golden Hexagon (GH) if he reaches age 57. The annual force of interest is 0.02. Find the variance of the present-value random variable for this policy.

Solution S3L27-5. We use the formula $Var(Z) = v^{2n}{*}_np_x{*}_nq_x$. We know that x = 44, n = 13, and

$v = e^{-0.02}$. Thus, $v^{2n} = v^{26} = e^{-0.02*26} = e^{-0.52}$.

We know from Solution S3L27-4 that ${}_{13}p_{44} = 37/50$.

We find ${}_{13}q_{44} = 1 - {}_{13}p_{44} = 13/50$.

Thus, $Var(Z) = e^{-0.52}(37/50)(13/50) = $ **Var(Z) = about 0.1143857534**.

Section 28

Endowment Insurance

As in Section 21, the following is defined to be the **present-value function.**

$\mathbf{z_t = Z = b_t v_t}$

$\mathbf{z_t = Z}$ is the present value, at policy issue, of the benefit payment.

$\mathbf{b_t}$ is the **benefit function.**

$\mathbf{v_t}$ is the **discount function. v** is the one-year discount factor by which a sum of money payable one year from now is multiplied to get its present value today. If the annual effective interest rate is r, then v = 1/(1+r).

A policy of **n-year endowment insurance** makes a payment either upon the beneficiary's death or upon the beneficiary's survival to the end of a term of n years. The earliest of these two times is the payment of death. For n-year endowment insurance that pays one unit in benefits, we have the following functions:

$\mathbf{b_t = 1 \text{ for } t \geq 0;}$

$\mathbf{v_t = v^t \text{ for } t \leq n;}$

$\mathbf{v_t = v^n \text{ for } t > n;}$

$\mathbf{Z = v^T \text{ if } T \leq n;}$

$\mathbf{Z = v^n \text{ if } T > n.}$

The actuarial present value of n-year endowment insurance is denoted by the symbol $\bar{A}_{x:n\neg}$.

n-year endowment insurance can be expressed as a sum of an n-year pure endowment and an n-year term life insurance policy. The actuarial present value of an n-year endowment insurance policy is the sum of the actuarial present values of the corresponding n-year pure endowment and n-year term life insurance policy. Thus,

$\mathbf{\bar{A}_{x:n\neg} = A^1{}_{x:n\neg} + \bar{A}^1{}_{x:n\neg}}$

We let Z_1 = the present-value random variable for an n-year term life insurance policy.

We let Z_2 = the present-value random variable for an n-year pure endowment.

We let Z_3 = the present-value random variable for an n-year endowment insurance policy.

Thus, $Z_3 = Z_1 + Z_2$ and $Var(Z_3) = Var(Z_1) + Var(Z_2) + 2Cov(Z_1, Z_2)$.

The covariance of $Cov(Z_1, Z_2)$ can be expressed as follows:

$$\mathbf{Cov(Z_1, Z_2) = -\bar{A}^1_{x:n\neg} * A^1_{x:n\neg}}$$

From Section 22, it is known that $Var(Z_1) = {}^2\bar{A}^1_{x:n\neg} - (\bar{A}^1_{x:n\neg})^2$.

From Section 27, it is known that $Var(Z_2) = {}^2A^1_{x:n\neg} - (A^1_{x:n\neg})^2$.

Thus, $Var(Z_3) = {}^2\bar{A}^1_{x:n\neg} - (\bar{A}^1_{x:n\neg})^2 + {}^2A^1_{x:n\neg} - (A^1_{x:n\neg})^2 - 2\bar{A}^1_{x:n\neg} * A^1_{x:n\neg}$

$$\mathbf{Var(Z_3) = {}^2\bar{A}^1_{x:n\neg} + {}^2A^1_{x:n\neg} - (\bar{A}^1_{x:n\neg} + A^1_{x:n\neg})^2}$$

Reminders:

For n-year term life insurance, $\mathbf{\bar{A}^1_{x:n\neg} = {}_0{}^n\!\int v^t * {}_tp_x * \mu_x(t)dt}$ and $\mathbf{E[Z^j] = {}_0{}^n\!\int e^{-\delta jt} * {}_tp_x * \mu_x(t)dt.}$

For n-year pure endowments, $\mathbf{A^1_{x:n\neg} = v^n * {}_np_x}$ and $\mathbf{Var(Z) = v^{2n} * {}_np_x * {}_nq_x}$.

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1997. Second Edition. Society of Actuaries: Itasca, Illinois. pp. 101-102.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L28-1. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is 0.09. Hasdrubal the Triceratops is currently 7 years old has a 2-year endowment insurance policy, which will pay him 1 Triceratops Currency Unit (TCU) upon death. Find the actuarial present value of this policy.

Solution S3L28-1. We use the formula $\bar{A}_{x:n\neg} = A^1_{x:n\neg} + \bar{A}^1_{x:n\neg}$.

We first need to find

$\bar{A}^1_{x:n\neg} = {}_0{}^n\!\int v^t * {}_tp_x * \mu_x(t)dt$ and

$A^1_{x:n\neg} = v^n * {}_np_x$

We know that $v = e^{-0.09}$, $x = 7$, and $n = 2$.

We find ${}_tp_x = s(x+t)/s(x) = s(7+t)/s(7) = e^{-0.34t}$. So ${}_np_x = {}_2p_7 = e^{-0.34*2}$.

We find $\mu_x(t) = -s'(x)/s(x) = 0.34e^{-0.34t}/e^{-0.34t} = 0.34$.

Thus, $\bar{A}^1{}_{7:2\neg} = {}_0{}^2\int e^{-0.09t}*e^{-0.34t}*0.34dt = {}_0{}^2\int 0.34e^{-0.43t}dt = (-34/43)e^{-0.43t} \mid {}_0{}^2 = (34/43)(1 - e^{-0.43*2}) =$

$\bar{A}^1{}_{7:2\neg} = 0.4561044$

Also, $A^1{}_{7:2\neg} = e^{-0.09*2}e^{-0.34*2} = e^{-0.86} = A^1{}_{7:2\neg} = 0.4231620823$.

Thus, $\bar{A}_{7:2\neg} = A^1{}_{7:2\neg} + \bar{A}^1{}_{7:2\neg} = 0.4231620823 + 0.4561044 =$ $\boldsymbol{\bar{A}_{7:2\neg}}$ **= about 0.8792664823**.

Problem S3L28-2. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is 0.09. Hasdrubal the Triceratops is currently 7 years old has a 2-year endowment insurance policy, which will pay him 1 Triceratops Currency Unit (TCU) upon death. If Z_1 is the present-value random variable for a 2-year term life insurance policy for Hasdrubal, and Z_2 is the present-value random variable for an n-year pure endowment for Hasdrubal, find the covariance of Z_1 and Z_2.

Solution S3L28-2. We use the formula $Cov(Z_1, Z_2) = -\bar{A}^1{}_{x:n\neg}*A^1{}_{x:n\neg}$.

We know from Solution S3L28-1 that $A^1{}_{7:2\neg} = 0.4231620823$ and $\bar{A}^1{}_{7:2\neg} = 0.4561044$.

Thus, $Cov(Z_1, Z_2) = -0.4561044*0.4231620823 =$ **about -0.1930060877**.

Problem S3L28-3. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is 0.09. Hasdrubal the Triceratops is currently 7 years old has a 2-year endowment insurance policy, which will pay him 1 Triceratops Currency Unit (TCU) upon death. Find the variance of the present-value random variable for this policy.

Solution S3L28-3. We use the formula $Var(Z_3) = Var(Z_1) + Var(Z_2) + 2Cov(Z_1, Z_2)$, with Z_3 being the present-value random variable for this policy and Z_1 and Z_2 as defined in Problem S3L28-2.

We know from Solution S3L28-2 that $Cov(Z_1, Z_2) = -0.1930060877$.

We can find $Var(Z_2)$ using the formula $Var(Z_2) = v^{2n}*{}_np_x*{}_nq_x$.

We know that $v = e^{-0.09}$, $x = 7$, and $n = 2$. We also know from Solution S3L28-1 that

${}_np_x = {}_2p_7 = e^{-0.34*2} = e^{-0.68}$. Thus, ${}_nq_x = 1 - {}_np_x = 1 - e^{-0.68}$.

Thus, $Var(Z_2) = e^{-0.09*4}*e^{-0.68}(1 - e^{-0.68}) = Var(Z_2) = 0.174388534$.

We need to find $Var(Z_1) = {}^2\bar{A}^1{}_{x:n\neg} - (\bar{A}^1{}_{x:n\neg})^2$.

We know from Solution S3L28-1 that $\bar{A}^1{}_{7:2\neg} = 0.4561044$.

We can find $E[Z_1{}^2] = {}^2\bar{A}^1{}_{7:2\neg}$ using the formula $E[Z^j] = {}_0{}^n\int e^{-\delta jt}*{}_tp_x*\mu_x(t)dt$ from Section 22.

We find ${}_tp_x = s(x+t)/s(x) = s(7+t)/s(7) = e^{-0.34t}$.

We find $\mu_x(t) = -s'(x)/s(x) = 0.34e^{-0.34t}/e^{-0.34t} = 0.34$.

Thus, $E[Z_1{}^2] = {}_0{}^2\int e^{-2*0.09t}*e^{-0.34t}*0.34dt = {}_0{}^2\int 0.34*e^{-0.52t}dt = (-34/52)e^{-0.52t} \Big|_0{}^2 = (34/52)(1 - e^{-0.52*2})$ $= {}^2\bar{A}^1{}_{7:2\neg} = 0.4227411695$.

Thus, $Var(Z_1) = 0.4227411695 - (0.4561044)^2 = Var(Z_1) = 0.2147099458$.

Now we can apply the formula $Var(Z_3) = Var(Z_1) + Var(Z_2) + 2Cov(Z_1 , Z_2)$.

$Var(Z_3) = 0.2147099458 + 0.174388534 + 2(-0.1930060877)$

$Var(Z_3)$ = about 0.0030863045. (That is an outstandingly predictable insurance policy!)

Problem S3L28-4. The life of a giant pin-striped cockroach has the following survival function associated with it: $s(x) = 1 - x/94$, for $0 \leq x \leq 94$ and 0 otherwise. Odoacer the Giant Pin-Striped Cockroach is currently 44 years old and has a 13-year endowment which will pay 1 Golden Hexagon (GH) if he reaches age 57. The annual force of interest is 0.02, and the actuarial present value of Odoacer's policy iscurrently 0.5705781735. Odoacer is not satisfied with this present value and seeks to turn his pure endowment into a 13-year endowment insurance policy. What will the present value of his new policy be?

Solution S3L28-4. We use the formula $\bar{A}_{x:n\neg} = A^1{}_{x:n\neg} + \bar{A}^1{}_{x:n\neg}$. We know that

$A^1{}_{44:13\neg} = 0.5705781735$.

We find $\bar{A}^1{}_{44:13\neg}$ using the formula $\bar{A}^1{}_{x:n\neg} = {}_0{}^n\int v^t*{}_tp_x*\mu_x(t)dt$.

We know that $x = 44$, $n = 13$, and $v = e^{-0.02}$.

We find ${}_tp_x = (1 - (44+t)/94)/(1-44/94) = (50 - t)/50$

We find $\mu_x(t) = -s'(x)/s(x) = (1/94)/(1 - x/94) = 1/(94 - x) = 1/(94 - (44+t)) = 1/(50 - t)$.

Interestingly enough, ${}_tp_x\mu_x(t) = ((50 - t)/50)/(50 - t) = (1/50)$.
Thus, $\bar{A}^1{}_{44:13\neg} = {}_0{}^{13}\int e^{-0.02t}(1/50)dt = (-50/50)e^{-0.02t} \Big|_0{}^{13} = (1 - e^{-0.02*13}) = \bar{A}^1{}_{44:13\neg} = 0.2289484142$.

Thus, $\bar{A}_{44:13\neg} = A^1{}_{44:13\neg} + \bar{A}^1{}_{44:13\neg} = 0.5705781735 + 0.2289484142 =$ **about 0.7995265877**.

Problem S3L28-5. The life of a giant pin-striped cockroach has the following survival function associated with it: $s(x) = 1 - x/94$, for $0 \leq x \leq 94$ and 0 otherwise. Odoacer the Giant Pin-Striped Cockroach is currently 44 years old and has a 13-year endowment which will pay 1 Golden Hexagon (GH) if he reaches age 57. The annual force of interest is 0.02, and the variance of Odoacer's policy is currently 0.1143857534. Odoacer is not satisfied with this variance and seeks to turn his pure endowment into a 13-year endowment insurance policy. What will the variance of his new policy be?

Solution S3L28-5. We know that x = 44, n = 13, and $v = e^{-0.02}$. We use the formula

$Var(Z_3) = Var(Z_1) + Var(Z_2) + 2Cov(Z_1, Z_2)$, where Z_1 is the present value of a 13-year term life insurance policy and Z_2 is the present value of a 13-year pure endowment.

We are given $Var(Z_2) = 0.1143857534$.

We use the formula $Cov(Z_1, Z_2) = -\bar{A}^1{}_{x:n\neg} * A^1{}_{x:n\neg}$, knowing from Solution S3L28-4 that

$A^1{}_{44:13\neg} = 0.5705781735$ and $\bar{A}^1{}_{44:13\neg} = 0.2289484142$.

Thus, $Cov(Z_1, Z_2) = -0.2289484142*0.5705781735 = Cov(Z_1, Z_2) = -0.130632968$.

We need to find $Var(Z_1) = {}^2\bar{A}^1{}_{x:n\neg} - (\bar{A}^1{}_{x:n\neg})^2$.

We can find $E[Z_1^2] = {}^2\bar{A}^1{}_{7:2\neg}$ using the formula $E[Z^j] = {}_0{}^n\int e^{-\delta jt} * {}_tp_x * \mu_x(t)dt$ from Section 22.

We find ${}_tp_x = (1 - (44+t)/94)/(1-44/94) = (50 - t)/50$

We find $\mu_x(t) = -s'(x)/s(x) = (1/94)/(1 - x/94) = 1/(94 - x) = 1/(94 - (44+t)) = 1/(50 - t)$.

Interestingly enough, ${}_tp_x\mu_x(t) = ((50 - t)/50)/(50 - t) = (1/50)$.
Thus, $E[Z_1^2] = {}_0{}^{13}\int e^{-0.02*2*t}*(1/50)dt = {}_0{}^{13}\int(1/50)e^{-0.04t}dt = (-1/2)e^{-0.04t} \Big|_0^{13} = (1/2)(1 - e^{-0.52})$
$= {}^2\bar{A}^1{}_{44:13\neg} = 0.202739726$.

Thus, $Var(Z_1) = 0.202739726 - 0.2289484142^2 = Var(Z_1) = 0.1503223497$

Now we can apply the formula $Var(Z_3) = Var(Z_1) + Var(Z_2) + 2Cov(Z_1, Z_2)$.

$Var(Z_3) = 0.1503223497 + 0.1143857534 + 2*-0.130632968$.

$Var(Z_3)$ = about 0.0034421671.

Section 29

Deferred Life Insurance

As in Section 21, the following is defined to be the **present-value function.**

$z_t = Z = b_t v_t$

$z_t = Z$ is the present value, at policy issue, of the benefit payment.

b_t is the **benefit function.**

v_t is the **discount function.** **v** is the one-year discount factor by which a sum of money payable one year from now is multiplied to get its present value today. If the annual effective interest rate is r, then v = 1/(1+r).

An **m-year deferred insurance policy** pays a benefit to the insured only if the insured person dies at least m years from the time the policy was issued. The following functions are associated with an m-year deferred insurance policy that pays one unit in benefits.

$b_t = 1$ for $t > m$;

$b_t = 0$ for $t \leq m$;

$v_t = v^t$ for $t > 0$;

$Z = v^T$ if $T > m$;

$Z = 0$ if $T \leq m$.

The actuarial present value of an m-year deferred insurance policy that pays one unit in benefits is denoted by $_{m|}\bar{A}_x$ and can be found via the following formula:

$_{m|}\bar{A}_x = {}_m{}^{\infty}\int v^t * {}_tp_x * \mu_x(t)dt$

The **rule of moments** applies to deferred life insurance policies as well:

$E[Z^j] \ @ \ \delta_t = E[Z] \ @ \ j\delta_t$

$E[Z^j] = {}_m{}^{\infty}\int (v^t)^j * {}_tp_x * \mu_x(t)dt$

$E[Z^j] = {}_m{}^{\infty}\int e^{-\delta jt} * {}_tp_x * \mu_x(t)dt$

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1997. Second Edition. Society of Actuaries: Itasca, Illinois. p. 103.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L29-1. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is 0.09. Spotarecirt the Triceratops is currently 11 years old has a 5-year deferred insurance policy, which will pay him 1 Triceratops Currency Unit (TCU) if he dies more than 5 years from now. Find the actuarial present value of this policy.

Solution S3L29-1. We use the formula ${}_{m|}\bar{A}_x = {}_m\int^{\infty} v^t * {}_tp_x * \mu_x(t)dt$.

Here, $x = 11$, $v = e^{-0.09}$, and $m = 5$.

We find ${}_tp_x = s(11 + t)/s(11) = e^{-0.34t}$.

We find $\mu_x(t) = -s'(x)/s(x) = 0.34e^{-0.34t}/e^{-0.34t} = 0.34$.

Thus, ${}_{5|}\bar{A}_{11} = {}_5\int^{\infty} e^{-0.09t} * e^{-0.34t} * 0.34dt = {}_5\int^{\infty} 0.34e^{-0.43t}dt = (-34/43)e^{-0.43t} \Big|_5^{\infty} = (34/43)e^{-0.43*5} =$

${}_{5|}\bar{A}_{11}$ = about 0.0921037527 TCU.

Problem S3L29-2. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is 0.09. Spotarecirt the Triceratops is currently 11 years old has a 5-year deferred insurance policy, which will pay him 1 Triceratops Currency Unit (TCU) if he dies more than 5 years from now. Find the variance of this policy.

Solution S3L29-2. If Z is the present-value random variable for this policy, then

$Var(Z) = E(Z^2) - E(Z)^2$. We know from Solution S3L29-1 that $E(Z) = 0.0921037527$.

We find $E(Z^2)$ using the formula $E[Z^j] = {}_m\int^{\infty} e^{-\delta jt} * {}_tp_x * \mu_x(t)dt$.

Here, $x = 11$, $m = 5$, $j = 2$, ${}_tp_x = e^{-0.34t}$, $\mu_x(t) = 0.34$, and $\delta = 0.09$. Thus,

$E[Z^2] = {}_5\int^{\infty} e^{-0.18t} * e^{-0.34t} * 0.34dt = {}_5\int^{\infty} 0.34e^{-0.52t}dt = (-34/52)e^{-0.52t} \Big|_5^{\infty} = (34/52)e^{-0.52*5} =$

$E[Z^2] = 0.0485634934$. Thus, $Var(Z) = E(Z^2) - E(Z)^2 = 0.0485634934 - 0.0921037527^2 =$

Var(Z) =about 0.0400803922.

Problem S3L29-3. The life of a giant pin-striped cockroach has the following survival function associated with it: $s(x) = 1 - x/94$, for $0 \leq x \leq 94$ and 0 otherwise. The annual force of interest is 0.02. Dwarvar the Giant Pin-Striped Cockroach is currently 30 years old and has a 22-year deferred life insurance policy which will pay 1 Golden Hexagon (GH) upon death, if he dies more than 22 years from now. Find the actuarial present value of Dwarvar's policy.

Solution S3L29-3. We use the formula $_{m|}\bar{A}_x = {}_m{}^{\infty}\int v^t * {}_tp_x * \mu_x(t)dt$.

Here, x = 30, v = $e^{-0.02}$, and m = 22.

We note that the upper bound of our integral is *not* infinity, since the most Dwarvar can live is

94 - 30 = 64 more years. Thus, 64 is the upper bound of the integral.

We find $_tp_x$ = (1 - (30+t)/94)/(1-30/94) = (64 - t)/64

We find $\mu_x(t)$ = -s'(x)/s(x) = (1/94)/(1 - x/94) = 1/(94 - x) = 1/(94 - (30+t)) = 1/(64 - t).

Conveniently enough, $_tp_x\mu_x(t)$ = ((64 - t)/64)/(64 - t) = (1/64).

Thus, $_{22|}\bar{A}_{30} = {}_{22}{}^{64}\int(1/64)e^{-0.02t}dt = (-50/64)e^{-0.02t} \Big|_{22}^{64} = (50/64)(e^{-0.02*22} - e^{-0.02*64}) = {}_{22|}\bar{A}_{30}$ = **about 0.285936813 GH.**

Problem S3L29-4. The life of a giant pin-striped cockroach has the following survival function associated with it: s(x) = 1 - x/94, for 0 ≤ x ≤ 94 and 0 otherwise. The annual force of interest is 0.02. Dwarvar the Giant Pin-Striped Cockroach is currently 30 years old and has a 22-year deferred life insurance policy which will pay 1 Golden Hexagon (GH) upon death, if he dies more than 22 years from now. Find the variance of Dwarvar's policy.

Solution S3L29-4. If Z is the present-value random variable for this policy, then

Var(Z) = $E(Z^2) - E(Z)^2$. We know from Solution S3L29-3 that E(Z) = 0.285936813.

We find $E(Z^2)$ using the formula $E[Z^j] = {}_m{}^{\infty}\int e^{-\delta jt} * {}_tp_x * \mu_x(t)dt$.

Here, x = 30, m = 22, j = 2, $_tp_x\mu_x(t)$ = 1/64 (from Solution S3L29-3), and δ = 0.02. Thus,

$E[Z^2] = {}_{22}{}^{64}\int(1/64)e^{-0.04t}dt = (-25/64)e^{-0.04t} \Big|_{22}^{64} = (25/64)(e^{-0.04*22} - e^{-0.04*64}) = E[Z^2]$ = about 0.1318274106.

Var(Z) = $E(Z^2) - E(Z)^2$ = 0.1318274106 - 0.285936813^2 = **Var(Z) = about 0.0500675496**.

Problem S3L29-5. For burgundy crickets, the survival function is s(x) = (1 - $0.0625x^2$) for 0 ≤ x ≤ 4 and 0 otherwise. Among burgundy crickets, interest is determined in an unusual manner, and the discount factor v^t is equal to (1/3)/(1/3 + (1/6)t). Tekcirc the Burgundy Cricket is 0 years old and has a 1-year deferred life insurance policy paying 1 Golden Hexagon (GH) upon death. Find the actuarial present value of this policy.

Solution S3L29-5. We use the formula $_{m|}\bar{A}_x = {}_m{}^{\infty}\int v^t * {}_tp_x * \mu_x(t)dt$.

Here, x = 0, v = (1/3)/(1/3 + (1/6)t), and m = 1.

$_tp_0$ = s(t) = (1 - 0.0625t^2)

μ_0(t) = -s'(x)/s(x) = 0.125x/(1 - 0.0625x^2) = 0.125t/(1 - 0.0625t^2)

Thus, $_tp_0\mu_0$(t) = (1 - 0.0625t^2)*0.125t/(1 - 0.0625t^2) = 0.125t.

Thus, v^t*$_tp_0\mu_0$(t) = ((1/3)/(1/3 + (1/6)t))0.125t = 2*0.125t/(2 + t) = 0.25t/(2 + t).

The upper bound of our integral will be 4, since the largest age a newborn burgundy cricket can attain is 4.

Thus, $_{1|}\bar{A}_0$ = $_1{}^4\int$(0.25t/(2 + t))dt

We use the Tabular Method of Integration by Parts:

Sign.....u.......dv

+......0.25t.....1/(2+t)

- ......0.25......ln(2+t)

+........0........(2+t)ln(2+t) - (2+t)

Thus, $_1{}^4\int$(0.25t/(2 + t))dt = [0.25tln(2+t) - 0.25(2+t)ln(2+t) + 0.25(2+t)] $|_1{}^4$ =

[0.25*4*ln(6) - 0.25(6)ln(6) + 0.25(6)] - [0.25ln(3) - 0.25(3)ln(3) + 0.25(3)] = **about 0.403264097 GH**.

Section 30

Annually Increasing Whole Life Insurance

As in Section 21, the following is defined to be the **present-value function.**

$\mathbf{z_t = Z = b_t v_t}$

$\mathbf{z_t = Z}$ is the present value, at policy issue, of the benefit payment.

$\mathbf{b_t}$ is the **benefit function.**

$\mathbf{v_t}$ is the **discount function. v** is the one-year discount factor by which a sum of money payable one year from now is multiplied to get its present value today. If the annual effective interest rate is r, then v = 1/(1+r).

An **annually increasing whole life insurance** policy will pay 1 unit in benefits if death occurs during the first year, 2 units in benefits if death occurs during the second year, and n units in benefits if death occurs during the nth year. The following functions characterize annually increasing whole life insurance.

$\mathbf{b_t = \lfloor t + 1 \rfloor}$ **for** $\mathbf{t \geq 0}$**;**

$\mathbf{v_t = v^t}$ **for** $\mathbf{t \geq 0}$**;**

$\mathbf{Z = \lfloor T + 1 \rfloor v^T}$ **for** $\mathbf{T \geq 0}$**.**

The function $\lfloor x \rfloor$ is the **greatest integer function,** where $\lfloor x \rfloor$ is the greatest integer that is less than or equal to x.

The actuarial present value of annually increasing whole life insurance is denoted as $(I\bar{A})_x$ and can be found via the following formula:
$\mathbf{E[Z] = (I\bar{A})_x = {}_0^{\infty}\int \lfloor t + 1 \rfloor v^t * {}_tp_x * \mu_x(t)dt}$

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1997. Second Edition. Society of Actuaries: Itasca, Illinois. p. 105.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L30-1. The lives of magenta hippopotami follow the survival function $s(x) = e^{-0.09x}$. The present-value discount factor among magenta hippopotami is computed in an unusual manner such that $v^t = 1/\lfloor t + 1 \rfloor$ .Sumatopoppih the Magenta Hippopotamus is 3 years old and has an annually increasing whole life insurance policy that pays a benefit of n Golden Hexagons

(GH) if death occurs in the nth year of the policy. Find the actuarial present value of such a policy.

Solution S3L30-1. We use the formula $(I\bar{A})_x = {}_0{}^{\infty}\int \lfloor t + 1 \rfloor v^t * {}_tp_x * \mu_x(t)dt$. Here, x = 3.

We find ${}_tp_3 = s(3 + t)/s(3) = e^{-0.09t}$.

We find $\mu_3(t) = -s'(x)/s(x) = 0.09e^{-0.09t}/e^{-0.09t} = 0.09$.

Moreover, we note that $\lfloor t + 1 \rfloor v^t = \lfloor t + 1 \rfloor (1/\lfloor t + 1 \rfloor) = 1$.

Thus, $(I\bar{A})_3 = {}_0{}^{\infty}\int 0.09e^{-0.09t}dt = -e^{-0.09t} \big|_0^{\infty} = e^{-0.09*0} - e^{-\infty} = 1 - 0 =$ **$(I\bar{A})_3$ = 1 GH**.

Problem S3L30-2. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is 0.09. Spartacus the Triceratops is currently 2 years old has a 2-year annually increasing life insurance policy, which will pay 1 Triceratops Currency Unit (TCU) if he dies within 1 year from now and 2 Triceratops Currency Units (TCU) if he dies between 1 and 2 years from now. Find the actuarial present value of this policy.

Solution S3L30-2. We use the formula $(I\bar{A})_x = {}_0{}^{\infty}\int \lfloor t + 1 \rfloor v^t * {}_tp_x * \mu_x(t)dt$. Here, however, the upper bound of the integral is not infinity but 2, since the policy only applies to the next two years.

We are given that x = 2 and $v = e^{-0.09}$.

We find ${}_tp_x = s(2 + t)/s(2) = e^{-0.34t}$.

We find $\mu_x(t) = -s'(x)/s(x) = 0.34e^{-0.34t}/e^{-0.34t} = 0.34$.

Thus, $(I\bar{A})_{2:2} = {}_0{}^2\int \lfloor t + 1 \rfloor\ e^{-0.09t} * e^{-0.34t} * 0.34dt = {}_0{}^2\int 0.34 \lfloor t + 1 \rfloor e^{-0.43t}dt$

We note that from t = 0 to t = 1, $\lfloor t + 1 \rfloor = 1$ and from t = 1 to t = 2, $\lfloor t + 1 \rfloor = 2$.

Thus, we can decompose the integral above into two integrals that are more convenient to work with.

${}_0{}^2\int 0.34 \lfloor t + 1 \rfloor e^{-0.43t}dt = {}_0{}^1\int 0.34e^{-0.43t}dt + {}_1{}^2\int 0.68e^{-0.43t}dt = (-34/43)e^{-0.43t} \big|_0^1 + (-68/43)e^{-0.43t} \big|_1^2 =$

$(34/43)(1 - e^{-0.43}) + (68/43)(e^{-0.43} - e^{-0.86}) =$ **$(I\bar{A})_2$ = about 0.635867154 TCU**.

Problem S3L30-3. From his current vantage point, Morgan the Mortal has a probability of 0.05 of dying in every year starting now, where $0 \le t \le 20$. He takes out annually increasing whole life insurance policy that pays n Golden Hexagons (GH) if he dies in the nth year. The annual force of interest is 0. Find the actuarial present value of Morgan's policy.

Solution S3L30-3. For every nth year between n = 1 and n = 20, inclusive, Morgan's policy will pay

n if he dies in that year. The probability of Morgan dying in the nth year is 0.05. So Morgan's expected value with regard to each year's potential payment is 0.05*n, which is also the expected present value of that year's payments, since the force of interest is zero.

Thus, the actuarial present value of Morgan's policy is ${}_{n=0}{}^{n=20}\Sigma 0.05*n = 0.05(20)(20+1)/2 =$ **10.5 GH.**

Problem S3L30-4. After getting away with massive life insurance fraud once, Imhotep the Immortal decides to engage in it on a repeated basis. He takes out an annually increasing whole life insurance policy that pays n Golden Hexagons (GH) if he dies in the nth year. He then fakes his own death on an annual basis, collects the payments, and then misrepresents his identity as that of some living policyholder who had a life insurance policy identical to his own. Thus, Imhotep expects to be able to collect n GH in every nth year. However, insurance companies have become wary of Imhotep's machinations, so they thwart his attempts with a probability of 0.32. The annual force of interest is 0.2, and Imhotep only wants to engage in this scheme during every year for which the current present value of his expected revenues is increasing. What is the actuarial present value of Imhotep's entire scheme to him?

Solution S3L30-4. During the nth year, Imhotep has 0.68 probability of collecting a payment of n, which will have a present value of $e^{-0.2n}$ to him now. We want to find the value of n that maximizes $0.68ne^{-0.2n}$, the present value of Imhotep's revenues from the nth year. Once that value of n is exceeded, Imhotep will move on to a different life insurance fraud scheme.

To find the maximum value of $f(n) = 0.68ne^{-0.2n}$, we first find f'(n).

$f'(n) = 0.68e^{-0.2n} - 0.136ne^{-0.2n}$.

At the maximum value for f(n), f'(n) = 0.

Thus, $0.68e^{-0.2n} - 0.136ne^{-0.2n} = 0$

$0.68e^{-0.2n} = 0.136ne^{-0.2n}$

0.68 = 0.136n and n = 5.

Thus, Imhotep will only engage in this scheme for 5 years. The actuarial present value of this scheme to him then is $0.68e^{-0.2} + 0.68*2e^{-0.2*2} + 0.68*3e^{-0.2*3} + 0.68*4e^{-0.2*4} + 0.68*5e^{-0.2*5} =$ **5.060912795 GH**.

Problem S3L30-5. The life of a giant pin-striped cockroach has the following survival function associated with it: s(x) = 1 - x/94, for $0 \le x \le 94$ and 0 otherwise. The annual force of interest is 0.02. Tullius the Giant Pin-Striped Cockroach is 77 years old and has a 3-year annually increasing life insurance policy that will pay n Golden Hexagons (GH) if he dies in the nth year of the policy, up to the policy's expiration. Find the actuarial present value of Tullius's policy.

Solution S3L30-5. We use the formula $(I\bar{A})_x = {}_0{}^{\infty}\int \lfloor t + 1 \rfloor v^t * {}_tp_x * \mu_x(t)dt$. Here, however, the upper bound of the integral is not infinity but 3, since the policy only applies to the next 3 years.

Here, $x = 77$, $v = e^{-0.02}$.

We find ${}_tp_x = (1 - (77+t)/94)/(1-77/94) = (17- t)/17$

We find $\mu_x(t) = -s'(x)/s(x) = (1/94)/(1 - x/94) = 1/(94 - x) = 1/(94 - (77+t)) = 1/(17 - t)$.

Conveniently enough, ${}_tp_x\mu_x(t) = ((17 - t)/17)/(17 - t) = (1/17)$.

Thus, we have $(I\bar{A})_{77:3} = {}_0{}^3\int \lfloor t + 1 \rfloor e^{-0.02t}*(1/17)dt$

We note that from $t = 0$ to $t = 1$, $\lfloor t + 1 \rfloor = 1$, from $t = 1$ to $t = 2$, $\lfloor t + 1 \rfloor = 2$, and from $t = 2$ to $t = 3$,

$\lfloor t + 1 \rfloor = 3$.

Thus, we can decompose the integral above into three integrals that are more convenient to work with.

${}_0{}^3\int \lfloor t + 1 \rfloor e^{-0.02t}*(1/17)dt = {}_0{}^1\int e^{-0.02t}*(1/17)dt + {}_1{}^2\int e^{-0.02t}*(2/17)dt + {}_2{}^3\int e^{-0.02t}*(3/17)dt =$

$(-50/17)e^{-0.02t} \big|_0^1 + (-100/17)e^{-0.02t} \big|_1^2 + (-150/17)e^{-0.02t} \big|_2^3 =$

$(50/17)(1 - e^{-0.02}) + (100/17)(e^{-0.02} - e^{-0.04}) + (150/17)(e^{-0.04} - e^{-0.06}) =$ **about 0.3402779756 GH.**

Section 31

Continuously Increasing Whole Life Insurance

As in Section 21, the following is defined to be the **present-value function.**

$\mathbf{z_t = Z = b_t v_t}$

$\mathbf{z_t = Z}$ is the present value, at policy issue, of the benefit payment.

$\mathbf{b_t}$ is the **benefit function.**

$\mathbf{v_t}$ is the **discount function. v** is the one-year discount factor by which a sum of money payable one year from now is multiplied to get its present value today. If the annual effective interest rate is r, then v = 1/(1+r).

A **countinuously increasing whole life insurance** policy will pay n unit in benefits if death occurs at time n, irrespective of whether n is a whole number or not. The following functions characterize continuously increasing whole life insurance.

$\mathbf{b_t = t \text{ for } t \geq 0;}$

$\mathbf{v_t = v^t \text{ for } t \geq 0;}$

$\mathbf{Z = Tv^T \text{ for } T \geq 0.}$

The actuarial present value of a continuously increasing whole life insurance policy is denoted as $(\bar{I}\bar{A})_x$ and can be found using the following formula:

$$\mathbf{(\bar{I}\bar{A})_x = {}_0^{\infty}\!\int t * v^t * {}_t p_x * \mu_x(t)dt = {}_0^{\infty}\!\int {}_{s|}\bar{A}_x \, ds}$$

Meaning of variables:

$\mathbf{{}_t p_x}$ = probability that life (x) will survive for t more years.

$\mathbf{\mu_x(t)}$ = force of mortality that life (x) will experience at age (x + t).

$\mathbf{{}_{s|}\bar{A}_x}$= the actuarial present value of an s-year deferred life insurance policy that pays one unit in benefits.

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1997. Second Edition. Society of Actuaries: Itasca, Illinois. pp. 106-107.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L31-1. An s-year deferred life insurance policy on the life of a 3-year-old rabbitskinned rabbit has the following actuarial present value: $e^{-0.02s}$. Find the actuarial present value of a continuously increasing whole life insurance policy on the life of a 3-year-old rabbitskinned rabbit.

Solution S3L31-1. We use the formula $(\bar{I}\bar{A})_x = {}_0{}^{\infty}\int {}_{s|}\bar{A}_x\, ds$. We are given ${}_{s|}\bar{A}_3 = e^{-0.02s}$.

Thus, $(\bar{I}\bar{A})_3 = {}_0{}^{\infty}\int e^{-0.02s} ds = (-50)e^{-0.02s} \Big|_0^{\infty}$ = $\mathbf{(\bar{I}\bar{A})_3 = 50.}$

Problem S3L31-2. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is 0.09. Anaxagoras the Triceratops is currently 10 years old has a continuously increasing whole life insurance policy, which will pay n Triceratops Currency Unit (TCU) if he dies at time n after the policy's inception. Find the actuarial present value of this policy.

Solution S3L31-2. We use the formula $(\bar{I}\bar{A})_x = {}_0{}^{\infty}\int t*v^t*{}_tp_x*\mu_x(t)dt$.

We are given that x = 10 and $v = e^{-0.09}$.

We find ${}_tp_x = s(10 + t)/s(10) = e^{-0.34t}$.

We find $\mu_x(t) = -s'(x)/s(x) = 0.34e^{-0.34t}/e^{-0.34t} = 0.34$.

Thus, $(\bar{I}\bar{A})_{10} = {}_0{}^{\infty}\int t*v^t*{}_tp_x*\mu_x(t)dt = {}_0{}^{\infty}\int t* e^{-0.09t}* e^{-0.34t} *0.34dt = {}_0{}^{\infty}\int 0.34t*e^{-0.43t}dt$

We use the Tabular Method of Integration by Parts:

Sign.....u.......dv

+......0.34t....$e^{-0.43t}$

-........0.34...$(-100/43)e^{-0.43t}$

+......0.......$(10000/1849)e^{-0.43t}$

Thus, $(\bar{I}\bar{A})_{10} = [0.34t(-100/43)e^{-0.43t} - 0.34(10000/1849)e^{-0.43t}] \Big|_0^{\infty}$

= 0.34(10000/1849) = $\mathbf{(\bar{I}\bar{A})_{10}}$ **=about 1.8388318 TCU.**

Problem S3L31-3. From his current vantage point, Miltiades the Mortal has a probability of 0.04 of dying in every year starting now, where $0 \leq t \leq 25$. He takes out a continuously increasing whole life insurance policy that pays n Golden Hexagons (GH) if he dies at time n. The annual force of interest is 0.08. Find the actuarial present value of Miltiades's policy.

Solution S3L31-3. We use the formula $(\bar{I}\bar{A})_x = {}_0{}^{\infty}\int t*v^t*{}_tp_x*\mu_x(t)dt$, noting that ${}_tp_x*\mu_x(t) = f_T(t)$ = 0.04 for $0 \leq t \leq 25$. Also, $v^t = e^{-0.08t}$. Thus,

$(\bar{I}\bar{A})_x = {}_0{}^{25}\int 0.04te^{-0.08t}$ dt, since the longest Miltiades can survive is 25 years.

We use the Tabular Method of Integration by Parts:

Sign.....u.......dv

+.......0.04t.....$e^{-0.08t}$

-........0.04....$(-100/8)e^{-0.08t}$

+.........0......$(10000/64)e^{-0.08t}$

Thus, $(\bar{I}\bar{A})_x = [0.04t(-100/8)e^{-0.08t} - 0.04(10000/64)e^{-0.08t}] \big|_0^{25} = [(-t/2)e^{-0.08t} - 6.25e^{-0.08t}] \big|_0^{25}$

$= (-12.5)e^{-0.08*25} - 6.25e^{-0.08*25} + 6.25 = (-12.5)e^{-2} - 6.25e^{-2} + 6.25 = 6.25 - 18.75e^{-2} =$

6.25 - 18.75(0.135335283) = 6.25 - 2.53753655625 = **about 3.7124634475 GH.**

Problem S3L31-4. The life of a giant pin-striped cockroach has the following survival function associated with it: s(x) = 1 - x/94, for $0 \leq x \leq 94$ and 0 otherwise. The annual force of interest is 0.02. Depirtsnip the Giant Pin-Striped Cockroach is 60 years old and has a continuously increasing whole life insurance policy that will pay n Golden Hexagons (GH) if he dies in year n. Find the actuarial present value of Depirtsnip's policy.

Solution S3L31-4. We use the formula $(\bar{I}\bar{A})_x = {}_0{}^{\infty}\int t*v^t*{}_tp_x*\mu_x(t)dt$, noting that ${}_tp_x*\mu_x(t) = f_T(t)$.

Since Depirtsnip has not died for the first 60 years of life, and deaths for giant pin-striped cockroaches are uniformly distributed, Depirtsnip has 1/34 probability of dying in each of the next 34 years. So ${}_tp_x*\mu_x(t) = 1/34$. The upper bound of our integral is not infinity, but 34, since Depirtsnip can live at most 34 more years.

We also know that $v^t = e^{-0.02t}$. Thus, $(\bar{I}\bar{A})_{60} = {}_0{}^{34}\int(1/34)te^{-0.02t}$.

We use the Tabular Method of Integration by Parts:

Sign.....u.......dv

+........(1/34)t...$e^{-0.02t}$

-........(1/34)...$-50e^{-0.02t}$

+.........0.......$2500e^{-0.02t}$

Thus, $(\bar{I}\bar{A})_{60} = [(-50/34)te^{-0.02t} - (2500/34)e^{-0.02t}] \Big|_0^{34} =$

$(2500/34) - 50e^{-0.68} - (2500/34)e^{-0.68} =$

$73.592412 - 123.592412e^{-0.68} = 73.592412 - 123.592412*0.506616992 = 73.592412 - 62.582099$ = **about 11.010313 GH**.

Problem S3L31-5. For irate rats, you are given that $(\bar{I}\bar{A})_6 = 5$ and $(\bar{I}\bar{A})_8 = 2$. Moreover,

${}_{s|}\bar{A}_x = e^{-(k\_x)s}$ for all values of x and s, where k_x is some value dependent on the value of x.

Find ${}_{2|}\bar{A}_6 - {}_{2|}\bar{A}_8$ for irate rats.

Solution S3L31-5. We use the formula $(\bar{I}\bar{A})_x = {}_0{}^{\infty}\!\int {}_{s|}\bar{A}_x\, ds$.

We find k_6 for x = 6. $(\bar{I}\bar{A})_6 = 5 = {}_0{}^{\infty}\!\int e^{-(k\_6)s}ds$

$5 = (-1/k_6)e^{-(k\_6)s} \Big|_0^{\infty}$

$5 = (1/k_6)$, so $k_6 = 0.2$

Thus, ${}_{s|}\bar{A}_6 = e^{-0.2s}$ and ${}_{2|}\bar{A}_6 = e^{-0.4}$.

We find k_8 for x = 8. $(\bar{I}\bar{A})_8 = 2 = {}_0{}^{\infty}\!\int e^{-(k\_8)s}ds$

$2 = (-1/k_8)e^{-(k\_8)s} \Big|_0^{\infty}$

$2 = (1/k_8)$, so $k_8 = 0.5$.

Thus, ${}_{s|}\bar{A}_8 = e^{-0.5s}$ and ${}_{2|}\bar{A}_8 = e^{-1}$.

Hence, $\mathbf{{}_{2|}\bar{A}_6 - {}_{2|}\bar{A}_8 = e^{-0.4} - e^{-1} = 0.302440605.}$

Section 32

Annually Decreasing Term Life Insurance

As in Section 21, the following is defined to be the **present-value function.**

$\mathbf{z_t = Z = b_t v_t}$

$\mathbf{z_t = Z}$ is the present value, at policy issue, of the benefit payment.

$\mathbf{b_t}$is the **benefit function.**

$\mathbf{v_t}$is the **discount function. v** is the one-year discount factor by which a sum of money payable one year from now is multiplied to get its present value today. If the annual effective interest rate is r, then v = 1/(1+r).

An **annually decreasing term life insurance** policy will pay n unit in benefits if death occurs during year 1, n-1 units in benefits if death occurs during year 2, and n-k+1 in benefits if death occurs in year k. The following functions characterize annually decreasing term life insurance.

$\mathbf{b_t = n - \lfloor t \rfloor \text{ for } t \leq n;}$

$\mathbf{b_t = 0, t > n;}$

$\mathbf{v_t = v^t \text{ for } t \geq 0;}$

$\mathbf{Z = v^T(n - \lfloor T \rfloor) \text{ for } T \leq n;}$

$\mathbf{0 \text{ for } T > n.}$

The actuarial present value of an annually decreasing term life insurance is denoted as $\mathbf{(D\bar{A})^1_{x:n\neg}}$ and can be found via the following formula.

$\mathbf{(D\bar{A})^1_{x:n\neg} = {}_0{}^n\int(n - \lfloor t \rfloor)v^t * {}_tp_x * \mu_x(t)dt}$

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1997. Second Edition. Society of Actuaries: Itasca, Illinois. p. 108.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L32-1. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is 0.09. Cato the Triceratops is currently 5 years old has a 2-year annually decreasing life insurance policy, which will pay 2 Triceratops Currency Unit (TCU) if he dies within 1 year from now and 1 Triceratops Currency

Units (TCU) if he dies between 1 and 2 years from now. Find the actuarial present value of this policy.

Solution S3L32-1. We use the formula $(D\bar{A})^1_{x:n\neg} = {}_0^n\int(n - \lfloor t \rfloor)v^t * {}_tp_x * \mu_x(t)dt$.

We are given that x = 5, n = 2, and $v = e^{-0.09}$.

We find ${}_tp_x = s(5 + t)/s(5) = e^{-0.34t}$.

We find $\mu_x(t) = -s'(x)/s(x) = 0.34e^{-0.34t}/e^{-0.34t} = 0.34$.

We note that from t = 0 to t = 1, $\lfloor t \rfloor = 0 \rightarrow n - \lfloor t \rfloor = 2$ and from t = 1 to t = 2,

$\lfloor t \rfloor = 1 \rightarrow n - \lfloor t \rfloor = 1$.

Thus, $(D\bar{A})^1_{5:2\neg} = {}_0^2\int(2 - \lfloor t \rfloor)e^{-0.09t}*e^{-0.34t}*0.34dt$.

We can decompose this integral as follows:
$(D\bar{A})^1_{5:2\neg} = {}_0^1\int(2*0.34)e^{-0.43t}dt + {}_1^2\int 0.34e^{-0.43t}dt$

$(D\bar{A})^1_{5:2\neg} = (-68/43)e^{-0.43t} \Big|_0^1 + (-34/43)e^{-0.43t} \Big|_1^2$

$(D\bar{A})^1_{5:2\neg} = (68/43)(1 - e^{-0.43}) + (34/43)(e^{-0.43} - e^{-0.86})$

$\mathbf{(D\bar{A})^1_{5:2\neg}}$ **= about 0.7324460461**

Problem S3L32-2. The life of a giant pin-striped cockroach has the following survival function associated with it: s(x) = 1 - x/94, for $0 \leq x \leq 94$ and 0 otherwise. The annual force of interest is 0.02. Lothar the Giant Pin-Striped Cockroach is 91 years old and has a 3-year annually decreasing life insurance policy that will pay 3-k+1 Golden Hexagons (GH) if he dies in the kth year of the policy, up to the policy's expiration. Find the actuarial present value of Tullius's policy.

Solution S3L32-2. We use the formula $(D\bar{A})^1_{x:n\neg} = {}_0^n\int(n - \lfloor t \rfloor)v^t * {}_tp_x * \mu_x(t)dt$, noting that ${}_tp_x * \mu_x(t) = f_T(t)$. Since Lothar has not died for the first 91 years of life, and deaths for giant pin-striped cockroaches are uniformly distributed, Lothar has 1/3 probability of dying in each of the next 3 years. So ${}_tp_x * \mu_x(t) = 1/3$.

We also know that $v^t = e^{-0.02t}$. Thus,

$(D\bar{A})^1_{91:3\neg} = {}_0^3\int(3 - \lfloor t \rfloor)(1/3)e^{-0.02t} = {}_0^1\int(3)(1/3)e^{-0.02t} + {}_1^2\int(2)(1/3)e^{-0.02t} + {}_2^3\int(1/3)e^{-0.02t}$

$(D\bar{A})^1_{91:3\neg} = (-50)e^{-0.02t} \Big|_0^1 + (-100/3)e^{-0.02t} \Big|_1^2 + (-50/3)e^{-0.02t} \Big|_2^3$

$(D\bar{A})^1_{91:3\neg} = 50(1 - e^{-0.02}) + (100/3)(e^{-0.02} - e^{-0.04}) + (50/3)(e^{-0.04} - e^{-0.06})$

$(D\bar{A})^1_{91:\overline{3}|}$ =about 1.954122566

Problem S3L32-3. For burgundy crickets, the survival function is $s(x) = (1 - 0.0625x^2)$ for $0 \le x \le 4$ and 0 otherwise. Among burgundy crickets, interest is determined in an unusual manner, and the discount factor v^t is equal to $2/(2 + t)$. Lysistrata the Burgundy Cricket is 1year old and has a 3-year annually decreasing term life insurance policy whose benefit in the last year of its existence is 1 Golden Hexagon (GH). Find the actuarial present value of this policy.

Solution S3L32-3. We use the formula $(D\bar{A})^1_{x:\overline{n}|} = {}_0{}^n\int(n - \lfloor t \rfloor)v^t * {}_tp_x * \mu_x(t)dt$.

Here, x = 1 and n = 3.

${}_tp_1 = s(x + t)/s(x) = s(1 + t)/s(1) = (1 - 0.0625(1+t)^2)/(1 - 0.0625(1)^2) =$

$(1 - 0.0625(1+t)^2)/(0.9375)$.

$\mu_1(t) = -s'(x)/s(x) = 0.125x/(1 - 0.0625x^2) = 0.125*(1+t)/(1 - 0.0625*(1+t)^2)$

Thus, ${}_tp_1*\mu_1(t) = [(1 - 0.0625(1+t)^2)/(0.9375)][0.125*(1+t)/(1 - 0.0625*(1+t)^2)] =$

$0.125*(1+t)/(0.9375) = (2/15)(1+t)$

Thus, $v^t*{}_tp_1*\mu_1(t) = [2/(2 + t)]\ (2/15)(1+t) = (4/15)(1+t)/(2 + t) = (4/15)(1 - 1/(2 + t))$

So $(D\bar{A})^1_{1:\overline{3}|} = {}_0{}^3\int(3 - \lfloor t \rfloor)(4/15)(1 - 1/(2 + t))dt$.

$(D\bar{A})^1_{1:\overline{3}|} = {}_0{}^1\int(3)(4/15)(1 - 1/(2 + t))dt + {}_1{}^2\int(2)(4/15)(1 - 1/(2 + t))dt + {}_2{}^3\int(4/15)(1 - 1/(2 + t))dt$

$(D\bar{A})^1_{1:\overline{3}|} = [0.8t - 0.8\ln(2+t)]\ \Big|_0^1 + [(8/15)t - (8/15)\ln(2+t)]\ \Big|_1^2 + [(4/15)t - (4/15)\ln(2+t)]\ \Big|_2^3$

$(D\bar{A})^1_{1:\overline{3}|} = 0.8 - 0.8\ln(3) + 0.8\ln(2) + (16/15) - (8/15)\ln(4) - (8/15) + (8/15)\ln(3) + 0.8 - (4/15)\ln(5) - (8/15) + (4/15)\ln(4)$

$(D\bar{A})^1_{1:\overline{3}|}$ = about 1.062692528.

Problem S3L32-4. Milton the Mortal takes out a 55-year annually decreasing term life insurance policy that would pay him \$100,000 if he died 55 years from now. He has a 0.3 probability of dying 15 years from now, a 0.3 probability of dying 43 years from now, a 0.2 probability of dying 65 years from now, and a 0.2 probability of dying 120 years from now. Milton can foresee that the annual effective rate of interest will be 0.04 for the next 40 years and 0.01 for every year thereafter. Find the actuarial present value of Maximus's policy to him. Round your answer to the nearest cent.

Solution S3L32-4. The policy will only pay Milton if he dies 15 years from now or 43 years from now. If he dies 15 years from now, he will collect 100000(55-15+1) = \$4,100,000.

If he dies 43 years from now, he will collect 100000(55-43+1) = $1,300,000.

Thus, the actuarial present value of Milton's policy is

$0.3*4100000/1.04^{15} + 0.3*1300000/(1.04^{40}*1.01^{3})$ = **about $761,819.02**

Problem S3L32-5. Imhotep the Immortal is scamming life insurance companies again. This time, he takes out a 6-year decreasing term life insurance policy which pays $1,000,000 in the last year of its existence. He plans to fake his death at the end of each year, collect the life insurance payment, and then misrepresent his identity as that of another living policyholder with an identical life insurance policy to his own. Insurance companies have become more sophisticated at preventing Imhotep's attempts at fraud, and they especially focus on guarding against the largest losses. Thus, every year for which Imhotep expects to get $n million by fraud, his probability of getting caught is 0.15n. The annual force of interest is 0.1. Find the expected present value of the fraudulent life insurance scheme to Imhotep. Round your answer to the nearest cent.

Solution S3L32-5. Duringevery year k, where $1 \leq k \leq 6$, Imhotep tries to collect $(7 - k) million. His probability of success is (1 - 0.15(7 - k)), and the discount factor applied to the kth year's amount is $e^{-0.1k}$.

Thus, the expected present value of Imhotep's scheme (in millions of dollars) is

$6*0.1*e^{-0.1} + 5*0.25*e^{-0.2} + 4*0.4*e^{-0.3} + 3*0.55*e^{-0.4} + 2*0.7*e^{-0.5} + 1*0.85*e^{-0.6}$ = about **$5,173,285.94.**

Section 33

Insurances Payable at the End of the Year of Death

When insurance policies pay at the end of the year of death, based solely on the number of complete years liked by the insured person prior to death. The random variable used in describing such policies is not the future-lifetime variable T, but rather the curate-future-lifetime random variable K. If the insured person dies in year k+1 of the insurance policy, then the curate-future-lifetime of the insured person is k. The present value of the benefit payment for such an insurance policy is z_{k+1} and is expressed as follows.

$z_{k+1} = b_{k+1}v_{k+1.}$

$\mathbf{z_{k+1} = Z}$ is the present value, at policy issue, of the benefit payment.

$\mathbf{b_{k+1}}$is the **benefit function.**

$\mathbf{V_{k+1}}$is the **discount function. v** is the one-year discount factor by which a sum of money payable one year from now is multiplied to get its present value today. If the annual effective interest rate is r, then v = 1/(1+r).

Suppose we have an **n-year term life insurance policy which pays one unit in benefits at the end of the year of death.** The actuarial present value for this insurance policy is denoted as $\mathbf{A^1{}_{x:n\neg}}$ and can be found as follows:

$\mathbf{E[Z] = A^1{}_{x:n\neg} = {}_{k=0}{}^{n-1}\sum v^{k+1} * {}_kp_x * q_{x+k}}$

If δ_{k+1} is the annual force of interest at time k+1, and the **rule of moments** holds for moments of E[Z]: $\mathbf{E[Z^j] \ @ \ \delta_{k+1} = E[Z] \ @ \ j\delta_{k+1}}$.

Thus, $\mathbf{E[Z^2] = {}^2A^1{}_{x:n\neg} = {}_{k=0}{}^{n-1}\sum e^{-2\delta(k+1)} * {}_kp_x * q_{x+k}}$

$\mathbf{Var(Z) = {}^2A^1{}_{x:n\neg} - (A^1{}_{x:n\neg})^2}$

The following **recursion relation** enables us to compute $A^1{}_{x:n\neg}$when $A^1{}_{x+1:n-1\neg}$is known:

$\mathbf{A^1{}_{x:n\neg} = vq_x + vp_x * A^1{}_{x+1:n-1\neg}}$

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1997. Second Edition. Society of Actuaries: Itasca, Illinois. pp. 109-110.

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Problem S3L33-1. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is currently 0.06. Honorius the Triceratops is currently 2 years old and takes out a 3-year term life insurance policy paying a benefit of 1 at the end of the year of death. Find the actuarial present value of this policy.

Solution S3L33-1. We use the formula $A^1_{x:n\neg} = {}_{k=0}{}^{n-1}\sum v^{k+1} * {}_kp_x * q_{x+k}$.

We know that $\delta = 0.06$, so $v = e^{-0.06}$. We are given that $x = 2$ and $n = 3$.

We find ${}_kp_2 = s(x + k)/s(x) = s(2 + k)/s(2) = e^{-0.34(2+k)}/e^{-0.34(2)} = e^{-0.34k}$

We find $q_{2+k} = 1 - s(3 + k)/s(2 + k) = 1 - e^{-0.34}$.

Thus, $A^1_{2:3\neg} = {}_{k=0}{}^{2}\sum e^{-0.06(k+1)} * e^{-0.34k} * (1 - e^{-0.34}) =$

$e^{-0.06} * (1 - e^{-0.34}) + e^{-0.12} * e^{-0.34} * (1 - e^{-0.34}) + e^{-0.18} * e^{-0.68} * (1 - e^{-0.34}) =$

$(1 - e^{-0.34})(e^{-0.06} + e^{-0.46} + e^{-0.86}) =$ **$A^1_{2:3\neg}$ = about 0.5753670393**.

Problem S3L33-2. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is currently 0.06. Honorius the Triceratops is currently 2 years old and takes out a 3-year term life insurance policy paying a benefit of 1 at the end of the year of death. Find the second moment of the present-value random variable for this policy.

Solution S3L33-2. We seek to find $E[Z^2] = {}^2A^1_{x:n\neg} = {}_{k=0}{}^{n-1}\sum e^{-2\delta(k+1)} * {}_kp_x * q_{x+k}$ for $x = 2$ and $n = 3$. We know from Solution S3L33-1 that $v = e^{-0.06}$, ${}_kp_2 = e^{-0.34k}$, and $q_{2+k} = 1 - e^{-0.34}$.

Thus, ${}^2A^1_{2:3\neg} = {}_{k=0}{}^{2}\sum e^{-0.12(k+1)} * e^{-0.34k} * (1 - e^{-0.34}) =$

$e^{-0.12} * (1 - e^{-0.34}) + e^{-0.24} * e^{-0.34} * (1 - e^{-0.34}) + e^{-0.36} * e^{-0.68} * (1 - e^{-0.34}) =$

$(1 - e^{-0.34})(e^{-0.12} + e^{-0.58} + e^{-1.04}) =$ **${}^2A^1_{2:3\neg}$ = about 0.5188922456**.

Problem S3L33-3. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is currently 0.06. Honorius the Triceratops is currently 2 years old and takes out a 3-year term life insurance policy paying a benefit of 1 at the end of the year of death. Find the variance this policy.

Solution S3L33-3. We use the formula $Var(Z) = {}^2A^1_{x:n\neg} - (A^1_{x:n\neg})^2$.

From Solutions S3L33-1 and S3L33-2, we know that ${}^2A^1_{2:3\neg} = 0.5188922456$ and $A^1_{2:3\neg} = 0.5753670393$. Thus, $Var(Z) = 0.5188922456 - (0.5753670393)^2 =$ **Var(Z) = about 0.1878450157**.

Problem S3L33-4. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is currently 0.06. Rex the Triceratops is currently 3 years old and takes out a 2-year term life insurance policy paying a benefit of 1 at the end of the year of death. Find the actuarial present value of this policy. (Hint: This does not have to be done directly! A past solution in this section will help you.)

Solution S3L33-4. We use the recursion formula $A^1_{x:n\neg} = vq_x + vp_x*A^1_{x+1:n-1\neg}$.

We already know from Solution S3L33-1 that $A^1_{2:3\neg} = 0.5753670393$.

Let x = 2 and n = 3. Then our desired value, $A^1_{3:2\neg}$, is $A^1_{x+1:n-1\neg}$ in this case.

We know that $v = e^{-0.06}$. Moreover, because $s(x) = e^{-0.34x}$ is the survival function pertaining to an exponential distribution, it is the case that $p_x = e^{-0.34}$and $q_x = (1 - e^{-0.34})$.

We can rearrange our formula to solve directly for $A^1_{x+1:n-1\neg}$:

$(A^1_{x:n\neg} - vq_x)/(vp_x) = A^1_{x+1:n-1\neg}$

Thus, $(0.5753670393 - e^{-0.06}(1 - e^{-0.34}))/(e^{-0.4}) =$ **$A^1_{3:2\neg}$ = about 0.4533991689**.

Problem S3L33-5. Tibbar the Rabbit is deciding between two term life insurance policies, each of whose benefits is payable at the end of the year of death. Tibbar is currently 12 years old and knows that his probability of surviving to the end of the next year is 0.88. The policies he chooses between each pay one Golden Hexagon (GH) in benefits. Policy A is a 5-year policy that Tibbar would enter into immediately (at age 12). Policy A has an actuarial present value of 0.46. Policy B is a 4-year policy that Tibbar would enter into one year from now (at age 13). Policy B has an actuarial present value of 0.43. Find the annual force of interest.

Solution S3L33-5. We are given that $p_{12} = 0.88$, and so $q_{12} = 1 - p_{12} = 0.12$. Moreover, we are given that $A^1_{12:5\neg} = 0.46$ and $A^1_{13:4\neg} = 0.43$. We need to find v.

We use the recursion formula $A^1_{x:n\neg} = vq_x + vp_x*A^1_{x+1:n-1\neg}$ for x = 12 and n = 5.

We rearrange the formula thus:
$A^1_{x:n\neg} = v(q_x + p_x*A^1_{x+1:n-1\neg})$
$v = A^1_{x:n\neg}/(q_x + p_x*A^1_{x+1:n-1\neg})$
Thus, $v = 0.46/(0.12 + 0.88*0.43)$
$v = 0.922953451$
$e^{-\delta} = 0.922953451$
$\delta = -\ln(0.922953451) =$ **δ = about 0.080176478**.

Section 34

Whole Life Insurance Payable at the End of the Year of Death

The actuarial present value of a *whole* life insurance policy for life (x) with one unit in benefits payable at the end of the year of death is denoted as $\mathbf{A_x}$. The following formulas pertain to A_x.

$\mathbf{A_x = \sum_{k=0}^{\infty} v^{k+1} * {}_kp_x * q_{x+k}}$

$\mathbf{l_x * A_x = \sum_{k=0}^{\infty} v^{k+1} * d_{x+k}}$

Recursion formula: $\mathbf{A_x = vq_x + vp_xA_{x+1}}$

$\mathbf{l_x*(1 + i)A_x = l_xA_{x+1} + d_x(1 - A_{x+1})}$

$\mathbf{A_{x+1} - A_x = iA_x - q_x(1 - A_{x+1})}$

In the last of these formulas, $\mathbf{q_x(1 - A_{x+1})}$ is the **annual cost of insurance,** and i is the annual effective interest rate.

Meaning of variables:

$\mathbf{{}_kp_x}$ **=** probability that life (x) will survive for k more years.

$\mathbf{q_{x+k}}$ **=** probability that life (x+k) will die within 1 year.

v = 1/(1+i) = one-year present value discount factor.

$\mathbf{l_x}$ **=** number of survivors in a deterministic survivorship group to age x. $\mathbf{l_x = l_0 * {}_xp_0}$, where the original number of members in a deterministic survivorship group is $\mathbf{l_0}$.

$\mathbf{d_x}$ **=** number of deaths aged x in a deterministic survivorship group.

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1997. Second Edition. Society of Actuaries: Itasca, Illinois. pp. 111-115.

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Problem S3L34-1. The actuarial present value of a whole life insurance policy for three-year-old ordinary green gremlins that pays one Golden Hexagon (GH) in benefits at the end of the year of death is 0.44. The actuarial present value of a whole life insurance policy for four-year-old ordinary green gremlins that pays one Golden Hexagon (GH) in benefits upon death is 0.39. A three-year-old ordinary green gremlin has only a 0.77 probability of surviving to age 4. Find the annual effective interest rate.

Solution S3L34-1. We use the formula $A_{x+1} - A_x = iA_x - q_x(1 - A_{x+1})$, where x = 3. We rearrange the formula thus:

$iA_x = A_{x+1} - A_x + q_x(1 - A_{x+1})$

$i = x+1 - A_x + q_x(1 - A_{x+1})]/A_x$

We are given that $q_3 = 1 - 0.77 = 0.23$, $A_3 = 0.44$ and $A_4 = 0.39$.

Thus, i = [0.39- 0.44+ 0.23(1 - 0.39)]/0.44 = **i = about 0.2052272727**.

Problem S3L34-2. 10-year-old white elephants have a 0.46 probability of reaching age 11. The annual force of interest is 0.05. The actuarial present value of a whole life insurance policy for an 11-year-old white elephant that pays one Golden Hexagon (GH) in benefits at the end of the year of death is 0.77. Find the actuarial present value of a whole life insurance policy for a 10-year-old white elephant that pays one Golden Hexagon (GH) in benefits at the end of the year of death.

Solution S3L34-2. We use the formula $A_x = vq_x + vp_xA_{x+1}$, where x = 10. We are given A_{11} = 0.77 and $p_{10} = 0.46$, so $q_{10} = 1 - 0.46 = 0.54$. Moreover, $v = e^{-0.05}$. Thus,

$A_{10} = e^{-0.05}(0.54 + 0.46*0.77)$ = **A_{10} = about 0.8505893514 GH**.

Problem S3L34-3. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is currently 0.06. Jefferson the Triceratops is currently 19 years old and takes out a whole insurance policy paying a benefit of 1 Golden Hexagon (GH) at the end of the year of death. Find the actuarial present value of this policy.

Solution S3L34-3. We use the formula $A_x = {}_{k=0}{}^{\infty}\sum v^{k+1} * {}_kp_x * q_{x+k}$.

We know that δ = 0.06, so $v = e^{-0.06}$. We are given that x = 19.

We find ${}_kp_{19} = s(x + k)/s(x) = s(19 + k)/s(19) = e^{-0.34(19+k)}/e^{-0.34(19)} = e^{-0.34k}$

We find $q_{19+k} = 1 - s(20 + k)/s(19 + k) = 1 - e^{-0.34}$.

Thus, $A_{19} = {}_{k=0}{}^{\infty}\sum e^{-0.06(k+1)} * e^{-0.34k} * (1 - e^{-0.34})$

$A_{19} = e^{-0.06}(1 - e^{-0.34})_{k=0}{}^{\infty}\sum e^{-0.06k} * e^{-0.34k}$

$A_{19} = e^{-0.06}(1 - e^{-0.34})_{k=0}{}^{\infty}\sum e^{-0.4k}$

$A_{19} = e^{-0.06}(1 - e^{-0.34})/(1 - e^{-0.4})$ = $\mathbf{A_{19} = 0.8233575754}$.

Problem S3L34-4. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is currently 0.06. Madison the Triceratops is currently 18 years old and takes out a whole insurance policy paying a benefit of 1 Golden Hexagon (GH) at the end of the year of death. Use a recursion formula and a previous solution to find the actuarial present value of this policy.

Solution S3L34-4. We already know from Solution S3L34-3 that for triceratopses, A_{19} = 0.8233575754. We seek to find A_{18}. We use the formula $A_x = vq_x + vp_xA_{x+1}$, where x = 18 and $v = e^{-0.06}$.

Since the deaths of triceratopses follow an exponential distribution, it follows that for every x,

$p_x = e^{-0.34}$ and $q_x = 1 - e^{-0.34}$.

Thus, $A_{18} = (e^{-0.06})(1 - e^{-0.34} + 0.8233575754e^{-0.34})$ = $\mathbf{A_{18} = 0.8233575754.}$

(This is as can be expected from an exponential distribution, where the values of p_x and q_x are each independent of the value of x. Thus, the value of the whole life insurance policy should not be expected to change on the basis of the age of the insured.)

Problem S3L34-5. There are 5470 7-year-old yellow bluebirds, of whom only 5012 will reach their 8th birthday. The annual effective interest rate is 0.03, and the actuarial present value for a whole insurance policy for 8-year-old yellow bluebirds paying a benefit of 1 Golden Hexagon (GH) at the end of the year of death is 0.83. Find the actuarial present value for a whole insurance policy for 7-year-old yellow bluebirds paying a benefit of 1 Golden Hexagon (GH) at the end of the year of death.

Solution S3L34-5. We use the formula $l_x*(1 + i)A_x = l_xA_{x+1} + d_x(1 - A_{x+1})$, which we rearrange thus:

$A_x = [l_xA_{x+1} + d_x(1 - A_{x+1})]/[l_x*(1 + i)]$. We are given that $l_7 = 5470$ and $l_8 = 5012$. Thus, $l_8 - l_7 = d_7 = 458$.

Moreover, i = 0.03 and $A_8 = 0.83$. Thus,

$A_7 = [5470*0.83 + 458(1 - 0.83)]/[5470*1.03]$ = $\mathbf{A_7}$ **= about 0.8196446637**.

Section 35

Endowment Insurance Payable at the End of the Year of Death

As in Section 33, the present value of the benefit payment for an insurance policy with the payment made at the end of the year of death is z_{k+1} and is expressed as follows.

$z_{k+1} = b_{k+1}v_{k+1.}$

$\mathbf{z_{k+1} = Z}$ is the present value, at policy issue, of the benefit payment.

$\mathbf{b_{k+1}}$is the **benefit function.**

$\mathbf{V_{k+1}}$is the **discount function. v** is the one-year discount factor by which a sum of money payable one year from now is multiplied to get its present value today. If the annual effective interest rate is r, then v = 1/(1+r).

Then for a policy of **n-year endowment insurance with one unit in benefits payable at the end of the year of death,** the following functions apply.

$\mathbf{b_{k+1} = 1}$ **for all positive integer values k;**

$\mathbf{v_{k+1} = v^{k+1}}$ **for all integer values of k such that $0 \le k \le n-1$;**

$\mathbf{v_{k+1} = v^n}$ **for all integer values of k such that $n \le k$;**

$\mathbf{Z = v^{K+1}}$ **for all integer values of K such that $0 \le K \le n-1$;**

$\mathbf{Z = v^n}$ **for all integer values of K such that $n \le K$.**

The actuarial present value of an n-year endowment insurance policy paying one unit in benefits at the end of the year of death is $\mathbf{A_{x:n\neg}}$ and can be found as follows.

$\mathbf{A_{x:n\neg}} = {}_{k=0}^{n-1}\sum \mathbf{v^{k+1} * {}_kp_x * q_{x+k}} + \mathbf{v^n * {}_np_x}$

$\mathbf{A_{x:n\neg} = A^1{}_{x:n\neg} + A_{x:n}{}^1{}_\neg}$,

Where $\mathbf{A^1{}_{x:n\neg}}$ is the actuarial present value of an n-year term life insurance policy paying one unit in benefits at the end of the year of death and $\mathbf{A_{x:n}{}^1{}_\neg}$ is the actuarial present value of an n-year pure endowment. (Please, please forgive me for this notation! I did not invent it, and I recognize the rather confusing nature of notation for two different concepts where the only difference in notation is in the horizontal placement of the superscript 1. I henceforth take the liberty of calling the actuarial present value of an n-year pure endowment $\boldsymbol{\acute{\mathring{A}}_{x:n}{}^{1}{}_{\neg}}$, which means that $\mathbf{A_{x:n\neg} = A^{1}{}_{x:n\neg} + \acute{\mathring{A}}_{x:n}{}^{1}{}_{\neg}}$.)

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1997. Second Edition. Society of Actuaries: Itasca, Illinois. p. 115.

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Problem S3L35-1. The actuarial present value of a 10-year term life insurance policy for newborn red bluefish paying one unit in benefits at the end of the year of death is 0.56. The actuarial present value of a 10-year endowment insurance policy for newborn red bluefish paying one unit in benefits at the end of the year of death is 0.88. Find the actuarial present value of a 10-year pure endowment for newborn red bluefish paying one unit in benefits.

Solution S3L35-1. We use the formula$A_{x:n\neg} = A^{1}{}_{x:n\neg} + \acute{\mathring{A}}_{x:n}{}^{1}{}_{\neg}$, which we rearrange thus:

$\acute{\mathring{A}}_{x:n}{}^{1}{}_{\neg} = A_{x:n\neg} - A^{1}{}_{x:n\neg}$
We are given that $A_{0:10\neg} = 0.88$ and $A^{1}{}_{0:10\neg} = 0.56$.

Thus, $\acute{\mathring{A}}_{0:10}{}^{1}{}_{\neg} = 0.88 - 0.56 =$ $\boldsymbol{\acute{\mathring{A}}_{0:10}{}^{1}{}_{\neg} = 0.32}$.

Problem S3L35-2. For 2-year-old white-whiskered wallabies, the actuarial present value of a 13-year term life insurance policy paying one unit in benefits at the end of the year of death is twice the actuarial present value of a 13-year pure endowment paying one unit in benefits, which is three times the present value of one dollar payable in 13 years at an annual force of interest of 0.23. Find the actuarial present value of a 13-year endowment insurance policy for 2-year-old white-whiskered wallabies, paying one unit in benefits at the end of the year of death.

Solution S3L35-2. The present value of one dollar payable in 13 years at an annual force of interest of 0.23 is $e^{-0.23*13} = 0.0502874367$.

Thus, $\acute{\mathring{A}}_{2:13}{}^{1}{}_{\neg} = 3*0.0502874367 = 0.1508623102$.

We are also given that $A^{1}{}_{2:13\neg} = 2*\acute{\mathring{A}}_{2:13}{}^{1}{}_{\neg} = 0.3017246203$.

Thus, $A_{2:13\neg} = A^{1}{}_{2:13\neg} + \acute{\mathring{A}}_{2:13}{}^{1}{}_{\neg} = 0.1508623102 + 0.3017246203 =$ **about 0.4525869305**.

Problem S3L35-3. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is currently 0.06. Alaric the Triceratops is currently 12 years old and takes out a 3-year endowment insurance policy paying a benefit of 1 at the end of the year of death. Find the actuarial present value of this policy.

Solution S3L35-3. We use the formula $A_{x:n\neg} = {}_{k=0}{}^{n-1}\sum v^{k+1} * {}_{k}p_{x} * q_{x+k} + v^{n} * {}_{n}p_{x}$.

Here, x = 12 and n = 3.

For an exponential distribution with survival function $s(x) = e^{-0.34x}$, ${}_kp_x = e^{-0.34k}$

and ${}_np_x = e^{-0.34n}$. Here, ${}_3p_x = e^{-0.34n} = e^{-1.02}$.

We know that $v = e^{-0.06}$ and so $v^n = e^{-0.06*3} = e^{-0.18}$.

So $v^n{*}{}_np_x = e^{-0.18}e^{-1.02} = e^{-1.2}$.

We find $q_{12+k} = 1 - s(13 + k)/s(12 + k) = 1 - e^{-0.34}$.

So ${}_{k=0}{}^{2}\sum e^{-0.06(k+1)}{*}e^{-0.34k}{*}(1 - e^{-0.34}) =$

$e^{-0.06}{*}(1 - e^{-0.34}) + e^{-0.12}{*}e^{-0.34}{*}(1 - e^{-0.34}) + e^{-0.18}{*}e^{-0.68}{*}(1 - e^{-0.34}) =$

$(1 - e^{-0.34})(e^{-0.06} + e^{-0.46} + e^{-0.86})$ = about 0.5753670393.

To get $A_{12:3\neg}$, we add 0.5753670393 + $e^{-1.2}$ = **$A_{12:3\neg}$ = about 0.8765612512**.

Problem S3L35-4. Mauricius the Mortal has a 0.36 probability of dying in 34.5 years, a 0.2 probability of dying in 44.2 years, a 0.14 probability of dying in 54.6 years, and a 0.3 probability of dying in 1004.7 years. The annual force of interest hereafter and forevermore is 0.04. Mauricius has a 50-year endowment insurance policy that pays 100,000 Golden Hexagons (GH) at the end of the year of death. Find the actuarial present value of this policy.

Solution S3L35-4. The values of the curate-future-lifetime random variable K for Mauricius are either 34, 44, 54, or 1004. Because this is a 50-year endowment insurance policy, the following present value factors correspond to these values of K:

$34 \rightarrow v^{35}$
$44 \rightarrow v^{45}$
$54 \rightarrow v^{50}$
$1004 \rightarrow v^{50}$

Thus, the probability of the present-value factor being v^{50} is 0.14 + 0.3 = 0.44.
Hence, the actuarial present value of Mauricius's policy is
$100000(0.36{*}e^{-0.04*35} + 0.2{*}e^{-0.04*45} + 0.44{*}e^{-0.04*50})$ = **about $18,138.22**

Problem S3L35-5. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is currently 0.06. For 12-year old triceratopses, find the value of n for which an n-year pure endowment with one unit in benefits half the actuarial present value of a 3-year pure endowment with one unit in benefits.

Solution S3L35-5. The value of an n-year pure endowment is $v^n{*}{}_np_x$, which for triceratopses is $e^{-0.06n}{*}e^{-0.34n} = e^{-0.4n}$. The value of a 3-year pure endowment, then, is $e^{-0.4*3} = e^{-1.2}$.
Then we want to find n such that $e^{-0.4n} = (1/2)e^{-1.2}$.
This means that $n = \ln((1/2)e^{-1.2})/-0.4$ = **n = about 4.732867951 years.**

Section 36

Exam-Style Questions on Life Insurance Policies

Before we attempt some exam-style questions pertaining to the material covered thus far in this study guide, it will be useful to know some special properties that hold under the assumption of uniform distributions of deaths (UDD) for fractional ages.

Under UDD for fractional ages, it is the case that

$\mu_{x+t} * {}_tp_x = q_x$ and

${}_tq_{x+s} = {}_tq_x/(1 - {}_sq_x)$

The problems in this section were designed to be similar to problems from past versions of the Casualty Actuarial Society's Exam 3L and the Society of Actuaries' Exam MLC. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L36-1.

Similar to Question 21 from the Casualty Actuarial Society's Spring 2008 Exam 3L.

Two whole life insurance policies for 60-year-old yak-rabbits are as follows:

Policy J assumes that the lives of yak-rabbits follow an exponential distribution with mean 8.

Policy K assumes that the lives of yak-rabbits follow a uniform distribution, and no yak-rabbit survives beyond age 90.

The annual force of interest in Yak-Rabbitland is 0.05. What is the absolute value of the difference between Policy J and Policy K?

Solution S3L36-1. We use the formula $\bar{A}_x = {}_0{}^\infty\!\int v^t * {}_tp_x * \mu_x(t)dt = {}_0{}^\infty\!\int v^t * f_T(t)dt$

For Policy J, an exponential distribution of lives implies that $\mu_x(t) = 1/\lambda$ (where λ is the mean of the exponential distribution) = 1/8 = 0.125 for all x.

Likewise, ${}_tp_x = e^{-0.125t}$ for all x.

We find $v^t = e^{-0.05t}$. Thus, $\bar{A}_{60} = {}_0{}^{\infty}\int 0.125*e^{-0.05t}*e^{-0.125t}dt = {}_0{}^{\infty}\int 0.125*e^{-0.175t}dt =$

$(-125/175)e^{-0.175t} \mid_0{}^{\infty} = 125/175 = \bar{A}_{60} = 0.714285714$ for Policy J.

For Policy K, since it is assumed that a 60-year-old yak-rabbit has at most 30 years left ti live and deaths are uniformly distributed, it follows that $f_T(t) = 1/30$.

Thus, $\bar{A}_{60} = {}_0{}^{30}\int(1/30)e^{-0.05t}dt = (-20/30)e^{-0.05t} \mid_0{}^{30} = (2/3)(1 - e^{-0.05*30})$ = about 0.517913227 for Policy K.

Thus, the absolute value of the difference between Policy J and Policy K is

0.714285714 - 0.517913227 = **about 0.196372487**

Problem S3L36-2.

Similar to Question 22 from the Casualty Actuarial Society's Spring 2008 Exam 3L.

For velociraptors, the force of mortality is a constant 0.12, and the force of interest is 0.09. For a velociraptor aged x, a whole life insurance policy has been issued. What is the 46th percentile of the distribution of the present value of this policy?

Solution S3L36-2. The present-value random variable of this policy is Z. We recall from Section 26 that $F_Z(z) = 1 - F_T(\ln(z)/\ln(v))$. Since the force of mortality given here is constant, we know that the lives of velociraptors are exponentially distributed with mean 1/0.12. Thus, $F_T(t) = 1 - e^{-0.12t}$. Thus,

$1 - F_T(\ln(z)/\ln(v)) = e^{-0.12(\ln(z)/\ln(v))}$.

Since the force of interest is 0.09, we have $v = e^{-0.09}$, so $\ln(v) = -0.09$. Thus,

$e^{-0.12(\ln(z)/\ln(v))} = e^{-0.12(\ln(z)/-0.09)} = e^{((4/3)\ln(z))} = e^{(\ln(z^{\wedge}(4/3)))} = F_Z(z) = z^{4/3}$. We want to find z such that $z^{4/3} = 0.46$. This is $z = 0.46^{3/4}$ = **about 0.558558125.**

Problem S3L36-3.

Similar to Question 30 from the Casualty Actuarial Society's Fall 2007 Exam 3.

The survival of white bulls follows a Weibull distribution, such that the force of mortality is $\mu_x = kx^4$.

You know that for white bulls, ${}_5q_1 = 0.64535$. Find ${}_{2|2}q_3$.

Solution S3L36-3. We first need to find the value of k. We know that a Weibull survival distribution looks as follows: $s(x) = \exp[-ux^{n+1}]$, where $u = k/(n + 1)$. Moreover, a Weibull force of mortality is of the form $\mu_x = kx^n$, so we know here that n = 4 and u = k/5.

We are given that ${}_5q_1 = 1 - s(6)/s(1) = 0.64535$ and $0.35465 = s(6)/s(1)$

We find $s(1) = \exp[-(k/5)1^5] = \exp[-(k/5)]$

We find $s(6) = \exp[-(k/5)6^5]$

Thus, $0.35465 = \exp[-(k/5)6^5]/\exp[-(k/5)]$

$0.35465 = \exp[-(k/5)(6^5 - 1)]$

$\ln(0.35465) = -(k/5)(6^5 - 1)$

$-5\ln(0.35465)/(6^5 - 1) = k$ = about 0.000666639158.

Now we can find ${}_{2|2}q_3 = (s(5) - s(7))/s(3) =$

$(\exp[-(k/5)5^5] - \exp[-(k/5)7^5])/\exp[-(k/5)3^5] =$

$(\exp[-(0.000666639158/5)5^5] - \exp[-(0.000666639158/5)7^5])/$

$\exp[-(0.000666639158/5)3^5] = (0.659251965 - 0.106369025)/0.968120551 =$ **${}_{2|2}q_3$ = about 0.571088941.**

Problem S3L36-4. Similar to Question 13 from the Casualty Actuarial Society's Fall 2006 Exam 3.

This question uses the **Illustrative Life Table, which can be found here**.

Using the Illustrative Life Table and assuming a uniform distribution of deaths for fractional ages, determine the truth or falsehood of each of these statements.

(a) $\mu_{34.5} * {}_{0.5}p_{34} = q_{34}$.

(b) ${}_{0.5}p_{34} * {}_{0.3}q_{34.5} = 0.00057$

(c) ${}_{0.3}q_{34.7} < {}_{0.3}q_{34.5}$

Solution S3L36-4.

Under a uniform distribution of deaths for fractional ages, it is the case that

$\mu_{x+t} * {}_tp_x = q_x$. (a) is the expression of this statement for x = 34 and t = 0.5. Thus, **(a) is true.**

We examine statement (b):

${}_{0.5}p_{34}*{}_{0.3}q_{34.5} = {}_{0.5}p_{34}*(1 - {}_{0.3}p_{34.5}) = {}_{0.5}p_{34} - {}_{0.8}p_{34} = {}_{0.8}q_{34} - {}_{0.5}q_{34} = 0.3q_{34}$(by the UDD assumption).

From the Illustrative Life Table, $1000q_{34} = 1.90$, so $0.3q_{34} = 1.90*0.3/1000 = 0.00057$, so **(b) is true.**

Under the UDD assumption, ${}_tq_{x+s} = {}_tq_x/(1 - {}_sq_x)$.
Thus, for x = 34, s = 0.5, t = 0.3, ${}_{0.3}q_{34.5} = {}_{0.3}q_{34}/(1 - {}_{0.5}q_{34})$.
For x = 34, s = 0.7, t = 0.3, ${}_{0.3}q_{34.5} = {}_{0.3}q_{34}/(1 - {}_{0.7}q_{34})$.
We know that ${}_{0.7}q_{34} > {}_{0.5}q_{34}$, since ${}_{0.7}q_{34} = {}_{0.5}q_{34} + {}_{0.2}q_{34.5}$.
Thus, $(1 - {}_{0.7}q_{34}) < (1 - {}_{0.5}q_{34})$
Thus, ${}_{0.3}q_{34}/(1 - {}_{0.7}q_{34}) > {}_{0.3}q_{34}/(1 - {}_{0.5}q_{34})$, so
${}_{0.3}q_{34.7} > {}_{0.3}q_{34.5}$ and **(c) is false.**

Problem S3L36-5. Similar to Question 33 from the Casualty Actuarial Society's Fall 2006 Exam 3.

Red rodents exhibit a constant force of mortality of 0.1, and the annual force of interest among red rodents is 0.03. Tnedor the Red Rodent has a 7-year deferred whole life insurance paying a benefit of 1 Golden Hexagon (GH) upon death. Find the variance Var(Z) of the present-value random variable for this policy.

Solution S3L36-5. We use the formula
${}_{m|}\bar{A}_x = {}_m^{\infty}\int v^t*{}_tp_x*\mu_x(t)dt$ to findthe actuarial present value of this policy:

A constant force of mortality corresponds to an exponential distribution with mean

$1/\mu_x$

So $\mu_x(t) = 0.1$ and ${}_tp_x = e^{-0.1t}$. Moreover, we can find $v^t = e^{-0.03t}$.

Thus, ${}_{7|}\bar{A}_x = {}_7^{\infty}\int e^{-0.03t}*e^{-0.1t}*0.1dt = {}_7^{\infty}\int 0.1e^{-0.13t}dt = (-10/13)e^{-0.13t} \big|_7^{\infty} = (10/13)e^{-0.13*7} = {}_{7|}\bar{A}_x =$ 0.309634018.

Now we find the second moment of the present value for this policy:

${}^2{}_{7|}\bar{A}_x = {}_7^{\infty}\int e^{-0.03*2t}*e^{-0.1t}*0.1dt = {}_7^{\infty}\int 0.1e^{-0.16t}dt = (-10/16)e^{-0.16t} \big|_7^{\infty} = (5/8)e^{-0.16*7} = {}^2{}_{7|}\bar{A}_x =$ 0.203924872.

Now we use the formula $Var(Z) = {}^2{}_{m|}\bar{A}_x - ({}_{m|}\bar{A}_x)^2 = 0.203924872 - 0.309634018^2 =$ **Var(Z) = 0.108051647**.

Section 37

Continuous Whole and Temporary Life Annuities

A **life annuity** is "a series of payments made continuously or at equal intervals... while a given life survives" (Bowers et. al, p. 133). Life annuities are different from **annuities-certain,** the subject of Exam 2/FM, in that payment is condition on the life's survival.

A **whole life annuity** makes payments to the annuitant until death. The present value of payments to be made for a whole life annuity is denoted by the symbol $\mathbf{Y = \bar{a}_{T\neg}}$ and has the following distribution function

(c. d. f.):

$\mathbf{F_Y(y) = F_T(-ln(1 - \delta y)/\delta)}$ **for** $\mathbf{0 < y < 1/\delta}$, where δ is the annual force of interest and T is the future-lifetime random variable for the annuitant.

$Y = \bar{a}_{T\neg}$ also has the following probability density function (p. d. f.):

$\mathbf{f_Y(y) = f_T(-ln(1 - \delta y)/\delta)/(1 - \delta y)}$ **for** $\mathbf{0 < y < 1/\delta}$.

The actuarial present value of a **continuous whole life annuity** (where payments are made continuously, with a momentary infinitesimal payment of dt made at every time t) is denoted by the symbol $\mathbf{E[Y] = \bar{a}_x}$ and has the following actuarial present value:

$$\mathbf{E[Y] = \bar{a}_x = {}_0{}^{\infty}\int v^t * {}_tp_x * dt = {}_0{}^{\infty}\int {}_tE_x * dt}$$

The symbol $\mathbf{{}_tE_x}$ is another way to express the actuarial present value of a t-year pure endowment, and we recall also that $\mathbf{{}_tE_x} = \mathbf{v^t * {}_tp_x}$.

The actuarial present value for a continuous **n-yeartemporary life annuity** that only makes payments at most up to time n (if the annuitant survives until time n) is denoted as $\mathbf{\bar{a}_{x:n\neg}}$ and can be found as follows:

$$\mathbf{\bar{a}_{x:n\neg} = {}_0{}^{n}\int v^t * {}_tp_x * dt = {}_0{}^{n}\int {}_tE_x * dt}$$

A useful relationship holds between $\bar{a}_x$ and $\bar{a}_{x+1}$.

$$\mathbf{\bar{a}_x = \bar{a}_{x:1\neg} + vp_x * \bar{a}_{x+1}}$$

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1997. Second Edition. Society of Actuaries: Itasca, Illinois. pp. 133-137.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L37-1. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is currently 0.06. Charlie the Triceratops has a continuous whole life annuity. What is the distribution function $F_Y(y)$of the present value payments to be made for such an annuity?

Solution S3L37-1. The lives of triceratopses are exponentially distributed.

Thus, $F_T(t) = 1 - e^{-0.34t}$. We use the formula $F_Y(y) = F_T(-\ln(1 - \delta y)/\delta)$ for $0 < y < 1/\delta$, where $\delta = 0.06$. Thus, $F_Y(y) = 1 - e^{0.34\ln(1 - \delta y)/\delta} = 1 - e^{0.34\ln(1 - 0.06y)/0.06} = 1 - e^{(17/3)\ln(1 - 0.06y)} =$

$1 - e^{\ln((1 - 0.06y)^\wedge(17/3))} =$ **$F_Y(y) = 1 - (1 - 0.06y)^{17/3}$ for $0 < y < 16.666666667$**.

Problem S3L37-2. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is currently 0.06. Charlie the Triceratops has a continuous whole life annuity. What is the probability density function $f_Y(y)$of the present value payments to be made for such an annuity?

Solution S3L37-2. We use the formula $f_Y(y) = f_T(-\ln(1 - \delta y)/\delta)/(1 - \delta y)$ for $0 < y < 1/\delta$.

The lives of triceratopses are exponentially distributed. Thus, $f_T(t) = 0.34e^{-0.34t}$ and so

$f_Y(y) = 0.34e^{-0.34(-\ln(1 - \delta y)/\delta)/(1 - \delta y)} = 0.34e^{(17/3)\ln(1 - 0.06y)}/(1 - 0.06y) =$

$0.34e^{\ln((1 - 0.06y)^\wedge(17/3))}/(1 - 0.06y) = 0.34*(1 - 0.06y)^{17/3}/(1 - 0.06y) =$

$f_Y(y) =0.34*(1 - 0.06y)^{14/3}$ for $0 < y < 16.666666667$.

Problem S3L37-3. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is currently 0.06. Charlie the Triceratops has a continuous whole life annuity. What is the actuarial present value $\bar{a}_x$ of Charlie's annuity?

Solution S3L37-3. We use the formula $\bar{a}_x = {}_0{}^{\infty}\int v^t*{}_tp_x*dt$. Since the lifetimes of triceratops are exponentially distributed, ${}_tp_x = e^{-0.34t}$ for all x. Moreover, $v^t = e^{-0.06t}$. Thus,

$\bar{a}_x = {}_0{}^{\infty}\int e^{-0.06t}*e^{-0.34t}*dt = {}_0{}^{\infty}\int e^{-0.4t}*dt = (-5/2)e^{-0.4t} \Big|_0^{\infty} =$ **$\bar{a}_x = 5/2 = 2.5$**.

Problem S3L37-4. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is currently 0.06. Desiderius the Triceratops has a continuous 4-year temporary life annuity. What is the actuarial present value $\bar{a}_{x:4\neg}$of Desiderius's annuity?

Solution S3L37-4. We use the formula $\bar{a}_{x:n\neg} = {}_0{}^{n}\int v^t*{}_tp_x*dt$. Since the lifetimes of triceratops are exponentially distributed, ${}_tp_x = e^{-0.34t}$ for all x. Moreover, $v^t = e^{-0.06t}$. Thus,

$\bar{a}_{x:4\neg} = {}_0{}^4\int e^{-0.06t} * e^{-0.34t} * dt = {}_0{}^4\int e^{-0.4t} * dt = (-5/2)e^{-0.4t} \mid_0{}^4 = (5/2)(1 - e^{-1.6}) =$ **$\bar{a}_{x:4\neg}$ = about 1.99525871**.

Problem S3L37-5. Seven-finned jumping fish can take out continuous whole and temporary life annuities. Each annuity pays 1*dt in benefits at each time t. A 5-year-old seven-finned jumping fish can get either a one-year temporary life annuity with present value 0.67 Golden Hexagons (GH) or a whole life annuity with present value 5.6 GH. The annual force of interest among seven-finned jumping fish is 0.2, and a 5-year-old seven-finned jumping fish has a probability of 0.77 of surviving to age 6. Find the actuarial present value of a continuous whole life annuity available to a 6-year-old seven-finned jumping fish.

Solution S3L37-5. We use the formula $\bar{a}_x = \bar{a}_{x:1\neg} + vp_x * \bar{a}_{x+1}$, which we rearrange thus:

$vp_x * \bar{a}_{x+1} = \bar{a}_x - \bar{a}_{x:1\neg}$

$\bar{a}_{x+1} = (\bar{a}_x - \bar{a}_{x:1\neg})/(vp_x)$

We are given that $\bar{a}_5 = 5.6$ and $\bar{a}_{5:1\neg} = 0.67$. Moreover, we know that $v = e^{-0.2}$ and $p_5 = 0.77$.

Thus, $\bar{a}_6 = (5.6 - 0.67)/(0.77e^{-0.2}) = 4.93/(0.77e^{-0.2}) =$ about **7.82015013 GH**.

(**Note:** This result is a genuine possibility. It is possible for the actuarial present values of life annuities to **increase** for older annuitants, if the rate at which older annuitants die for some reason happens to be less than the rate at which younger annuitants die. Once a seven-finned jumping fish has gotten over the mortality spike during the sixth year of its life, its life expectancy can actually increase.)

Section 38

Variances and Relationships Among Present Values of Continuous Whole and Temporary Life Annuities

The following relationship holds with respect to actuarial present values of continuous whole life annuities and of whole life insurance policies.

$\mathbf{1 = \delta\bar{a}_x + \bar{A}_x}$

We can also determine the variance of the present value of a continuous whole life annuity:

$\mathbf{Var(\bar{a}_{T\neg}) = ({}^2\bar{A}_x - (\bar{A}_x)^2)/\delta^2}$

If the force of mortality is a constant value μ, then

$\mathbf{\bar{a}_x = 1/(\delta + \mu)}$

$\mathbf{\bar{A}_x = \mu/(\delta + \mu)}$

$\mathbf{{}^2\bar{A}_x = \mu/(2\delta + \mu)}$

$\mathbf{Var(\bar{a}_{T\neg}) = \mu/[(2\delta + \mu)(\delta + \mu)^2]}$

$\mathbf{Pr(\bar{a}_{T\neg} > \bar{a}_x) = (\mu/(\delta + \mu))^{\mu/\delta}}$

If Y is the present value of an n-year **temporary life annuity,** then

$\mathbf{Var(Y) = (2/\delta)(\bar{a}_{x:n\neg} - {}^2\bar{a}_{x:n\neg}) - (\bar{a}_{x:n\neg})^2}$

Meaning of Symbols:

$\mathbf{\bar{A}_x}$ **=** the actuarial present value of a whole life insurance policy for life (x).

$\mathbf{{}^2\bar{A}_x}$ = the second moment of the actuarial present value of a whole life insurance policy for life (x).

$\mathbf{\bar{a}_x}$ = the actuarial present value of a continuous whole life annuity for life (x).

$\mathbf{\bar{a}_{T\neg}}$ **=** the present-value random variable for a continuous whole life annuity for an annuitant with a future-lifetime random variable of T.

δ = the annual force of interest.

$\bar{a}_{x:n\neg}$= the actuarial present value of a continuous n-year temporary life annuity for life (x).

$_tE_x$= the actuarial present value of a t-year pure endowment for life (x).

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1997. Second Edition. Society of Actuaries: Itasca, Illinois. pp. 136-139

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Problem S3L38-1. A 34-year-old blue pterodactyl can get a whole life insurance policy paying a benefit of 1 at death for 0.56 Golden Hexagons (GH) or a continuous whole life annuity with actuarial present value $\bar{a}_{34} = 26$. Find the annual force of interest.

Solution S3L38-1. We use the formula $1 = \delta\bar{a}_x + \bar{A}_x$.

Then $1 - \bar{A}_x = \delta\bar{a}_x$

$(1 - \bar{A}_x)/\bar{a}_x = \delta$

We are given that $\bar{A}_{34} = 0.56$

Thus, $\delta = (1 - 0.56)/26 =$ **δ = about 0.0169230769**.

Problem S3L38-2. A whole life insurance policy on the life of a 50-year-old giant mongoose has an actuarial present value of 0.53. Under a force of interest of 0.047, the second moment of the present value of this policy is 0.43. Find the variance of the present value of a continuous whole life annuity for a 50-year-old giant mongoose.

Solution S3L38-2. We use the formula $\text{Var}(\bar{a}_{T\neg}) = ({}^2\bar{A}_x - (\bar{A}_x)^2)/\delta^2$.

We are given that $\bar{A}_{50} = 0.53$, ${}^2\bar{A}_{50} = 0.43$, and $\delta = 0.047$.

Thus, $\text{Var}(\bar{a}_{T\neg}) = (0.43 - (0.53)^2)/0.047^2 =$ **$\text{Var}(\bar{a}_{T\neg})$ = about 67.4966048**.

Problem S3L38-3. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is currently 0.06. Charlie the Triceratops has a continuous whole life annuity whose actuarial present value is denoted by $\bar{a}_x$. What is the variance of the present value of this annuity?

Solution S3L38-3. The lifetimes of triceratopses are exponentially distributed, which means that triceratopses exhibit a constant force of mortality - in this case, 0.34. We can thus use the formula $\text{Var}(\bar{a}_{T\neg}) = \mu/[(2\delta + \mu)(\delta + \mu)^2]$ for $\mu = 0.34$ and $\delta = 0.06$. Thus,

$\text{Var}(\bar{a}_{T\neg}) = 0.34/[(2*0.06 + 0.34)(0.34 + 0.06)^2] = 0.34/0.0736 =$ **$\text{Var}(\bar{a}_{T\neg})$ = about 4.619565217**.

Problem S3L38-4. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is currently 0.06. Charlie the Triceratops has a continuous whole life annuity whose actuarial present value is denoted by $\bar{a}_x$ = 2.5. What is the probability that the actuarial present value of this annuity will exceed 2.5?

Solution S3L38-4.

Since triceratopses have a constant force of mortality, we can use the formula

$\Pr(\bar{a}_{T¬} > \bar{a}_x) = (\mu/(\delta + \mu))^{\mu/\delta}$ with $\mu = 0.34$ and $\delta = 0.06$.

Thus, $\Pr(\bar{a}_{T¬} > 2.5) = (0.34/(0.06 + 0.34))^{0.34/0.06} = (0.85)^{17/3}$ =

$\Pr(\bar{a}_{T¬} > 2.5)$ = about 0.3981443701.

Problem S3L38-5. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is currently 0.06. Desiderius the Triceratops has a continuous 4-year temporary life annuity whose actuarial present value is $\bar{a}_{x:4¬}$. What is the variance of the present value of Desiderius's annuity?

Solution S3L38-5. We use the formula

$\mathrm{Var}(Y) = (2/\delta)(\bar{a}_{x:n¬} - {}^2\bar{a}_{x:n¬}) - (\bar{a}_{x:n¬})^2$

We can find $\bar{a}_{x:4¬}$. We use the formula $\bar{a}_{x:n¬} = {}_0{}^n\int v^t * {}_tp_x * dt$. Since the lifetimes of triceratopses are exponentially distributed, ${}_tp_x = e^{-0.34t}$ for all x. Moreover, $v^t = e^{-0.06t}$. Thus,

$\bar{a}_{x:4¬} = {}_0{}^4\int e^{-0.06t} * e^{-0.34t} * dt = {}_0{}^4\int e^{-0.4t} * dt = (-5/2)e^{-0.4t} \Big|_0^4 = (5/2)(1 - e^{-1.6}) = \bar{a}_{x:4¬}$ = about 1.99525871.

To find ${}^2\bar{a}_{x:4¬}$, we can use the Rule of Moments and apply the resulting formula

${}^2\bar{a}_{x:n¬} = {}_0{}^n\int v^{2t} * {}_tp_x * dt$.

Thus, ${}^2\bar{a}_{x:4¬} = {}_0{}^4\int e^{-0.12t} * e^{-0.34t} * dt = {}_0{}^4\int e^{-0.46t} * dt = (-100/46)e^{-0.46t} \Big|_0^4 = (100/46)(1 - e^{-1.84})$ = about 1.828657769.

Hence, $\mathrm{Var}(Y) = (2/0.06)(1.99525871 - 1.828657769) - (1.99525871)^2$

Var(Y) = about 1.572307369.

Section 39

Continuous Deferred Life Annuities and Certain and Life Annuities

A continuous n-year **deferred life annuity** begins to make continuous payments n years after the present time. The actuarial present value of an n-year deferred life annuity is denoted by $_{n|}\bar{a}_x$ and can be found as follows:

$$_{n|}\bar{a}_x = {}_n\int^{\infty} v^t * {}_tp_x * dt = {}_n\int^{\infty} {}_tE_x * dt$$

If Y is the present value of such an n-year deferred life annuity, then

$$Var(Y) = (2/\delta)v^{2n}{}_np_x(\bar{a}_{x+n} - {}^2\bar{a}_{x+n}) - ({}_{n|}\bar{a}_x)^2$$

A continuous n-year **certain and life annuity** makes *guaranteed* payments for the first n years of its existence, *no matter what happens to the annuitant.* After the first n-years, this annuity behaves the same as a life annuity. The actuarial present value on a continuous n-year certain and life annuity is denoted by the symbol $\bar{a}^{---}_{x:n\neg}$ (with the line drawn *directly over* the "x:n¬" whenever one's writing and publishing interface permits) and can be found as follows.

$\bar{a}^{---}_{x:n\neg} = \bar{a}_{n\neg} + {}_n\int^{\infty} v^t * {}_tp_x * dt$, where $\bar{a}_{n\neg}$ is the symbol representing the present value of a continuous n-year annuity-certain. We recall that $\bar{a}_{n\neg} = (1 - e^{-n\delta})/\delta$.

Moreover, the following relationships hold.

$$\bar{a}^{---}_{x:n\neg} = \bar{a}_{n\neg} + (\bar{a}_x - \bar{a}_{x:n\neg})$$

$$\bar{a}^{---}_{x:n\neg} = \bar{a}_{n\neg} + {}_{n|}\bar{a}_x$$

Meaning of Symbols:

$\bar{A}_x$ = the actuarial present value of a whole life insurance policy for life (x).

$^2\bar{A}_x$ = the second moment of the actuarial present value of a whole life insurance policy for life (x).

$\bar{a}_x$ = the actuarial present value of a continuous whole life annuity for life (x).

$\bar{a}_{T\neg}$ = the present-value random variable for a continuous whole life annuity for an annuitant with a future-lifetime random variable of T.

δ = the annual force of interest.

$\bar{a}_{x:n\neg}$= the actuarial present value of a continuous n-year term life annuity for life (x).

${}_tE_x$= the actuarial present value of a t-year pure endowment for life (x).

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1997. Second Edition. Society of Actuaries: Itasca, Illinois. pp. 136-140.

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Problem S3L39-1. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is currently 0.06. Diocletian the Triceratops has a continuous 3-year deferred life annuity. What is the actuarial present value ${}_{3|}\bar{a}_x$ of Diocletian's annuity?

Solution S3L39-1. We use the formula ${}_{n|}\bar{a}_x = {}_n{}^{\infty}\int v^t * {}_tp_x * dt$.

Since the lifetimes of triceratopses are exponentially distributed, ${}_tp_x = e^{-0.34t}$ for all x. Moreover, $v^t = e^{-0.06t}$. Thus, ${}_{3|}\bar{a}_x = {}_3{}^{\infty}\int e^{-0.06t} * e^{-0.34t} * dt = {}_3{}^{\infty}\int e^{-0.4t} * dt = (-5/2)e^{-0.4t} \Big|_3^{\infty} = (5/2)(e^{-1.2}) =$

${}_{3|}\bar{a}_x$ = about **0.7529855298**.

Problem S3L39-2. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is currently 0.06. Diocletian the Triceratops has a continuous 3-year deferred life annuity. What is the variance of Diocletian's annuity?

Solution S3L39-2. We use the formula $Var(Y) = (2/\delta)v^{2n}{}_np_x(\bar{a}_{x+n} - {}^2\bar{a}_{x+n}) - ({}_{n|}\bar{a}_x)^2$. Since the lifetimes of triceratopses are exponentially distributed, triceratopses exhibit a constant force of mortality. Conveniently enough, for constant forces of mortality, the actuarial present value of a life annuity *does not depend on the initial age of the annuitant*! Thus, $\bar{a}_{x+n} = \bar{a}_x$. We use the formula $\bar{a}_x = 1/(\delta + \mu) = 1/(0.06 + 0.34) = 1/0.4 = 2.5$ for triceratopses. Thus, $\bar{a}_{x+3} = 2.5$ as well.

Moreover, from Solution S3L39-1, we know that ${}_{3|}\bar{a}_x = 0.7529855298$ and that ${}_3p_x = e^{-0.34*3} =$

$e^{-1.02}$. Here, 2n = 6, so $v^{2n} = e^{-0.06*6} = e^{-0.36}$.

We have only to find ${}^2\bar{a}_{x+3}$, which is the same as ${}^2\bar{a}_x$ for triceratopses.

By the Rule of Moments, if : $\bar{a}_x = 1/(\delta + \mu)$, then ${}^2\bar{a}_x = 1/(2\delta + \mu) = 1/(0.12 + 0.34)$ = about 2.173913043.

Thus, $Var(Y) = (2/0.06)e^{-0.36} * e^{-1.02}(2.5 - 2.173913043) - (0.7529855298)^2 =$

Var(Y) = about 2.167562282.

Problem S3L39-3. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is currently 0.06. Jorgax the Triceratops has a continuous 3-year certain and life annuity. What is the actuarial present value

$\bar{a}^{---}{}_{x:3\neg}$ of Jorgax's annuity?

Solution S3L39-3. We use the formula $\bar{a}^{---}{}_{x:n\neg} = \bar{a}_{n\neg} + {}_{n|}\bar{a}_x$.
We know from Solution S3L39-1 that ${}_{3|}\bar{a}_x = 0.7529855298$.

We need to find $\bar{a}_{3\neg}$ using the formula $\bar{a}_{n\neg} = (1 - e^{-n\delta})/\delta$.

$\bar{a}_{3\neg} = (1 - e^{-0.18})/0.06 = \bar{a}_{3\neg}$ = about 2.745496476.

Thus, $\bar{a}^{---}{}_{x:3\neg} = 2.745496476 + 0.7529855298 = \bar{a}^{---}{}_{x:3\neg} =$ **about 3.498482006**.

Problem S3L39-4. For 5-year-old jumping giraffes, the following assets giving the same continuous payoff have the following actuarial present values:

- A continuous whole life annuity: 45 Golden Hexagons (GH)

- A continuous 6-year temporary life annuity: 4.9 GH

- A continuous 6-year certain and life annuity: 45.7 GH.

Find the actuarial present value of a continuous 6-year annuity-certain for 5-year-old jumping giraffes.

Solution S3L39-4. We use the formula $\bar{a}^{---}{}_{x:n\neg} = \bar{a}_{n\neg} + (\bar{a}_x - \bar{a}_{x:n\neg})$, which we rearrange as follows:

$\bar{a}_{n\neg} = \bar{a}^{---}{}_{x:n\neg} - \bar{a}_x + \bar{a}_{x:n\neg}$.

We are given that $\bar{a}_5 = 45$, $\bar{a}_{5:6\neg}$ ---$_{5:6\neg}$ = 45.7.

Thus, $\bar{a}_{6\neg} = 45.7 - 45 + 4.9 = \bar{a}_{6\neg} =$ **5.6 GH**.

Problem S3L39-5. The future lifetimes of 11-year-old pygmy whales are uniformly distributed with probability of death 1/55 in every subsequent year until age 66. The annual force of interest is 0.1. Ymgyp the Pygmy Whale has a continuous 20-year certain and life annuity whose actuarial present value is $\bar{a}^{---}{}_{11:20\neg}$. Find $\bar{a}^{---}{}_{11:20\neg}$.

Solution S3L39-5. We use the formula $\bar{a}^{---}{}_{x:n\neg} = \bar{a}_{n\neg} + {}_n{}^{\infty}\!\int v^t * {}_tp_x * dt$.

The probability that Ymgyp will survive to year t is ${}_tp_x = (55 - t)/55$.

The discount factor is $v^t = e^{-0.1t}$.

Here, n = 20 and $\bar{a}_{20\neg}$ = (1 - $e^{-20\delta}$)/δ = (1 - e^{-2})/0.1 = 8.646647168.

Thus, $\bar{a}^{---}_{11:20\neg}$ = 8.646647168 + $_{20}{}^{\infty}\int e^{-0.1t}$*((55 - t)/55)*dt.

$\bar{a}^{---}_{11:20\neg}$ 20$^{\infty}\int e^{-0.1t}$ dt - $_{20}{}^{\infty}\int$(1/55)t$e^{-0.1t}$ dt.

$_{20}{}^{\infty}\int e^{-0.1t}$ dt = 10e^{-2} = about 1.353352832.

To find $_{20}{}^{\infty}\int$(1/55)t$e^{-0.1t}$ dt, we use the Tabular Method of integration by parts:

Sign.......u........dv

+......(1/55)t......$e^{-0.1t}$

-.......(1/55).....-10$e^{-0.1t}$

+..........0.......100$e^{-0.1t}$

Thus, $_{20}{}^{\infty}\int$(1/55)t$e^{-0.1t}$ dt = [(-10/55)t$e^{-0.1t}$ - (100/55)$e^{-0.1t}$] $\Big|_{20}^{\infty}$ =

(200/55)e^{-2}+(100/55)e^{-2} = about 0.738192454 .

Thus, $\bar{a}^{---}_{11:20\neg}$= 8.646647168 + 1.353352832 - 0.738192454.

So $\bar{a}^{---}_{11:20\neg}$= **about 9.261807546**.

Section 40

Actuarial Accumulated Values and Derivatives of Actuarial Present Values of Continuous Life Annuities

The **actuarial accumulated value** at the end of the term of an n-year temporary life annuity of 1 per year payable continuously while (x) survives is denoted by the symbol $\check{\mathbf{s}}_{\mathbf{x:n}\neg}$ (with a *straight* line drawn directly over the "s" whenever circumstances permit) and can be found as follows.

$\check{s}_{\mathbf{x:n}\neg} = \bar{\mathbf{a}}_{\mathbf{x:n}\neg} / {}_{\mathbf{n}}\mathbf{E}_{\mathbf{x}}$
$\check{s}_{\mathbf{x:n}\neg} = {}_{\mathbf{0}}{}^{\mathbf{n}}\int(\mathbf{1}/{}_{\mathbf{n-t}}\mathbf{E}_{\mathbf{x+t}})\mathbf{dt}$

We recall that ${}_{\mathbf{n}}\mathbf{E}_{\mathbf{x}} = \mathbf{v}^{\mathbf{n}} * {}_{\mathbf{n}}\mathbf{p}_{\mathbf{x}}$.

It is possible to calculate the derivatives of actuarial present values of continuous life annuities through the following formulas.

$\mathbf{d(\bar{a}_x)/dx = [\mu(x) + \delta]\bar{a}_x - 1}$
$\mathbf{\partial(\bar{a}_{x:n\neg})/\partial x = [\mu(x) + \delta]\bar{a}_{x:n\neg} - (1 - {}_nE_x)}$
$\mathbf{\partial({}_{n|}\bar{a}_x)/\partial n = -v^n * {}_np_x}$

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1997. Second Edition. Society of Actuaries: Itasca, Illinois. pp. 140-141.

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Problem S3L40-1. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is currently 0.06. Desiderius the Triceratops has a continuous 4-year temporary life annuity. At the end of four years, if Desiderius survives, what will be the actuarial accumulated value $\check{s}_{x:4\neg}$ of Desiderius's annuity?

Solution S3L40-1. We use the formula $\check{s}_{x:n\neg} = \bar{a}_{x:n\neg} / {}_nE_x$. To find $\bar{a}_{x:4\neg}$, we use the formula $\bar{a}_{x:n\neg} = {}_0{}^n\int v^t * {}_tp_x * dt$. Since the lifetimes of triceratops are exponentially distributed, ${}_tp_x = e^{-0.34t}$ for all x. Moreover, $v^t = e^{-0.06t}$. Thus,
$\bar{a}_{x:4\neg} = {}_0{}^4\int e^{-0.06t} * e^{-0.34t} * dt = {}_0{}^4\int e^{-0.4t} * dt = (-5/2)e^{-0.4t} \Big|_0{}^4 = (5/2)(1 - e^{-1.6}) = \bar{a}_{x:4\neg}$ = about 1.99525871.
Now we find ${}_4E_x = v^4 * {}_4p_x = e^{-0.24}e^{-1.36} = e^{-1.6}$
Thus, $\check{s}_{x:4\neg} = 1.99525871/e^{-1.6} = \check{s}_{\mathbf{x:4}\neg}$ **= about 9.882581086**.

Problem S3L40-2. The lives of blue bears exhibit a survival function $s(x) = 1 - x/67$ for $0 \leq x \leq 67$. Carlos the Blue Bear is 34 years old and has a continuous 6-year temporary life annuity. The annual force of interest is currently 0.035. At the end of six years, if Carlos survives, what will be the actuarial accumulated value $\check{s}_{34:6\neg}$ of his annuity? Set up the appropriate integral and use any calculator to evaluate it.

Solution S3L40-2. We use the formula $\check{s}_{x:n\neg} = {}_0{}^n\int(1/{}_{n-t}E_{x+t})dt$.
First, we try to find ${}_{6-t}E_{34+t} = v^{6-t} * {}_{6-t}p_{34+t}$.
Since we have a constant force of interest of 0.035, $v^{6-t} = e^{-0.035(6-t)} = e^{-0.21}e^{0.035t}$.
${}_{6-t}p_{34+t} = s(40)/s(34+t) = (27/67)/((33 – t)/67) = {}_{6-t}p_{34+t} = 27/(33-t)$.
Thus**, $\check{s}_{34:6\neg} = {}_0{}^6\int[e^{0.21}e^{-0.035t}(33-t)/27]dt$**
$\check{s}_{34:6\neg}$**= about 7.444296275.**

Problem S3L40-3. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is currently 0.06. Find Let f(n) = $\partial({}_{n|}\bar{a}_x)/\partial n$ for triceratopses. Find f(7).

Solution S3L40-3. We use the formula $\partial({}_{n|}\bar{a}_x)/\partial n = -v^n * {}_np_x$. For triceratopses, ${}_np_x = e^{-0.34n}$ and $v^n = e^{-0.06n}$
Thus, $f(n) = -v^n * {}_np_x = -e^{-0.4n}$ and $f(7) = -e^{-0.4*7} = -e^{-2.8}$ = **about -0.0608100626**.

Problem S3L40-4. For orange-spotted flying hippopotami, you are given that $\bar{a}_{6:3\neg} = 2.55$, ${}_3E_6 = 0.88$, and the force of mortality for 6-year-old orange-spotted flying hippopotami is 0.06. Let $g(x) = \partial(\bar{a}_{x:3\neg})/\partial x$. The annual force of interest is 0.09. Find g(6).

Solution S3L40-4. We use the formula $\partial(\bar{a}_{x:n\neg})/\partial x = [\mu(x) + \delta]\bar{a}_{x:n\neg} - (1 - {}_nE_x)$.
Thus, $g(6) = [\mu(6) + \delta]\bar{a}_{6:3\neg} - (1 - {}_3E_6) = [0.06 + 0.09]2.55 – (1 – 0.88) =$ **g(6) = 0.2625**.

Problem S3L40-5. A 15-year-old yaac (short for Yet Another Absurdist Creature) can get a continuous whole life annuity paying 1*dt at every instant for a price of 6.64 Golden Hexagons (GH). Yaac force of mortality follows the equation $\mu(x) = 0.1x^{1/8}$ for all x between 12 and 16, thereafter resorting to a constant force of mortality. Define $h(x) = d(\bar{a}_x)/dx$. You are given h(15) = 0.2. Find the annual force of interest.

Solution S3L40-5. We use the formula $d(\bar{a}_x)/dx = [\mu(x) + \delta]\bar{a}_x – 1$, rearranging it thus:
$(d(\bar{a}_x)/dx + 1)/\bar{a}_x = \mu(x) + \delta$
$\delta = (d(\bar{a}_x)/dx + 1)/\bar{a}_x - \mu(x)$
$\delta = (h(x) + 1)/\bar{a}_x - \mu(x)$
$\delta = (h(15) + 1)/\bar{a}_{15} – \mu(15)$
$\delta = (0.2 + 1)/6.64 – 0.1*15^{1/8}$
δ = about 0.0404378364

Section 41

Discrete Whole Life Annuities

A discrete whole-life annuity-due makes payments every year at the beginning of each year, so long as the annuitant survives. For life (x), a discrete whole-life annuity-due paying a benefit of 1 per year has actuarial present value $\ddot{a}_x$, which can be determined as follows:
$\mathbf{\ddot{a}_x = {}_{k=0}{}^{\infty}\Sigma v^k * {}_k p_x}$

If the discrete whole life annuity is an **annuity-immediate,** making payments at the beginning of each year, then the actuarial present value of such an annuity is denoted by $\mathbf{a_x}$ and can be found as follows:
$\mathbf{a_x = {}_{k=1}{}^{\infty}\Sigma v^k * {}_k p_x}$

The following recursive relationship exists between $\ddot{a}_x$ and $\ddot{a}_{x+1}$.
$\mathbf{\ddot{a}_x = 1 + v * p_x * \ddot{a}_{x+1}}$

Moreover, if we know A_x, the actuarial present value of a whole life insurance policy with benefits payable at the end of the year of death (Reminder: $A_x = {}_{k=0}{}^{\infty}\sum v^{k+1} * {}_k p_x * q_{x+k}$), we can figure out $\ddot{a}_x$ as follows:
$\mathbf{\ddot{a}_x = (1 - A_x)/d}$

Here, d is the **annual rate of discount,** which compares to the annual effective interest rate i as follows: d = i/(1+i). Thus, $\mathbf{\ddot{a}_x = (1+i)(1 - A_x)/i}$**.**

The variance of the present value of a discrete whole-life annuity-due is denoted as $Var(\ddot{a}_{K+1\neg})$ and can be found as follows:

$\mathbf{Var(\ddot{a}_{K+1\neg}) = ({}^2A_x – (A_x)^2)/d^2}$

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1997. Second Edition. Society of Actuaries: Itasca, Illinois. pp. 143-144, 146.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L41-1. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is currently 0.06. Triton the Triceratops is currently 7 years old and takes out a discrete whole life annuity-due paying a benefit of 1 Golden Hexagon (GH) at the beginning of each year. Find the actuarial present value of this annuity.

Solution S3L41-1. We use the formula $\ddot{a}_x = {}_{k=0}{}^{\infty}\Sigma v^k * {}_k p_x$. Since the lifetimes of triceratopses are exponentially distributed, ${}_k p_x = e^{-0.34k}$ for all x. Moreover, $v^k = e^{-0.06k}$. Thus,
$\ddot{a}_7 = {}_{k=0}{}^{\infty}\Sigma e^{-0.06k} * e^{-0.34k} = {}_{k=0}{}^{\infty}\Sigma e^{-0.4k} = 1/(1-e^{-0.4}) =$ $\mathbf{\ddot{a}_7 =}$ **about 3.033244782 GH**.

Problem S3L41-2. The life of a yellow whale has the following survival function associated with it: $s(x) = e^{-0.07x}$. The annual force of interest is currently 0.02. Horatius the Yellow Whale is currently 54 years old and takes out a discrete whole life annuity-immediate paying a benefit of 1 Golden Hexagon (GH) at the beginning of each year. Find the actuarial present value of this annuity.

Solution S3L41-2. We use the formula $a_x = {}_{k=1}{}^{\infty}\Sigma v^k * {}_kp_x$. Since the lifetimes of yellow are exponentially distributed, ${}_kp_x = e^{-0.07k}$ for all x. Moreover, $v^k = e^{-0.02k}$. Thus,
$a_{54} = {}_{k=1}{}^{\infty}\Sigma e^{-0.02k} * e^{-0.07k} = {}_{k=1}{}^{\infty}\Sigma e^{-0.09k} = e^{-0.09}/(1 - e^{-0.09}) =$ **a_{54} = about 10.6186101 GH**.

Problem S3L41-3. A 10-year-old chicken-cricket can obtain a discrete whole life annuity-due paying a benefit of 1 Golden Hexagon (GH) at the beginning of each year for 45 GH. The annual force of interest is 0.07, and only 89% of 10-year-old chicken-crickets survive to age 11. Find the actuarial present value of a discrete whole life annuity-due offering the same conditions to an 11-year-old chicken-cricket.

Solution S3L41-3. We use the formula $\ddot{a}_x = 1 + v*p_x*\ddot{a}_{x+1}$, which we rearrange as follows:
$v*p_x*\ddot{a}_{x+1} = \ddot{a}_x - 1$
$\ddot{a}_{x+1} = (\ddot{a}_x - 1)/(v*p_x)$
For x = 10, we are given that $\ddot{a}_{10} = 45$, $v = e^{-0.07}$, and $p_{10} = 0.89$.
Thus, $\ddot{a}_{11} = (45 - 1)/(e^{-0.07}*0.89) = \ddot{a}_{11}$ = about **53.02287638 GH**. (This means that a chicken-cricket who survives to age 11 is a fortunate chicken-cricket indeed, having gotten over the mortality spike during the tenth year of life.)

Problem S3L41-4. A 15-year-old heptapus (think octopus with seven tentacles) can get a whole life insurance policy with a benefit of 1 Golden Hexagon (GH) payable at the end of the year of death for 0.67 GH. The annual effective rate of interest is 0.04. Find the actuarial present value of a discrete whole life annuity-due paying annual benefits of 1 GH and available to a 15-year-old heptapus.

Solution S3L41-4. We use the formula $\ddot{a}_x = (1+i)(1 - A_x)/i$ for x = 15, i = 0.04, and $A_{15} = 0.67$.
Thus, $\ddot{a}_{15} = (1.04)(1 - 0.67)/0.04 =$ **$\ddot{a}_{15}$ = 8.58 GH**.

Problem S3L41-5. Under an annual force of interest of 0.05, the actuarial present value of a whole life insurance policy with a benefit of 1 Golden Hexagon (GH) payable at the end of the year of death for a 20-year-old squawking fish is 0.44 GH. The second moment of the present value of this policy is 0.31 GH. Find the variance of the present value of a discrete whole life annuity-due paying annual benefits of 1 GH and available to a 20-year-old squawking fish.

Solution S3L41-5. We use the formula $Var(\ddot{a}_{K+1\neg}) = ({}^2A_x - (A_x)^2)/d^2$.
For x = 20, we are given that ${}^2A_{20} = 0.31$ and $A_{20} = 0.44$.
We find $d = r/(1+r) = (e^{\delta} - 1)/e^{\delta} = (e^{0.05} - 1)/e^{0.05} = d = 0.0487705755$
Thus, $Var(\ddot{a}_{K+1\neg}) = (0.31 - (0.44)^2)/0.0487705755^2 =$ **about 48.93698619**.

Section 42

Discrete Temporary Life Annuities

It is possible to find the actuarial present value of a discrete whole life annuity-immediate as follows:

$a_x = (1 - (1+i)A_x)/i$, where i is the annual effective interest rate and A_x is the actuarial present value of a whole life insurance policy with benefits payable at the end of the year of death.

The actuarial present value of a **discrete n-year temporary life annuity-due** is denoted as **$\ddot{a}_{x:n\neg}$** and can be found via the following formula.

$$\ddot{a}_{x:n\neg} = {}_{k=0}{}^{n-1}\Sigma v^k * {}_k p_x$$

The actuarial present value of a **discrete n-year temporary life annuity-immediate** is denoted as **$a_{x:n\neg}$** and can be found via the following formula.

$$a_{x:n\neg} = {}_{k=1}{}^{n}\Sigma v^k * {}_k p_x$$

The following relationships hold:

$$\ddot{a}_{x:n\neg} = 1 + a_{x:n-1\neg}$$

If Y is the present-value random variable for a discrete n-year temporary annuity-due, then

$Var(Y) = [{}^2A_{x:n\neg} - (A_{x:n\neg})^2]/d^2$, where d = r/(1+r) and **$A_{x:n\neg}$** is the actuarial present value of an n-year endowment insurance policy paying one unit in benefits at the end of the year of death.

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1997. Second Edition. Society of Actuaries: Itasca, Illinois. pp. 145-148.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L42-1. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is currently 0.06. Eduardo the Triceratops is currently 14 years old and takes out a discrete 5-year life annuity-due paying a benefit of 1 Golden Hexagon (GH) at the beginning of each year. Find the actuarial present value of this annuity.

Solution S3L42-1. We use the formula $\ddot{a}_{x:n\neg} = {}_{k=0}{}^{n-1}\Sigma v^k * {}_k p_x$. Since the lifetimes of triceratopses are exponentially distributed, ${}_k p_x = e^{-0.34k}$ for all x. Moreover, $v^k = e^{-0.06k}$. Thus, for n = 5,

$\ddot{a}_{14:5\neg} = {}_{k=0}{}^{4}\Sigma e^{-0.06k} * e^{-0.34k} = {}_{k=0}{}^{4}\Sigma e^{-0.4k} = 1 + e^{-0.4} + e^{-0.8} + e^{-1.2} + e^{-1.6} =$ **about 2.62273974 GH**.

Problem S3L42-2. Odaroloc the Colorado Beetle has a whole life insurance policy with benefits payable at the end of the year of death, whose actuarial present value is 0.92 Golden Hexagons (GH). The annual effective interest rate is 0.03. Odaroloc wishes to purchase a discrete whole life annuity-immediate from an actuarially fair insurance provider. How much will he have to pay for the annuity?

Solution S3L42-2. We use the formula $a_x = (1 - (1+i)A_x)/i$, where we are given i = 0.03 and $A_x = 0.92$. Thus, $a_x = (1 - 1.03*0.92)/0.03$ = **a_x = about 1.746666666667 GH**.

Problem S3L42-3. The life of a yellow whale has the following survival function associated with it: $s(x) = e^{-0.07x}$. The annual force of interest is currently 0.02. Wolley the Yellow Whale is currently 34 years old and takes out a discrete 17-year temporary life annuity-immediate paying a benefit of 1 Golden Hexagon (GH) at the end of each year. Find the actuarial present value of this annuity.

Solution S3L42-3. We use the formula $a_{x:n\neg} = {}_{k=1}{}^{n}\Sigma v^k * {}_kp_x$ for n = 17. Since the lifetimes of yellow whales are exponentially distributed, ${}_kp_x = e^{-0.07k}$ for all x. Moreover, $v^k = e^{-0.02k}$. Thus,

$a_{34:17\neg} = {}_{k=1}{}^{17}\Sigma e^{-0.02k} * e^{-0.07k} = {}_{k=1}{}^{17}\Sigma e^{-0.09k} = e^{-0.09}(1 - e^{-0.09*17})/(1 - e^{-0.09})$ = **$a_{34:17\neg}$ = about 8.319302275 GH.**

Problem S3L42-4. A 12-year-old swimming squirrel can get a 4-year discrete temporary life annuity-immediate for 34 Golden Hexagons (GH). It can also get a 15-year discrete temporary life annuity-due for twice the actuarial present value of a 5-year discrete temporary life annuity-due. How much will the swimming squirrel have to pay for the 15-year annuity?

Solution S3L42-4. We use the formula $ä_{x:n\neg} = 1 + a_{x:n-1\neg}$ to find $ä_{12:5\neg} = 1 + a_{12:4\neg} = 1 + 34 = 35$.

We are also given $ä_{12:15\neg} = 2ä_{12:5\neg} = 2*35$ = **$ä_{12:15\neg}$ =70 GH.**

Problem S3L42-5. Elteeb the Colorado Beetle can get a 6-year endowment insurance policy paying one unit in benefits at the end of the year of death for 0.67 Golden Hexagons (GH). The second moment of the present value of this policy is 0.55. The annual effective interest rate is 0.03. Find the variance of the present value of a discrete 6-year temporary annuity-due available to Elteeb.

Solution S3L42-5. We use the formula $Var(Y) = [{}^2A_{x:n\neg} - (A_{x:n\neg})^2]/d^2$ =

$[{}^2A_{x:n\neg} - (A_{x:n\neg})^2]/(i/(1+i))^2$

We are given i = 0.03, ${}^2A_{x:7\neg} = 0.55$, and $A_{x:7\neg} = 0.67$.

Thus, $Var(Y) = (0.55 - 0.67^2)/(0.03/1.03)^2$ = **Var(Y) = about 119.1744333**.

Section 43

Discrete Deferred Life Annuities and Certain and Life Annuities

The actuarial present value of a **discrete n-year deferred whole life annuity-due** is denoted as $_{n|}\ddot{a}_x$ and can be found via the following formula.

$$_{n|}\ddot{a}_x = \sum_{k=n}^{\infty} v^k * {}_kp_x$$

The actuarial present value of a **discrete n-year deferred whole life annuity-immediate** is denoted as $_{n|}a_x$ and can be found via the following formula.

$$_{n|}a_x = \sum_{k=n+1}^{\infty} v^k * {}_kp_x$$

The actuarial present value of a **discrete n-year certain and whole life annuity-due** is denoted as $\ddot{a}_{\overline{x:n\neg}}$ (with a line drawn *directly over* the subscripts whenever possible) and can be found via the following formula.

$$\ddot{a}_{\overline{x:n\neg}} = \ddot{a}_{n\neg} + \sum_{k=n}^{\infty} v^k * {}_kp_x = \ddot{a}_{n\neg} + {}_{n|}\ddot{a}_x$$

(We recall from annuity theory that $\ddot{a}_{n\neg} = (1+i)(1 - (1+i)^{-n})/i$, where i is the annual effective interest rate.)

The actuarial present value of a **discrete n-year certain and whole life annuity-immediate** is denoted as $a_{\overline{x:n\neg}}$ (with a line drawn *directly over* the subscripts whenever possible) and can be found via the following formula.

$$a_{\overline{x:n\neg}} = a_{n\neg} + \sum_{k=n+1}^{\infty} v^k * {}_kp_x = a_{n\neg} + {}_{n|}a_x$$

(We recall from annuity theory that $a_{n\neg} = (1 - (1+i)^{-n})/i$.)

The **actuarial accumulated value** of an n-year temporary annuity-due paying 1 every year while life (x) survives is denoted by $\ddot{s}_{x:n\neg}$ ("s" with two dots directly over it. The proxy symbol used here is the closest I could find.). It can be found as follows:

$$\ddot{s}_{x:n\neg} = \ddot{a}_{x:n\neg}/{}_nE_x = \sum_{k=0}^{n+1}(1/{}_{n-k}E_{x+k})$$

We recall that $_nE_x = v^n * {}_np_x$.

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1997. Second Edition. Society of Actuaries: Itasca, Illinois. pp. 145-148.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L43-1. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is currently 0.06. Montezuma the Triceratops is currently 5 years old and takes out a discrete 2-year deferred whole life annuity-due paying a benefit of 1 Golden Hexagon (GH) at the beginning of each year. Find the actuarial present value of this annuity.

Solution S3L43-1. We use the formula $_{n|}\ddot{a}_x = {}_{k=n}{}^{\infty}\Sigma v^k * {}_kp_x$. . Since the lifetimes of triceratopses are exponentially distributed, $_kp_x = e^{-0.34k}$ for all x. Moreover, $v^k = e^{-0.06k}$. Thus, for n = 2, $_{2|}\ddot{a}_5 = {}_{k=2}{}^{\infty}\Sigma e^{-0.06k} * e^{-0.34k} = {}_{k=2}{}^{\infty}\Sigma e^{-0.4k} = e^{-0.8}/(1 - e^{-0.4})$ = **about 1.362924736 GH**.

Problem S3L43-2. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is currently 0.06. Montezuma II the Triceratops is currently 5 years old and takes out a discrete 2-year certain and whole life annuity-due paying a benefit of 1 Golden Hexagon (GH) at the beginning of each year. Find the actuarial present value of this annuity.

Solution S3L43-2. We use the formula $\ddot{a}^{---}{}_{x:n\neg} = \ddot{a}_{n\neg} + {}_{k=n}{}^{\infty}\Sigma v^k * {}_kp_x = \ddot{a}_{n\neg} + {}_{n|}\ddot{a}_x$. We know from Solution S3L43-2 that $_{2|}\ddot{a}_5 = 1.362924736$.We now find $\ddot{a}_{2\neg} = (1+i)(1 - (1+i)^{-2})/i$.
We find $i = e^{0.06} - 1$. Thus, $\ddot{a}_{2\neg} = (e^{0.06})(1 - (e^{-0.12}))/(e^{0.06} - 1) = \ddot{a}_{2\neg}$ = about 1.941764535.
Thus, $\ddot{a}^{---}{}_{5:2\neg}$ = 1.941764535 + 1.362924736 = **about 3.30468927 GH**.

Problem S3L43-3. The life of a yellow whale has the following survival function associated with it: $s(x) = e^{-0.07x}$. The annual force of interest is currently 0.02. Oywell the Yellow Whale is currently 15 years old and takes out a discrete 7-year deferred whole life annuity-immediate paying a benefit of 1 Golden Hexagon (GH) at the end of each year. Find the actuarial present value of this annuity.

Solution S3L43-3. We use the formula $_{n|}a_x = {}_{k=n+1}{}^{\infty}\Sigma v^k * {}_kp_x$ for n = 7. Since the lifetimes of yellow whales are exponentially distributed, $_kp_x = e^{-0.07k}$ for all x. Moreover, $v^k = e^{-0.02k}$. Thus, $_{7|}a_{15} = {}_{k=8}{}^{\infty}\Sigma e^{-0.02k} * e^{-0.07k} = {}_{k=8}{}^{\infty}\Sigma e^{-0.09k} = e^{-0.72}/(1 - e^{-0.09}) = {}_{7|}a_{15}$ = **about 5.655384677 GH**.

Problem S3L43-4. The life of a yellow whale has the following survival function associated with it: $s(x) = e^{-0.07x}$. The annual force of interest is currently 0.02. Oywell II the Yellow Whale is currently 15 years old and takes out a discrete 7-year certain and whole life annuity-immediate paying a benefit of 1 Golden Hexagon (GH) at the end of each year. Find the actuarial present value of this annuity.

Solution S3L43-4. We use the formula $a^{---}{}_{x:n\neg} = a_{n\neg} + {}_{k=n+1}{}^{\infty}\Sigma v^k * {}_kp_x = a_{n\neg} + {}_{n|}a_x$.
From Solution S3L43-3, it is known that $_{7|}a_{15} = 5.655384677$.
We find $i = e^{0.02} - 1$. Thus, $a_{7\neg} = (1 - e^{-0.14})/(e^{0.02} - 1)$ = about 6.466985083 GH.
Hence, $a^{---}{}_{15:7\neg}$ = 5.655384677 + 6.466985083 = **about 12.12236976 GH.**

Problem S3L43-5. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is currently 0.06. Eduardo the

Triceratops is currently 14 years old and takes out a discrete 5-year life annuity-due paying a benefit of 1 Golden Hexagon (GH) at the beginning of each year. Assuming that Eduardo survives to the end of the 5-year period, what will be the actuarial accumulated value of his annuity then?

Solution S3L43-5. We use the formula $\ddot{s}_{x:n\neg} = \ddot{a}_{x:n\neg}/{}_nE_x$.
To find $\ddot{a}_{14:5\neg}$, we use the formula $\ddot{a}_{x:n\neg} = {}_{k=0}{}^{n-1}\Sigma v^k * {}_kp_x$. Since the lifetimes of triceratopses are exponentially distributed, ${}_kp_x = e^{-0.34k}$ for all x.
Moreover, $v^k = e^{-0.06k}$. Thus, for n = 5,
$\ddot{a}_{14:5\neg} = {}_{k=0}{}^4\Sigma e^{-0.06k} * e^{-0.34k} = {}_{k=0}{}^4\Sigma e^{-0.4k} = 1 + e^{-0.4} + e^{-0.8} + e^{-1.2} + e^{-1.6}$ = about 2.62273974 GH.
We find ${}_5E_{14} = v^5 * {}_5p_x = e^{-0.06*5}e^{-0.34*5} = e^{-0.4*5} = e^{-2}$.
Thus, our desired answer is $2.62273974e^2$ = **about 19.37957107 GH**.

Section 44

The Equivalence Principle and Fully Continuous Benefit Premiums for Whole, Term, and Endowment Life Insurance Policies

A life insurance policy is frequently purchased by means of a life annuity consisting of **contract premiums** that are specified in the insurance contract. The determination of contract premiums is based on some manner of **premium principle.**

The **equivalence principle** is a kind of premium principle. It requires that the expected loss of an insurance policy to the insurer be equal to zero. If L is the loss random variable for the insurer, then, under the equivalence principle,

E[L] = 0 and **E[present value of benefits] = E[present value of benefit premiums].**

(Note: the term **benefit premiums** will be used to refer to contract premiums satisfying the equivalence principle.)

We will first work with **fully continuous premiums**, where level annual benefit premiums (of the same amount per year) are paid on a continuous basis.

For a whole life insurance policy that pays a unit in benefits upon the death of life (x) and that has actuarial present value $\bar{A}_x$, the annual fully continuous benefit premium is denoted as $\mathbf{P^-(\bar{A}_x)}$ (with the line drawn *directly over* the "P" whenever possible) and can be found as follows:

$\mathbf{P^-(\bar{A}_x) = \bar{A}_x/\bar{a}_x}$

Recall: $\bar{A}_x = {}_0{}^{\infty}\int v^t * {}_tp_x * \mu_x(t)dt$ and $\bar{A}_x = \mu/(\mu + \delta)$ for constant force of mortality μ and constant force of interest δ. Moreover, $\bar{a}_x = {}_0{}^{\infty}\int v^t * {}_tp_x * dt$ and $\bar{a}_x = 1/(\mu + \delta)$ for constant force of mortality μ and constant force of interest δ.

Thus, for constant force of mortality μ and constant force of interest δ, $P^-(\bar{A}_x) =$

$\bar{A}_x/\bar{a}_x = [\mu/(\mu + \delta)]/[\ 1/(\mu + \delta)] =$ $\mathbf{P^-(\bar{A}_x) = \mu.}$

The variance of the loss Var[L] under this kind of benefit premium can be found as follows:
$\mathbf{Var[L] = ({}^2\bar{A}_x - (\bar{A}_x)^2)/(\delta\bar{a}_x)^2}$

For an n-year term life insurance policy that pays a unit in benefits upon the death of life (x) and that has actuarial present value $\bar{A}^1_{x:n\neg}$, the fully continuous annual benefit premium is denoted as

$\mathbf{P^-(\bar{A}^1_{x:n\neg}) = \bar{A}^1_{x:n\neg} / \bar{a}_{x:n\neg}}$

Recall: $\bar{A}^1_{x:n\neg} = {}_0{}^{\infty}\int z_t * f_T(t)dt = {}_0{}^{n}\int v^t * {}_tp_x * \mu_x(t)dt$ and $\bar{a}_{x:n\neg} = {}_0{}^{n}\int v^t * {}_tp_x * dt$.

For an n-year endowment insurance policy that pays a unit in benefits upon the death of life (x) and that has actuarial present value $\bar{A}_{x:n\neg}$, the fully continuous annual benefit premium is denoted as

$\mathbf{P^-(\bar{A}_{x:n\neg}) = \bar{A}_{x:n\neg} / \bar{a}_{x:n\neg}}$.

Recall: $\bar{A}_{x:n\neg} = {}_0{}^{n}\int v^t * {}_tp_x * \mu_x(t)dt + v^n * {}_np_x$.

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1997. Second Edition. Society of Actuaries: Itasca, Illinois. pp. 167-173.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L44-1. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is 0.07. François the Triceratops is currently 3 years old has a whole life insurance policy, which will pay him 1 Triceratops Currency Unit (TCU) upon death. Under the equivalence principle, what is the annual fully continuous level benefit premium that François will pay for this policy?

Solution S3L44-1. Since triceratops lifetimes are exponentially distributed, triceratopses have a constant force of mortality of 0.34. We thus use the formula $\mathbf{P^-(\bar{A}_x) = \mu = 0.34}$ **TCU.**

Problem S3L44-2. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is 0.07. François the Triceratops is currently 3 years old has a whole life insurance policy, which will pay him 1 Triceratops Currency Unit (TCU) upon death. Under the equivalence principle, what is the variance of the loss to the insurer for this policy?

Solution S3L44-2. We use the formula $Var[L] = ({}^2\bar{A}_x - (\bar{A}_x)^2)/(\delta\bar{a}_x)^2$. Since triceratopses exhibit a constant force of mortality, we have $\bar{A}_x = \mu/(\mu + \delta) = 0.34/0.41 = 34/41$. Moreover, $\bar{a}_x = 1/(\mu + \delta) = 1/0.41 = 100/41$. Now we need to find ${}^2\bar{A}_x = \mu/(\mu + 2\delta)$ for a constant force of mortality.

Thus, ${}^2\bar{A}_x = 0.34/0.48 = 34/48$.

Hence, $Var[L] = (34/48 - (34/41)^2)/(0.07*100/41)^2$ = **17/24 = about 0.70833333333**.

Problem S3L44-3. The life of a giant pin-striped cockroach has the following survival function associated with it: $s(x) = 1 - x/94$, for $0 \le x \le 94$ and 0 otherwise. Hcaorkcoc the Giant Pin-Striped Cockroach is currently 56 years old and has a whole life insurance policy which will pay

10 Golden Hexagons (GH) upon death. The annual force of interest is 0.02. Under the equivalence principle, what is the annual fully continuous level benefit premium that Hcaorkcoc will pay for this policy?

Solution S3L44-3. Here, $P^{-}(\bar{A}_x) = 10\bar{A}_x/\bar{a}_x$.

We use the formula $\bar{A}_x = {}_0^{\infty}\int v^t * {}_tp_x * \mu_x(t)dt$.

We want to find $10\bar{A}_{56}$. Since no giant pin-striped cockroach lives past the age of 94, our integral's upper bound will be 94-56 = 38, because Hcaorkcoc will not live for more than 38 additional years.

We find ${}_tp_{56} = s(x + t)/s(x) = s(56 + t)/s(56) = (1 - (56+t)/94)/(1 - 56/94) = (38 - t)/38$

We find $\mu_{56}(t) = -s'(x)/s(x) = (1/94)/(1 - x/94) = (-1/94)/[(94 - x)/94] = 1/(94 - x) =$

$1/(94 - (56+t)) = 1/(38 - t)$. Conveniently enough, ${}_tp_{56} * \mu_{56}(t) = ((38 - t)/38)(1/(38 - t)) = 1/38$.
We find $v^t = e^{-0.02t}$.

Thus, $10\bar{A}_{56} = 10*{}_0^{38}\int e^{-0.02t}*(1/38)dt$

$10\bar{A}_{56} = 10*(-50/38)e^{-0.02t} \Big|_0^{38} = 10(50/38)(1 - e^{-0.76}) = 10\bar{A}_{56}$ = about 7.004389118 GH.

Now we find $\bar{a}_{56} = \bar{a}_{56:38\neg} = {}_0^{38}\int v^t * {}_tp_x * dt$, since Hcaorkcoc will not live for more than 38 additional years.

Thus, $\bar{a}_{56} = {}_0^{38}\int e^{-0.02t}*(38 - t)/38*dt = {}_0^{38}\int e^{-0.02t}dt - {}_0^{38}\int (1/38)te^{-0.02t}dt = 26.61667865 - 11.63862424$

Thus, $\bar{a}_{56}$ = 14.97805441 GH and so $P^{-}(\bar{A}_{56}) = 10\bar{A}_{56}/\bar{a}_{56} = 7.004389118/14.97805441 =$ **about 0.4676434553 GH.**

Problem S3L44-4. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is 0.07. Jerry the Triceratops is currently 3 years old has a 6-year term life insurance policy, which will pay him 1 Triceratops Currency Unit (TCU) upon death. Under the equivalence principle, what is the annual fully continuous level benefit premium that Jerry will pay for this policy?

Solution S3L44-4.

We use the formula $P^{-}(\bar{A}^1_{x:n\neg}) = \bar{A}^1_{x:n\neg} / \bar{a}_{x:n\neg}$.

We find $\bar{A}^1_{x:n\neg} = {}_0^{n}\int v^t * {}_tp_x * \mu_x(t)dt$.

Since triceratopses exhibit a constant force of mortality, we have $\mu_x(t) = 0.34$. Moreover, since triceratops lifetimes are exponentially distributed, we have ${}_tp_x = e^{-0.34t}$. Moreover, $v^t = e^{-0.07t}$.

Thus, $\bar{A}^1_{3:6\neg} = {}_0{}^6\int 0.34e^{-0.07t}*e^{-0.34t}dt = {}_0{}^6\int 0.34e^{-0.41t}dt = (-34/41)e^{-0.41t} \mid_0^6 = (34/41)(1 - e^{-0.41*6}) =$ about 0.7584197968 TCU.

We find $\bar{a}_{3:6\neg} = {}_0{}^n\int v^t*{}_tp_x*dt = {}_0{}^6\int e^{-0.07t}*e^{-0.34t}dt = {}_0{}^6\int e^{-0.41t}dt = (-100/41)e^{-0.41t} \mid_0^6$

$= (100/41)(1 - e^{-0.41*6})$

But $P^-(\bar{A}^1_{3:6\neg}) = \bar{A}^1_{3:6\neg} / \bar{a}_{3:6\neg} = [(34/41)(1 - e^{-0.41*6})]/[(100/41)(1 - e^{-0.41*6})] = 34/100 =$ **0.34 = μ**.

Note: We have just seen an example showing that for a constant force of mortality, it does not matter whether the life insurance policy in question is a whole or a term life insurance policy. The benefit premium will be μ in either case.

Problem S3L44-5. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is 0.09. Hasdrubal the Triceratops is currently 7 years old has a 2-year endowment insurance policy, which will pay him 1 Triceratops Currency Unit (TCU) upon death. Under the equivalence principle, what is the annual fully continuous level benefit premium that Hasdrubal will pay for this policy?

Solution S3L44-5. We use the formula $P^-(\bar{A}_{x:n\neg}) = \bar{A}_{x:n\neg} / \bar{a}_{x:n\neg}$.
We first find $\bar{A}_{x:n\neg} = {}_0{}^n\int v^t*{}_tp_x*\mu_x(t)dt + v^n*{}_np_x$.
We first need to find
$\bar{A}^1{}_{x:n\neg} = {}_0{}^n\int v^t*{}_tp_x*\mu_x(t)dt$ and
$A^1{}_{x:n\neg} = v^n*{}_np_x$
We know that $v = e^{-0.09}$, $x = 7$, and $n = 2$.
We find ${}_tp_x = s(x+t)/s(x) = s(7+t)/s(7) = e^{-0.34t}$. So ${}_np_x = {}_2p_7 = e^{-0.34*2}$.
We find $\mu_x(t) = -s'(x)/s(x) = 0.34e^{-0.34t}/e^{-0.34t} = 0.34$.
Thus, $\bar{A}^1{}_{7:2\neg} = {}_0{}^2\int e^{-0.09t}*e^{-0.34t}*0.34dt = {}_0{}^2\int 0.34e^{-0.43t}dt = (-34/43)e^{-0.43t} \mid_0^2 = (34/43)(1 - e^{-0.43*2}) =$
$\bar{A}^1{}_{7:2\neg} = 0.4561044$
Also, $A^1{}_{7:2\neg} = e^{-0.09*2}e^{-0.34*2} = e^{-0.86} = A^1{}_{7:2\neg} = 0.4231620823$.
Thus, $\bar{A}_{7:2\neg} = A^1{}_{7:2\neg} + \bar{A}^1{}_{7:2\neg} = 0.4231620823 + 0.4561044 = \bar{A}_{7:2\neg} =$ about 0.8792664823.
Now we find $\bar{a}_{x:n\neg} = {}_0{}^n\int v^t*{}_tp_x*dt$.
$\bar{a}_{7:2\neg} = {}_0{}^2\int e^{-0.09t}*e^{-0.34t}dt = (-100/43)e^{-0.43t} \mid_0^2 = (100/43)(1 - e^{-0.43*2}) = 1.341483529$ GH.
Thus, $P^-(\bar{A}_{7:2\neg}) = \bar{A}_{7:2\neg} / \bar{a}_{7:2\neg} = 0.8792664823/1.341483529 =$ **$P^-(\bar{A}_{7:2\neg}) = 0.6554433677$ GH**.

Note: Even though the force of mortality in the above problem was constant, we have seen an example where the fully continuous benefit for an *endowment insurance* policy is not equal to μ.

Section 45

Fully Continuous Benefit Premiums for Pure Endowments and Exam-Style Questions on Life Insurance and Life Annuities

This section will be a mix of exam-style questions on previously covered material and questions covering new material.

We continue to examine fully continuous benefit premiums under the equivalence principle for various kinds of life insurance policies.

For an n-year pure endowment which pays a benefit of 1 to life x, the fully continuous benefit premium $\mathbf{P}^{-}({}_nE_x)$ can be found as follows.

$\mathbf{P}^{-}({}_nE_x) = {}_nE_x / \bar{a}_{x:n¬}$

Recall: ${}_nE_x = v^n * {}_np_x$ and $\bar{a}_{x:n¬} = {}_0{}^n\int v^t * {}_tp_x * dt$.

Moreover, $\bar{a}_{x:n¬} = (1 - e^{-(\delta+\mu)n})/(\mu + \delta)$ for constant force of mortality μ and constant force of interest δ.

In Section 37, we introduced the formula

$\mathbf{\bar{a}_x = \bar{a}_{x:1¬} + vp_x * \bar{a}_{x+1}}$

An analogous formula holds for any number of years n:

$\mathbf{\bar{a}_x = \bar{a}_{x:n¬} + v^n * {}_np_x * \bar{a}_{x+n}}$

But $v^n * {}_np_x$ is the same as ${}_nE_x$. Thus,

$\mathbf{\bar{a}_x = \bar{a}_{x:n¬} + {}_nE_x * \bar{a}_{x+n}}$

The recursion relation

$A^1_{x:n¬} = vq_x + vp_x * A^1_{x+1:n-1¬}$ from Section 34 can also be generalized to m time periods:

$\mathbf{A^1_{x:n¬} = A^1_{x:m¬} + v^m * {}_mp_x * A^1_{x+m:n-m¬}}$

Moreover, if one is given a life table and required to calculate the actuarial present values of term life insurance policies, the following identity is useful.

$\mathbf{A^1{}_{x:m\neg} = {}_{n=0}{}^{m}\Sigma v^n({}_{n-1\,|}\,q_x)}$

Moreover, the following relationship holds between the actuarial present values of term and whole life insurance policies.

$\mathbf{A_x = A^1{}_{x:m\neg} + v^m * {}_m p_x * A_{x+m}}$

Some of the problems in this section were designed to be similar to problems from past versions of the Casualty Actuarial Society's Exam 3L and the Society of Actuaries' Exam MLC. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1997. Second Edition. Society of Actuaries: Itasca, Illinois. pp. 173.

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Problem S3L45-1. The life of a triceratops has the following survival function associated with it: $s(x) = e^{-0.34x}$. The annual force of interest in Triceratopsland is 0.09. Otto the Triceratops is currently 7 years old has a 2-year pure endowment, which pays a benefit of 1 Triceratops Currency Unit (TCU). Under the equivalence principle, what is the annual fully continuous level benefit premium that Hasdrubal will pay for this policy?

Solution S3L45-1. We use the formula $\bar{P}({}_nE_x) = {}_nE_x/\ \bar{a}_{x:n\neg}$.

Since triceratopses exhibit a constant force of mortality, we have $\mu_x(t) = 0.34$ and therefore $\bar{a}_{x:n\neg} = (1 - e^{-(\delta+\mu)n})/(\mu + \delta)$ For n = 2, $\bar{a}_{7:2\neg} = (1 - e^{-(0.43)2})/(0.43) =$

$(1 - e^{-0.86})/(0.43)$ = about 1.341483529 TCU.

Moreover, since triceratops lifetimes are exponentially distributed, we have ${}_tp_x = e^{-0.34t}$.

Moreover, $v^t = e^{-0.07t}$.

Thus, ${}_2E_7 = v^n * {}_np_x = e^{-0.07*2}e^{-0.34*2} = e^{-0.86}$.

Thus, $P({}_2E_7) = e^{-0.86}/1.341483529 = P({}_2E_7)$ = **about 0.3154433677 TCU**.

Problem S3L45-2.

Similar to Question 38 from the Casualty Actuarial Society's Fall 2007 Exam 3.

You know the following about the actuarial present values of continuous whole life annuities and a 5-year pure endowment.

1) ${}_5E_4 = 0.35$

2) $\bar{a}_4 = 3.25$

3) $\bar{a}_9 = 7.776$

Find $\bar{a}_{4:5\neg}$, the value of a 5-year continuous temporary life annuity for life (4).

Solution S3L45-2. We use the formula $\bar{a}_x = \bar{a}_{x:n\neg} + {}_nE_x*\bar{a}_{x+n}$, which we rearrange thus:

$\bar{a}_{x:n\neg} = \bar{a}_x - {}_nE_x*\bar{a}_{x+n}$

Thus, $\bar{a}_{4:5\neg} = 3.25 - 0.35*7.776 =$ $\mathbf{\bar{a}_{4:5\neg} = 0.5284}$.

Problem S3L45-3.

Similar to Question 34 from the Casualty Actuarial Society's Fall 2006 Exam 3.

This question uses the **Illustrative Life Table, which can be found here**.

Mortality occurs according to the Illustrative Life Table. There is a 6-year term life insurance policy on a life currently aged 54.

For the next three years, the annual effective interest rate will be 0.07.

Thereafter, the annual effective interest rate will be 0.09.

Find $A^1_{54:6\neg}$, the actuarial present value of term life insurance policy in question.

Solution S3L45-3.

Since the interest rates are different for different time periods, we will need to use the formula $A^1_{x:n\neg} = A^1_{x:m\neg} + v^m*{}_mp_x*A^1_{x+m:n-m\neg}$.

Here, $A^1_{54:6\neg} = A^1_{54:3\neg} + v^3*{}_3p_{57}*A^1_{57:3\neg}$.

The "v" in this formula is the three-year discount factor using the interest rate 0.09, the rate that takes effect 3 years from now. Thus, $v^3 = 1/1.09^3$

Moreover, we can use the Illustrative Life Table to find

${}_3p_{57} = l_{60}/l_{57} = 8188074/8479908 = 0.9655852398$.

Thus, $v^3*{}_3p_{57} = 0.9655852398/1.09^3 = 0.7456089708$.

Hence, we are left with $A^1_{54:6\neg} = A^1_{54:3\neg} + 0.7456089708*A^1_{57:60\neg}$.

Now we find $A^1_{54:3\neg}$ using the formula $A^1_{x:m\neg} = {}_{n=0}{}^{m}\Sigma v^n({}_{n-1\,|}\,q_x)$.

Thus, $A^1_{54:3\neg} = vq_{54} + v^2({}_{1\,|}\,q_{54}) + v^3({}_{2\,|}\,q_{54})$

$A^1_{54:3\neg} = 0.00824/1.09 + (1- 0.00824)(0.00896)/1.09^2 + (1- 0.00824)(1 - 0.00896)(0.00975)/1.09^3$
$= A^1_{54:3\neg} = 0.0224387938$.

Therefore, $A^1_{54:6\neg} = 0.0224387938 + 0.7456089708*A^1_{57:3\neg}$.

Now we need to find $A^1_{57:3\neg}$. We use the formula $A_x = A^1_{x:m\neg} + v^m*{}_mp_x*A_{x+m}$.

In this case,

$A_{57} = A^1_{57:3\neg} + v^3*{}_3p_{57}*A_{60}$.

$A^1_{57:3\neg} = A_{57} - v^3*{}_3p_{57}*A_{60}$

We already know that $v^3*{}_3p_{57} = 0.7456089708$.

From the Illustrative Life Table, we know that $A_{57} = 0.32984$ and $A_{60} = 0.36913$.

Thus, $A^1_{57:3\neg} = 0.32984 - 0.7456089708*0.36913 = A^1_{57:3\neg} = 0.0546133606$.

Therefore, $A^1_{54:6\neg} = 0.0224387938 + 0.7456089708*0.0546133606 =$

$A^1_{54:6\neg}$ = about 0.0631590054.

Problem S3L45-4.

Similar to Question 39 from the Casualty Actuarial Society's Fall 2006 Exam 3.

Real Risky Insurance Co. collects a premium P at the beginning of the year. It also has a surplus of \$56 at the beginning of the year. All funds that the company has at the beginning of the year are invested at an annual effective interest rate of 20%.

The following losses may occur at the end of the year:
Loss of \$0 with probability 0.2
Loss of \$89 with probability 0.5
Loss of \$78000 with probability 0.3.

What is the smallest premium P that Real Risky Insurance Co. can charge in order to keep its probability of ruin during this year at most 30%?

Note: This question requires only a basic knowledge of probability theory and interest rates. The rest is just a matter of thinking the problem through and applying the theory to the given context. Try solving the problem yourself before you examine the solution here.

Solution S3L45-4. To keep the probability of ruin during this year at most 30%, the company needs to make sure that a loss of $89 will not bankrupt it. (A loss of $0 will already not bankrupt it.) Thus, it needs to have at least $89 in funds at the end of the year.

The company has $(56 + P) to invest at 20% interest.
Thus, at the lowest value of P, 89 = 1.2(56 + P), and so P = 89/1.2 - 56 =
P = about $18.166666667

Problem S3L45-5.

Similar to Question 39 from the Casualty Actuarial Society's Fall 2006 Exam 3.

Ridiculously Risky Insurance Co. collects a premium of $40 at the beginning of the year. It also has a surplus of $56 at the beginning of the year. Because of excessive risk-taking in this world, all financial markets have collapsed, and thus the annual effective interest rate is 0.

The following losses may occur at the end of the year:
Loss of $0 with probability 0.2
Loss of $89 with probability 0.5
Loss of $100 with probability 0.3.

What is the probability that Ridiculously Risky Insurance Co. will avoid bankruptcy for the next 2 years?

Note: This question requires only a basic knowledge of probability theory and interest rates. The rest is just a matter of thinking the problem through and applying the theory to the given context. Try solving the problem yourself before you examine the solution here.

Solution S3L45-5.

During the first year, the company has 56 + 40 = $96 to pay losses with.

It will only be bankrupted during the first year if its loss is $100.

If the year 1 loss is 0, then at the beginning of year 2, the company will collect $40 and will have $136, thus rendering it solvent in the event of *any* loss in year 2.

This is one component of the desired probability and is equal to 0.2.

If the year 1 loss is 89, then the company will have 96 - 89 + 40 = $47 at the beginning of year 2. It then will only survive year 2 if the year 2 loss is 0.

This is the second component of the desired probability and is equal to 0.5*0.2 = 0.1.

Thus, the total probability of the company surviving for the next two years is 0.2 + 0.1 = **0.3.**

Section 46

Exam Style Questions on Life Table Functions for Independent Lives and Multiple Causes of Decrement

When there are multiple causes of decrement (i.e. some kind of failure or bad event happening to the entity in question), we can express life table functions pertaining to *all* causes of decrement with the superscript $^{(\tau)}$. If assign numbers 1 through n to each of the causes of decrement, then we can express life table functions pertaining some cause of decrement I with the superscript $^{(i)}$, where i can be an integer from 1 through n.

If you are given each ${}_mq_x^{(i)}$, then the following holds: $\mathbf{{}_mq_x^{(\tau)} = {}_{i=1}^{n}\sum {}_mq_x^{(i)}}$.

Moreover, if you are given each $\mu_x^{(i)}$, then the following holds: $\boldsymbol{\mu_x^{(\tau)} = {}_{i=1}^{n}\sum\mu_x^{(i)}}$.

The following useful relationship holds between any $\mathbf{{}_np_x}$ and $\boldsymbol{\mu_{x+t}}$:

$$\mathbf{{}_np_x = \exp[-{}_0^n\textstyle\int\mu_{x+t}dt]}.$$

Some of the problems in this section were designed to be similar to problems from past versions of the Casualty Actuarial Society's Exam 3L and the Society of Actuaries' Exam MLC. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1997. Second Edition. Society of Actuaries: Itasca, Illinois. pp. 308-311.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L46-1.

Similar to Question 14 from the Casualty Actuarial Society's Fall 2006 Exam 3.

Two independent lives are currently aged 70 and 90. Mortality follows DeMoivre's Law with ω = 150. (This is the future - when life expectancy has been increased dramatically due to technological advances.)

Let ${}_{30|10}\varphi_{70:90}$ be the probability that the first of the two deaths occurs between 30 and 40 years from now.

Let ${}_{30|10}\lambda_{70:90}$ be the probability that the last of the two deaths occurs between 30 and 40 years from now.

Find ${}_{30|10}\lambda_{70:90}$ - ${}_{30|10}\varphi_{70:90}$.

Solution S3L46-1.

DeMoivre's Law corresponds to the *uniform distribution*, with ω being the limited age.

A life aged 70 has probability 1/(150-70) = 1/80 of dying each year.

A life aged 90 has probability 1/(150-90) = 1/60 of dying each year.

We find ${}_{30|10}\varphi_{70:90}$:

For a first death to occur between 30 and 40 years from now, it follows that both lives must have survived during the next 30 years. The probability that this is the case is

(50/80)(30/60) = 0.3125

If both lives survive for 30 years, the probability of both surviving another 10 years is (40/50)(20/30) = 8/15.

The probability that at least one of the survivors to age 30 will perish between 30 and 40 years from now is 1 - 8/15 = 7/15.

Therefore, ${}_{30|10}\varphi_{70:90}$ = 0.3125(7/15) = 7/48

Now we find ${}_{30|10}\lambda_{70:90}$:

${}_{30|10}\lambda_{70:90}$ is the complement of the sum of the following probabilities.

P(both lives die before 30 years from now) = (30/80)(30/60) = 0.1875

P(both lives die after 40 years from now) = (40/80)(20/60) = 0.166666667

P(no lives die before 30 years from now and only one life dies between 30 and 40 years from now) = 0.3125*(10/50)(20/30) + 0.3125*(40/50)(10/30) = 0.125

P(one life dies before 30 years from now and one life dies after 40 years from now) = (30/80)(20/60) + (40/80)(30/60) = 0.375

Thus, ${}_{30|10}\lambda_{70:90}$ = 1 - 0.1875 - 0.166666667 - 0.125 - 0.375 = 0.1458333333333 = 7/48.

Thus, ${}_{30|10}\lambda_{70:90}$ - ${}_{30|10}\varphi_{70:90}$ = 7/48 - 7/48 = **0**.

Problem S3L46-2.

Similar to Question 15 from the Casualty Actuarial Society's [Fall 2006 Exam 3](#).

Two government departments are operated by Bureaucrat X and Bureaucrat Y. Department A only operates so long as both bureaucrats are alive. Department B operates so long as at least one bureaucrat is alive. Bureaucrat mortality is independent for each bureaucrat and exhibits the property that ${}_tq_0 = t^2/400$ for $t < 20$. Find the absolute value of the difference in the expected operational lifetimes of Departments A and B.

Solution S3L46-2.

We use the following formula from Section 10: $\mathring{e}_x = {}_0{}^{\infty}\int {}_tp_x dt$. Here, the upper bound of our integral is 20, since no bureaucrat will survive past 20 years from now.

Since ${}_tq_0 = t^2/400$, ${}_tp_0 = 1 - t^2/400$.

For Department A, the relevant function to be integrated is $({}_tp_0)^2$, since *both* bureaucrats must be alive for the department to function.
Thus, for Department A, $E(A) = {}_0{}^{20}\int({}_tp_0)^2dt = {}_0{}^{20}\int(1 - t^2/400)^2dt = 10.6666666667$ years.

We know from basic probability theory that (X or Y) = X + Y - (X and Y).

Thus, E(X or Y) = E(X) + E(Y) - E(X and Y).

Department B will keep operating if either X or Y are alive. Thus, the expected lifetime for this department will be ($\mathring{e}_0$ for X) + ($\mathring{e}_0$ for Y) - ($\mathring{e}_0$ for X and Y).

We already found ($\mathring{e}_0$ for X and Y) - the expected lifetime of department A - to be 10.6666666667 years.

We also know that ($\mathring{e}_0$ for X) = ($\mathring{e}_0$ for Y) = ${}_0{}^{20}\int(1 - t^2/400)dt = 13.333333333$ years.

Hence, E(B) = 2*13.333333333 - 10.6666666667 = 16 years.

We want to find E(B) - E(A) = 16 - 10.6666666667 = **16/3 = 5.333333333 years.**

Problem S3L46-3.

Similar to Question 16 from the Casualty Actuarial Society's [Fall 2006 Exam 3](#).

Pink slugs aged x can perish from Bad Thing 1 or Bad Thing 2. The following 2-decrement table expresses the probabilities associated with perishing from each Bad Thing.

x........ $q_x^{(1)}$........$q_x^{(2)}$

2.......0.09........0.04

3.......0.14........0.02

4.......0.01........0.04

5.......0.07........0.10

Use this 2-decrement table to find ${}_4q_2^{(\tau)}$.

Solution S3L46-3.

We use the formula ${}_mq_x^{(\tau)} = {}_{i=1}{}^{n}\sum {}_mq_x^{(i)}$ to find the following:

$q_2^{(\tau)} = 0.09 + 0.04 = 0.13$

$q_3^{(\tau)} = 0.14 + 0.02 = 0.16$

$q_4^{(\tau)} = 0.01 + 0.04 = 0.05$

$q_5^{(\tau)} = 0.07 + 0.10 = 0.17$

Now to find ${}_4q_2^{(\tau)}$, we need to find $1 - {}_4p_2^{(\tau)} = 1 - (p_2^{(\tau)})(p_3^{(\tau)})(p_4^{(\tau)})(p_5^{(\tau)}) =$

$1 - (1 - 0.13)(1 - 0.16)(1 - 0.05)(1 - 0.17) =$ $\mathbf{{}_4q_2^{(\tau)} = 0.4237642}$.

Problem S3L46-4.

Similar to Question 16 from the Casualty Actuarial Society's Fall 2006 Exam 3.

Pink slugs aged x can perish from Bad Thing 1 or Bad Thing 2. The following 2-decrement table expresses the probabilities associated with perishing from each Bad Thing.

x........ $q_x^{(1)}$........$q_x^{(2)}$

2.......0.09........0.04

3.......0.14........0.02

4.......0.01........0.04

5.......0.07........0.10

Use this 2-decrement table to find ${}_{3|}q_2^{(2)}$. Hint: You *do* need to take into account the values of $q_x^{(1)}$. Think about how this might be done.

Solution S3L46-4. ${}_{3|}q_2^{(2)}$ is the probability that a pink slug will perish from Bad Thing 2 between 3 and 4 years from now. This implies that the pink slug needs to have survived *both Bad Things* until turning 5.

Hence, using some of the values calculated in Solution S3L46-3,

${}_{3|}q_2^{(2)} = (p_2^{(\tau)})(p_3^{(\tau)})(p_4^{(\tau)})(q_5^{(2)}) = (1 - 0.13)(1 - 0.16)(1 - 0.05)0.10 = \mathbf{{}_{3|}q_2^{(2)} = 0.069426}$.

Problem S3L46-5.

Similar to Question 17 from the Casualty Actuarial Society's Fall 2006 Exam 3.

Mortimer the Mortal can only die of shark attacks or falling elephants. His force of mortality due to shark attacks is $\mu_{x+t}^{(1)} = 0.5t$. His force of mortality due to falling elephants is $\mu_{x+t}^{(2)} = 0.2t$. Mortimer takes out a 30-year term life insurance policy paying 30 Golden Hexagons (GH) if he dies of a shark attack and 20 GH if he dies of a falling elephant. Elephants have fallen on all the buildings containing financial markets, so the interest rate is 0. Find the actuarial present value of this policy to Mortimer.

Solution S3L46-5. We use the formula $\mu_x^{(\tau)} = {}_{i=1}{}^{n}\sum\mu_x^{(i)}$ to find $\mu_{x+t}^{(\tau)} = 0.5t + 0.2t = 0.7t$.

We use the formula ${}_np_x = \exp[-{}_0{}^n\int\mu_{x+t}dt]$.

Thus, ${}_np_x^{(\tau)} = \exp[-{}_0{}^n\int 0.7t*dt] = \exp[-0.35n^2]$.

Now we use the formula $\bar{A}^1{}_{x:n\neg} = {}_0{}^n\int v^t*{}_tp_x*\mu_x(t)dt$.

We know that ${}_tp_x^{(\tau)} = \exp[-0.35t^2]$. Moreover, $v^t = 1$ for all t, since the interest rate is 0.

Thus, $\bar{A}^{30}{}_{x:n\neg}{}^{(1)}$due to shark attacks is $30*{}_0{}^n\int v^t*{}_tp_x^{(\tau)}*\mu_{x+1}^{(1)}dt =$

$30*{}_0{}^n\int 0.5t*\exp[-0.35t^2]dt = 30*(0.5/0.7)(-\exp[-0.35t^2])\,|\,{}_0{}^{30} =$

$30(0.5/0.7)(1 - \exp[-0.35*900])$ = about 21.42857143 GH.

$\bar{A}^{20}{}_{x:n\neg}{}^{(2)}$due to falling elephants is $20*{}_0{}^n\int v^t*{}_tp_x^{(\tau)}*\mu_{x+1}^{(2)}dt =$

$20*{}_0{}^n\int 0.2t*\exp[-0.35t^2]dt = 20*(0.2/0.7)(-\exp[-0.35t^2])\,|\,{}_0{}^{30} =$

$20(0.2/0.7)(1 - \exp[-0.35*900])$ = about 5.714285714 GH.

Thus, the actuarial present value of this policy to Mortimer is 21.42857143 + 5.714285714 = **about 27.14285714 GH.**

Section 47

The Pareto and Weibull Probability Distributions

The **Pareto distribution** is characterized by the following properties.

Survival function for Pareto Distribution: $\mathbf{s(x) = \theta^{\alpha}/(x + \theta)^{\alpha}}$ for some specified parameters α and θ with $\alpha \geq 1$ and $\theta \geq 0$. Note that different notation is often used for these parameters, and $(x + \theta)$ is often expressed as a variable in itself (denoted, for instance, as y, such that $s(y) = \theta^{\alpha}/(y)^{\alpha}$, for $y \geq x$).

Cumulative distribution function for Pareto Distribution: $\mathbf{F(x) = 1 - \theta^{\alpha}/(x + \theta)^{\alpha}}$
Probability density function for Pareto Distribution: $\mathbf{f(x) = \alpha\theta^{\alpha}/(x + \theta)^{\alpha+1}}$

We introduced Weibull's Law of Mortality in Section 25. The Weibull distribution can also be expressed via different notation, however. We will focus on one alternative expression of this distribution here.

Survival function for Weibull Distribution: $\mathbf{s(x) = \exp[-(x/\theta)^{\tau}]}$ for some specified parameters $\boldsymbol{\theta}$ and $\boldsymbol{\tau}$, for $\theta > 0$ and $\tau > 0$.

Cumulative distribution function for Weibull Distribution: $\mathbf{F(x) = 1 - \exp[-(x/\theta)^{\tau}]}$
Probability density function for Weibull Distribution: $\mathbf{f(x) = \tau x^{\tau-1}\exp[-(x/\theta)^{\tau}]/\theta^{\tau}}$

The earlier problems in this section are designed to acquaint students with this new notation. Then, it will be possible to undertake some exam-style questions.

Some of the problems in this section were designed to be similar to problems from past versions of the Casualty Actuarial Society's Exam 3L and the Society of Actuaries' Exam MLC. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

Source: Broverman, Sam. Actuarial Exam Solutions - CAS Exam 3 - Fall 2006.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L47-1. The operational lifetimes of squigglewidgets in years follow a Pareto distribution with $\theta = 0.36$ and $\alpha = 3$. You have 94000 operational squigglewidgets. After 6 years, how many operational squigglewidgets would you expect to have? Round down to the nearest whole squigglewidget.

Solution S3L47-1. We use the formula $s(x) = \theta^\alpha/(x + \theta)^\alpha$ for $x = 6$, $\theta = 0.36$, and $\alpha = 3$. Thus, $s(6) = 0.36^3/6.36^3$ = about 0.0001813577651. Since we have 94000 squigglewidgets to begin with, at time 6, we can expect to have 94000s(6) = 94000*0.0001813577651 = 17.04762992 or **about 17 squigglewidgets.**

Problem S3L47-2. The number of millennia, starting at present, for which Prometheus Tower can be expected to stand is modeled by a Weibull distribution $\theta = 4$ and $\tau = 2$. What is the probability that Prometheus Tower will still be standing 1500 years from now?

Solution S3L47-2. We use the formula $s(x) = \exp[(-x/\theta)^\tau]$ for $x = 1.5$, $\theta = 4$, and $\tau = 2$. Thus, $s(1.5) = \exp[-(1.5/4)^2] = \exp[-0.140625]$ = **s(1.5) = 0.8688150563**.

Problem S3L47-3. The lifetimes of gray gremlins follow a Pareto distribution with $\theta = 4$. You know that a newborn gray gremlin has a probability of 0.1051 of surviving to age 15. Find α for this Pareto distribution.

Solution S3L47-3. We use the formula $s(x) = \theta^\alpha/(x + \theta)^\alpha$ for $x = 15$, $\theta = 4$, and $s(15) = 0.1051$. Thus, $0.1051 = (4/19)^\alpha$ and $\alpha*\ln(4/19) = \ln(0.1051)$. Thus, $\alpha = \ln(0.1051)/\ln(4/19) =$ **α = about 1.445849747**.

Problem S3L47-4. You are given the following facts about a probability density function corresponding to a particular Weibull distribution: $\theta = 0.022$ and $\tau = 0.4$. Find f(6) for this distribution. Hint: The answer will be very small!

Solution S3L47-4. We use the formula $f(x) = \tau x^{\tau-1}\exp[-(x/\theta)^\tau]/\theta^\tau$. Thus, $f(6) = 0.4*6^{-0.6}\exp[-(6/0.022)^{0.4}]/0.022^{0.4}$ = **about 0.00005068503794**.

Problem S3L47-5.
Similar to Question 18 from the Casualty Actuarial Society's Fall 2006 Exam 3.
A reckless real estate speculator can incur losses in accordance with a spliced loss distribution consisting of a Weibull distribution for modeling losses less than \$1,500,000 and a Pareto distribution modeling losses greater than \$1,500,000. The Weibull distribution has parameters $\theta_1 = 500{,}000$ and $\tau = 2$. The Pareto distribution has parameters $\theta_2 = 2{,}000{,}000$ and $\alpha = 3$. The probability of the speculator's losses being less than \$1,500,000 is 0.7. Find the probability that his losses are less than \$3,000,000.

Solution S3L47-5. First, we consider the probability of a loss greater than \$1,500,000 being also greater than \$3,000,000. This is s(3000000)/s(1500000) and is modeled by the given Pareto distribution. We use the formula $s(x) = \theta^\alpha/(x + \theta)^\alpha$ for $\theta = 2{,}000{,}000$ and $\alpha = 3$. Thus, $s(3000000)/s(1500000) = (3000000/5000000)^3/(1500000/3500000)^3 = (3/5)^3/(5/7)^3 = (21/25)^3 = 0.592704$. This is the *conditional* probability of a loss greater than \$1,500,000 being also greater than \$3,000,000. To get the absolute probability of a loss being greater than \$3,000,000, we multiply 0.592704 by 0.3, the probability of a loss being greater than \$1,500,000. This probability is 0.1778112. Our desired probability is the complement of this result, 1 - 0.1778112 = **0.8221888.**

Section 48

The Negative Binomial Distribution and Some Associated Mixture Distributions

A **negative binomial** random variable is "a discrete random variable M with parameters

$r > 0$ and $\beta > 0$ for which

1. The only possible values are the non-negative integers 0, 1, 2, ..., etc.
2. $P[M = k] = ([r(r + 1)...(r + k - 1)]/k!)(\beta^k/(1+ \beta)^{k+r})$

3. $E[M] = r\beta$

4. $Var[M] = r\beta(1+ \beta)$" (Daniel, p. 11).

In a mixture distribution where β is itself a random variable with p.d.f. f(β) and with β ranging from a to b, the following formula applies.

$$\mathbf{P[M = k] = {}_a{}^b\!\int P[M = k \mid \beta]*f(\beta)d\beta}$$

Sources: Broverman, Sam. Actuarial Exam Solutions - CAS Exam 3 - Fall 2006.

Daniel, James W. 2008. Poisson Processes (and Mixture Distributions).

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L48-1. The number of great fleas sitting on the point of a needle follows a negative binomial distribution with r = 0.7 and β = 2. Find the probability that exactly 3 great fleas are sitting on the point of the needle.

Solution S3L48-1. We need to use the formula

$P[M = k] = ([r(r + 1)...(r + k - 1)]/k!)(\beta^k/(1+ \beta)^{k+r})$, for k = 3, r = 0.7, and β = 2.

Thus, $P[M = 3] = ((0.7)(1.7)(2.7)/3!)(2^3/3^{3.7})$ = **about 0.0735361383**.

Problem S3L48-2. The number of great fleas sitting on the point of a needle follows a negative binomial distribution with r = 0.7 and β = 2. Find the expected value of the number of great fleas that are sitting on the point of the needle.

Solution S3L48-2. We use the formula $E[M] = r\beta = 0.7*2 =$ **1.4 great fleas.**

Problem S3L48-3. The number of great fleas sitting on the point of a needle follows a negative binomial distribution with r = 0.7 and β = 2. Find the variance of the number of great fleas that are sitting on the point of the needle.

Solution S3L48-3. We use the formula Var[M] = rβ(1+ β) = 0.7*2(1 + 2) = **4.2.**

Problem S3L48-4. The number of magenta watermelons that an orange hippopotamus eats on a given day follows a negative binomial distribution with r = 4 and β being distributed uniformly on the interval between 1 and 5. Find the probability that this orange hippopotamus finds 2 magenta watermelons today. Set up the appropriate integral and then use any calculator to evaluate it.

Solution S3L48-4.
We first attempt to apply the formula
We need to use the formula
$P[M = k] = ([r(r + 1)...(r + k -1)]/k!)(\beta^k/(1+ \beta)^{k+r})$.
In this case, for k = 2 and r = 4, we have
$P[M = 2 \mid \beta] = (4*5/2!)(\beta^2/(1+ \beta)^6) = 10\beta^2/(1+ \beta)^6$
β itself is a random variable with probability density function 1/(5-1) = ¼.
Thus, $P[M = 2] = {}_1{}^5\int P[M = 2 \mid \beta]*f(\beta)d\beta = {}_1{}^5\int(1/4)(10\beta^2/(1+ \beta)^6)d\beta =$
$\mathbf{{}_1{}^5\int(5/2)(\beta^2/(1+ \beta)^6)d\beta}$ **= about 0.0387088477**.

Problem S3L48-5.

Similar to Question 18 from the Casualty Actuarial Society's Fall 2006 Exam 3.

The number of white cucumbers that a cucumber-hunter finds on a given day follows a negative binomial distribution with r = 2 and β being distributed uniformly on the interval between 3 and 6. Find the probability that this cucumber-hunter finds at least one white cucumber today.

Solution S3L48-5. We want to find P[M ≥ 1] = 1 - P[M = 0].
To find P[M = 0], we use the formula $P[M = k] = ([r(r + 1)...(r + k -1)]/k!)(\beta^k/(1+ \beta)^{k+r})$.
Here, k = 0 and r = 2, so [r(r + 1)...(r + k -1)] will be the smallest of r and (r + k - 1), which will be r + 0 - 1 = 2 - 1 = 1. Likewise, k! = 0! = 1, and $\beta^k = \beta^0 = 1$.
So, we have $P[M = 0 \mid \beta] = 1/(1+ \beta)^{0+r} = 1/(1+ \beta)^2$.
β itself is a random variable with probability density function 1/(6-3) = 1/3.
$P[M = 0] = {}_3{}^6\int P[M = 0 \mid \beta]*f(\beta)d\beta = {}_3{}^6\int(1/3)(1+ \beta)^{-2}*d\beta = -(1/3)(1+ \beta)^{-1} \Big|_3^6 =$
$(1/3)(4)^{-1}-(1/3)(7)^{-1}$= 1/12 - 1/21 = 1/28. Our desired answer is 1 - 1/28 = **27/28 = about 0.9642857143.**

Section 49

The Lognormal Probability Distribution, Markov Chains, and Assorted Exam-Style Questions

Calculations performed with the **Lognormal Distribution** are highly similar to those performed with the Normal Distribution, discussed in Dr. Marcel Finan's free study guide for Exam 1/P.

Let X be a random variable that follows a Lognormal Distribution with parameters μ and σ. Then, for some value of k,
$\Pr(X \leq k) = N[(\ln(k) - \mu)/\sigma]$
$\Pr(X > k) = 1 - N[(\ln(k) - \mu)/\sigma]$

A note on the function N(x): N(x) is called the *cumulative normal distribution function.* It is the probability that a randomly chosen number in the standard normal distribution (where mean = 0 and variance = 1) is less than x. It is impossible to directly integrate the normal probability distribution function to find N(x). On the exam, you will be given a table of values for N(x), so such integration will not be necessary. Here, however, we will use the Microsoft Excel function NormSDist. Try entering "=NormSDist(1)" into a cell in MS Excel. The result should be 0.84134474. Using this method will give us greater accuracy than using a table would.

Markov chains are illustrated by matrices which show the probability of moving from one state to another. Consider a 2-by-2 matrix pertaining to a Markov chain:

...X...Y
X(a....b)
Y(c....d)

This matrix represents the probabilities of moving between states X and Y within a single time period. The starting states appear on the vertical axis, while the states at the end of the period appear on the horizontal axis. For instance, a is the probability of an entity in state X staying in state X. b is the probability of an entity in state X moving to state Y. Note that a + b must equal 1 if X and Y are the only possible states.

If the matrices for each time period are the same, we call the resulting Markov chain **homogeneous.** If the matrices for each time period are not the same, we call the Markov chain **non-homogeneous.**

Source: Broverman, Sam. Actuarial Exam Solutions - CAS Exam 3 - Fall 2006.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L49-1. Gunnar the Gambler is playing a game of cards where his winnings follow a lognormal distribution with $\mu = 5$ and $\sigma = 2$. Find the probability that his winnings are less than or equal to \$4000.

Solution S3L49-1. We use the formula $\Pr(X \leq k) = N[(\ln(k) - \mu)/\sigma]$, where $k = 4000$, $\mu = 5$, and $\sigma = 2$. Thus, $\Pr(X \leq 5000) = N[(\ln(4000) - 5)/2] = N[1.64702482]$.
We use the function "=NormSDist(1.64702482)" to find our desired probability, **0.950223528.**

Problem S3L49-2. In a certain class, the number of spelling errors per paper received by the professor follows a lognormal distribution with $\mu = 3$ and $\sigma = 1.5$. Find the probability that a given paper will have more than 140 spelling errors in it.

Solution S3L49-2. We use the formula $\Pr(X > k) = 1 - N[(\ln(k) - \mu)/\sigma]$ where $k = 140$, $\mu = 3$, and $\sigma = 1.5$. Thus, $\Pr(X > 140) = 1 - N[(\ln(140) - 3)/1.5] = 1 - N[1.294428282]$. We use the function "=1 - NormSDist(1.294428282)" to find our desired probability, **0.097758763**.

Problem S3L49-3. Similar to Question 20 from the Casualty Actuarial Society's Fall 2006 Exam 3.

Lucretia owns a mine, where the only possible accidents are either due to cave-ins or noxious fumes, and only one accident occurs per year. The financial loss to Lucretia from a cave-in is three times the loss to her from noxious fumes. Noxious fumes have a likelihood of occurring that is four times greater than the probability that a cave-in will occur. The annual loss from noxious fumes follows a lognormal distribution with $\mu = 7$ and $\sigma = 4$. Find the probability that Lucretia's loss will be greater than \$3000 this year.

Solution S3L49-3. Let X be the random variable denoting the loss from noxious fumes and let Y be the random variable denoting the loss from cave-ins. We know that $Y = 3X$ and
$\Pr(X > k) = 1 - N[(\ln(k) - \mu)/\sigma]$.

One accident will occur this year, and since noxious fumes are 4 times as likely as cave-ins, it follows that noxious fumes have a 4/5 probability of occurring and cave-ins have a 1/5 probability of occurring.

We want to find $(4/5)\Pr(X > 3000) + (1/5)\Pr(Y > 3000) =$
$(4/5)\Pr(X > 3000) + (1/5)\Pr(3X > 3000) =$
$(4/5)\Pr(X > 3000) + (1/5)\Pr(X > 1000) =$
$(4/5)(1 - N[(\ln(3000) - 7)/4]) + (1/5)(1 - N[(\ln(1000) - 7)/4]) =$
$0.8(1 - N[0.2515918919]) + 0.2(1 - N[-0.0230611803])$
We use the function
"=0.8*(1 - NormSDist(0.2515918919)) + 0.2*(1 - NormSDist(-0.0230611803))" to find our desired probability, **0.422382463**.

Problem S3L49-4. Similar to Question 21 from the Casualty Actuarial Society's Fall 2006 Exam 3. Jawaharlal is currently 65 years old and has a survival function $s_1(x) = (1 - x/140)^{1/2}$.

Andrew is currently 89 years old and has a survival function $s_2(y) = 1 - y/120$. The two lives are completely independent.

Find the conditional probability that, given that both Jawaharlal and Andrew survive during the next 15 years, Jawaharlal will survive to the beginning of the 16th year while Andrew will not.

Solution S3L49-4. The probability that Jawaharlal will survive to age 81, given that he survives to age 80 is ${}_1p^{(x)}{}_{80} = s_1(81)/s_1(80) = (1 - 81/140)^{1/2}/(1 - 80/140)^{1/2} = 0.991631652$.
The probability that Andrew will not survive to age 105, given that he survives to age 104 can be found considering that Andrew's future lifetime is uniformly distributed, with a maximum of 120. At age 104, therefore, Andrew will have a 1/(120-104) = 1/16 probability of senselessly perishing in each year. Our desired probability is the product of the two probabilities we found: 0.991631652*(1/16) = **about 0.0619769783.**

Problem S3L49-5. The following non-homogeneous Markov Chain illustrates the transitions among states X, Y, and Z during three time periods.

Period 1:
...X...Y....Z
X[1.0..0....0..]
Y[0.4..0.3..0.3]
Z[0....0....1..]

Period 2:
...X...Y....Z
X[1.0..0....0..]
Y[0.2..0.7..0.1]
Z[0....0....1..]

Period 3:
...X...Y....Z
X[1.0..0....0..]
Y[0.5..0...0.5]
Z[0....0....1..]

Find the probability that an entity in state Y at the onset of Period 1 will be in state X at the end of Period 3.

Solution S3L49-5. We note that an entity in state X will always stay in state X, and the same holds for an entity in state Z.

Our desired probability is then equal to the sum of the following probabilities:
Pr(Y at onset → X after Period 1) = 0.4
Pr(Y at onset → Y after Period 1→ X after Period 2) = 0.3*0.2 = 0.06
Pr(Y at onset → Y after Period 1→ Y after Period 2→ X after Period 3) = 0.3*0.7*0.5 = 0.105
We add up the above probabilities to get our answer: 0.4 + 0.06 + 0.105 = **0.565.**

Section 50

Probability Generating Functions, Poisson Processes, and Assorted Exam-Style Questions

The **probability generating function** for a discrete probability distribution is denoted $\mathbf{P_N(t)}$ for the random variable N, where $Pr(N = n) = p_n$. $\mathbf{P_N(t)}$ has the following form.

$\mathbf{P_N(t) = p_0 + p_1 t + p_2 t^2 + ... + p_n t^n + ...}$

We can perform some derivative calculations for this functions.

$P_N'(t) = p_1 + 2p_2 t + ... + np_n t^{n-1} + ...$

$P_N''(t) = 2*1p_2 + 2*3p_3 t + 3*4p_4 t^2 ... + n*(n-1)p_n t^{n-2} + ...$

We note that $P_N'(1) = p_1 + 2p_2 + 3p_3 + ... + np_n + ...$, which by definition is the same as E[N]. Thus, $\mathbf{P_N'(1) = E[N]}$.

We also note that $P_N''(1) = 2*1p_2 + 2*3p_3 + 3*4p_4 ... + n*(n-1)p_n + ...$

We can add $0 = 0(-1)p_0 + 1*0p_1$ to this quantity to get

$P_N''(1) = 0(-1)p_0 + 1*0p_1 + 2*1p_2 + 2*3p_3 + 3*4p_4 ... + n*(n-1)p_n + ...$

This, by definition is $E[N(N-1)] = E[N^2] - E[N]$.

Thus, $\mathbf{P_N''(1) = E[N^2] - E[N]}$.

A **Poisson random variable** M with mean Λ is defined as follows (according to Daniel 2008):

50.1. M can only assume whole-number values.

50.2. $Pr[M = k] = e^{-\Lambda}\Lambda^k/k!$

50.3. $E[M] = Var[M] = \Lambda$

A **Poisson process** N, together with a **rate function** λ, can be described as follows (according to Daniel 2008):

50.4. N(0) = 0 and N is non-decreasing for whole-number values.

50.5. Any two non-overlapping time increments are independent with respect to N. What happens in one of those non-overlapping time increments does not affect what happens in the others.

50.6. "For all $t \geq 0$ and $h > 0$, the increment $N(t + h) - N(t)$ is a Poisson random variable with mean $\Lambda = {}_t{}^{t+h}\int\lambda(z)dz$."

50.7. The function $m(t) = {}_t{}^{t+h}\int\lambda(z)dz$ is the **mean value function** or **operational time.**

50.8. Where λ is equal to a constant, N is a **homogeneous poisson process.**

Sources: Broverman, Sam. Actuarial Exam Solutions - CAS Exam 3 - Fall 2006.

Daniel, James W. 2008. Poisson Processes (and Mixture Distributions).

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L50-1. Similar to Question 23 from the Casualty Actuarial Society's Fall 2006 Exam 3.

The number of blue rhinoceroses that spontaneously grow wings on a given day follows a negative binomial distribution with mean 12 and variance 60. Find the probability that at least 13 but fewer than 15 blue rhinoceroses will grow wings today.

Solution S3L50-1. In a negative binomial distribution, $E[M] = r\beta$ and $Var[M] = r\beta(1+ \beta)$.

Thus, $Var[M]/E[M] = (1+ \beta)$. In our case, $60/12 = (1+ \beta) = 5$, so $\beta = 4$.

Since $r\beta = 12$, it follows that $r = 12/4 = 3$.

Now we can use the formula $P[M = k] = ([r(r + 1)...(r + k -1)]/k!)(\beta^k/(1+ \beta)^{k+r})$.

We want to find $P[M = 13] + P[M = 14]$.

$P[M = 13] = [(3*4*5*6*7*8*9*10*11*12*13*14*15)/13!](4^{13}/5^{16}) = 2*14*15(4^{13}/5^{16}) =$ 0.1847179535.

$P[M = 14] = [(3*4*5*6*7*8*9*10*11*12*13*14*15*16)/14!](4^{14}/5^{17}) = 2*15*16(4^{14}/5^{17}) =$ 0.168884986.

So our desired answer is 0.1847179535 + 0.168884986 = **about 0.3536029395.**

Problem S3L50-2. Similar to Question 25 from the Casualty Actuarial Society's Fall 2006 Exam 3.

You know the following about derivatives of the probability generating function for a discrete probability distribution:
P'(1) = 68

P''(1) = 19400.
Find Var(N), where N is the random variable for the distribution.

Solution S3L50-2. We know that $P_N'(1) = E[N]$ and $P_N''(1) = E[N^2] - E[N]$.

Thus, $19400 = E[N^2] - 68$ and $E[N^2] = 19468$.

Now we use the formula $Var(N) = E[N^2] - (E[N]^2) = 19468 - 68^2 =$ **Var(N) = 14844**.

Problem S3L50-3. Similar to Question 27 from the Casualty Actuarial Society's Fall 2006 Exam 3. Sam the Speaker is always interrupted by Harold the Heckler when Sam gives a public speech. The length of the time period for which Sam can speak without being heckled is exponentially distributed with a mean of 5 minutes. Find the probability that, after the beginning of his speech, Sam will be heckled at least once within the first 3 minutes.

Solution S3L50-3. The survival function (in terms of minutes) for an exponential distribution with a mean of 5 minutes is $s(x) = e^{-x/5}$. Sam "survives" the first three minutes of the speech if he does not get heckled at all during that time. We want to find the *complement* of s(3), the probability that Sam gets heckled at least once. Thus, we want to find $1 - s(3) = 1 - e^{-3/5}$ **= about 0.4511883639**.

Problem S3L50-4. Similar to Question 28 from the Casualty Actuarial Society's Fall 2006 Exam 3.

The number of flies that a flycatcher catches per hour follows a non-homogeneous process with the following rate function, with t in hours:

$\lambda(t) = 56t + 3$ for $0 \leq t < 45$.

$\lambda(t) = 4t^2 - 34t + 4$ for $45 \leq t \leq 90$.

$\lambda(t) = 845$ for $t > 90$.

How many flies can the flycatcher be expected to catch between hours 17 and 100, inclusive?

Solution S3L50-4. We want to find $\Lambda = {}_{17}{}^{100}\int \lambda(z)dz =$

$\Lambda = {}_{17}{}^{45}\int(56z + 3)dz + {}_{45}{}^{90}\int(4z^2 - 34z + 4)dz + 845*10$ (since, during the last 10 hours, the flycatcher can expect to catch 845 flies in each hour).

Thus, Λ = 48692 + 747405 + 8450 = **804,547 flies.**

Problem S3L50-5. Similar to Question 26 from the Casualty Actuarial Society's Fall 2006 Exam 3.

Which of these statements about Poisson processes are true?

(a) What happens from time 3 to time 7 is independent of what happens from time 7 to time 16.

(b) What happens from time 3 to time 7 is independent of what happens from time 5 to time 13.

(c) The rate function for all Poisson processes is constant.

(d) For all Poisson random variables M, defined as M = N(t + h) - N(t), it is the case that

Var[M] = $\int_t^{t+h}\lambda(z)dz$.

Solution S3L50-5.

(a) is true, because [3, 7] and [7, 16] are not overlapping time intervals, even though they touch at an endpoint. So Condition 50.5 is met.

(b) is false, because [3, 7] and [5, 13] are overlapping time intervals, so independence of these intervals cannot be the case. Condition 50.5 is not met in this case.

(c) is false. Non-homogeneous Poisson processes have non-constant rate functions. We just saw an example of this in Problem S3L50-4.

(d) is true, because Var[M] = Λ by Condition 50.3, and $\Lambda = \int_t^{t+h}\lambda(z)dz$ by Condition 50.6.

Section 51

Binomial, Pareto, and Compound Probability Distributions

The following additional facts about the Pareto probability distribution are useful.

If random variable X follows a Pareto distribution with parameters θ and α and survival function **$s(x) = \theta^\alpha/(x + \theta)^\alpha$**, then the following is the case.

$E[X] = \theta/(\alpha - 1)$

$E[X^2] = 2\theta^2/[(\alpha - 1)(\alpha - 2)]$

You may also be asked to work with **truncated expected values** for the Pareto distribution. These are denoted by **E[X Λ K]**, which means the expected value of X, where X can assume any value less than or equal to K. The following holds under the Pareto distribution.

$E[X \Lambda K] = [\theta/(\alpha - 1)][1 - [\theta/(K + \theta)]^{(\alpha - 1)}]$

Now we will briefly review the **binomial probability distribution.** If random variable N follows a binomial distribution with parameters m and q, then the following is the case.

$Pr[N = n] = C(m, n)q^n(1 - q)^{m-n}$

$E[N] = mq$

$Var[N] = mq(1 - q)$

Whenever we have a random variable S which follows a **compound distribution,** where some random variable N is chosen and then some different random variable X is chosen based on the choice on N, then the following is the case.

$Var[S] = E[N]Var[X] + Var[N](E[X]^2)$

For instance, N could represent the frequency of losses and X could represent the severity of losses that occur. We will see another possible application of a compound distribution in one of the problems of this section.

Source: Broverman, Sam. Actuarial Exam Solutions - CAS Exam 3 - Fall 2006.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L51-1. The number of frogs on a lily pad follows a Pareto distribution with $\theta = 32$ and $\alpha = 5$. Find the expected value of the number of frogs on this lily pad.

Solution S3L51-1. We use the formula $E[X] = \boldsymbol{\theta/(\alpha - 1) = 32/(5 - 1) = 32/4 = 8}$ **frogs.**

Problem S3L51-2. The number of lily pads in a pond follows a binomial distribution with m = 12400 and q = 0.24. Find the variance of the number of lily pads in this pond.

Solution S3L51-2. We use the formula Var[N] = mq(1 - q) = 12400*0.24*0.76 = **Var[N] = 2261.76**.

Problem S3L51-3. The number of lily pads in a pond follows a binomial distribution with m = 12400 and q = 0.24. The number of frogs on any lily pad follows a Pareto distribution with θ = 32 and α = 5. Find the variance of the total number of frogs that are sitting on lily pads throughout the pond.

Solution S3L51-3. We want to find Var[S], where S follows a compound distribution where the binomial distribution of N is considered first, followed by the Pareto distribution of X.
We already know from Solutions S3L51-1 and S3L51-2 that E[X] = 8, Var[N] = 2261.76.
We, moreover, find E[N] = mq = 12400*0.24 = 2976.
Then we find $E[X^2] = 2\theta^2/[(\alpha - 1)(\alpha - 2)] = 2*32^2/(4*3) = 170.666666667$
Thus, $Var[X] = E[X^2] - (E[X]^2) = 170.666666667 - 64 = 106.666666667$
Now we use the formula
$Var[S] = E[N]Var[X] + Var[N](E[X]^2) =$
2976*106.666666667 +2261.76*64 = **Var[S] = 462192.64**.

Problem S3L51-4. Similar to Question 29 from the Casualty Actuarial Society's [Fall 2006 Exam 3].

The frequency of losses in spaceship accidents (N) follows a binomial distribution with m = 21000 and q = 0.55. The severity of each loss (X) follows a Pareto distribution with θ = 2340 and α = 9. Find the standard deviation of the compound distribution of the random variable S, where S denotes the aggregate losses in spaceship accidents.

Solution S3L51-4. We first find E[N] = mq = 21000*0.55 =11550.

Then we find Var[N] = mq(1 - q) = 11550*0.45 = 5197.5.

Then we find $E[X] = \theta/(\alpha - 1) = 2340/8 = 292.5$

Then we find $E[X^2] = 2\theta^2/[(\alpha - 1)(\alpha - 2)] = 2*5475600/(8*7) = 195557.1429$

Then we find $Var[X] = E[X^2] - (E[X]^2) = 195557.1429 - 292.5^2 = 110000.8929$

Now we find Var[S] = E[N]Var[X] + Var[N](E[X]^2) =

11550*110000.8929 + 5197.5*292.5^2 = about 1715188922.

We find SD[S] = (1715188922)$^{1/2}$ = **SD[S] = about 41414.83939**.

Problem S3L51-5. Similar to Question 30 from the Casualty Actuarial Society's Fall 2006 Exam 3.

There are two insurance policies designed to insure against losses due to catastrophic cake splatterings. Policy C has no limits or deductibles and pays the full loss to the claimant. Policy D has a deductible of 800 Golden Hexagons (GH) and a limit of 7800 GH. That is, Policy D will pay all claims between 800 GH and 7800 GH. In year 0, losses due to catastrophic cake splatterings follow a Pareto Distribution with α = 5 and θ = 6000. Government officials are printing money like crazy in order to buy up all the cakes and throw them at the helpless populace. Therefore, the annual inflation rate is 56%. Find the absolute value of the difference between the expected losses under policies C and D in year 5.

Solution S3L51-5. As a result of inflation θ under the Pareto Distribution will increase from 6000 to 6000*1.56^5 = 55433.74787. α will be unchanged.
Under Policy C, the expected loss in year 5 will simply be θ/(α - 1) = 55433.74787/4 = 13858.43697 GH.

Under Policy D, only the loss amounts between 800 and 7800 GH will be paid (a great deal for the insurance company due to the inflation that happened).

Thus, the expected loss under Policy D is E[X for X between 800 and 7800] =

E[X Λ 7800] - E[X Λ 800] =

[θ/(α - 1)][1 - [θ/(7800 + θ)]$^{(\alpha - 1)}$] - [θ/(α - 1)][1 - [θ/(800 + θ)]$^{(\alpha - 1)}$] =

[13858.43697][1 - [55433.74787/(7800 + 55433.74787)]^4] -

[13858.43697][1 - [55433.74787/(800 + 55433.74787)]^4] =

5673.496468 - 771.9492408 = 4901.547227.

The difference between the expected losses under these two policies is therefore

13858.43697 - 4901.547227 = **about 8956.889739 GH.**

Section 52

Fully Discrete Benefit Premiums, Infinite Divisibility Property for Some Probability Distributions, and Assorted Exam-Style Questions

The negative binomial distribution and the Poisson distribution both exhibit the **infinite divisibility property.**

Let X follow a negative binomial distribution with parameters r and β. When we consider only a *part* of the distribution of X, say over the range (a, b), then the variable X distributed over (a, b) *also* follows a negative binomial distribution with parameters r and Pr(a < X < b)*β.

Let T follow a Poisson distribution with parameter λ. When we consider only a *part* of the distribution of T, say over the range (a, b), then the variable T distributed over (a, b) *also* follows a Poisson distribution with parameter Pr(a < X < b)*λ.

The following relationships hold between actuarial present values of whole and term life insurance policies, respectively, where for one of the policies ($\bar{A}$) benefits are paid at the time of death and for the other (A) benefits are paid at the end of the year of death. In order for these relationships to hold, deaths for fractional ages must be uniformly distributed. Here, i is the annual effective rate of interest and δis the annual force of interest.

$\bar{A}_x = (i/\delta)A_x$
$\bar{A}^1_{x:n\neg} = (i/\delta)A^1_{x:n\neg}$

For **fully discrete benefit premiums** where benefits on the associated life insurance policy are always paid at the end of the year of death, the following formulas hold, where "P" denotes the fully discrete benefit premium and the subscripts and superscripts of P correspond with the subscripts and superscripts of the associated life insurance policy.

$P_x = A_x/\ddot{a}_x$
$P^1_{x:n\neg} = A^1_{x:n\neg}/\ddot{a}_{x:n\neg}$
$P_{x:n\neg} = A_{x:n\neg}/\ddot{a}_{x:n\neg}$

When using life tables, the following identities will also prove useful.

${}_nE_x = {}_mE_x * {}_{n-m}E_{x+m}$, where m is any number less than n.
$A^1_{x:n\neg} = A_x - {}_nE_x * A_{x+n}$
$\ddot{a}_{x:n\neg} = \ddot{a}_x - {}_nE_x * \ddot{a}_{x+n}$

Some of the problems in this section were designed to be similar to problems from past versions of the Casualty Actuarial Society's Exam 3L and the Society of Actuaries' Exam MLC. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

Sources: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1997. Second Edition. Society of Actuaries: Itasca, Illinois. pp. 120, 181-183.

Broverman, Sam. Actuarial Exam Solutions - CAS Exam 3 - Fall 2006.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L52-1.

Similar to Question 31 from the Casualty Actuarial Society's Fall 2006 Exam 3.

The frequency of a firm's annual losses due to defective paper clips follows a negative binomial distribution with r = 5 and β = 8. The severity of each loss follows a Pareto distribution with θ = 1550 and α = 3. The firm is insured for these losses under a policy with a deductible of $1000. What is the variance of the number of payments the insurance company will have to make because of this firm's losses due to defective paper clips?

Solution S3L52-1. The insurance company will only have to make a payment for a loss that is greater than $1000. Since we are given the severity distribution of each loss, we can find the probability that any random loss will be greater that $1000. This probability is

$s(1000) = \theta^\alpha/(1000 + \theta)^\alpha = (1550/2550)^3$ = about 0.2245817973.

Since the negative binomial distribution is infinitely divisible, the frequency of losses that the insurance company will have to pay for follows a negative binomial distribution with r = 5 and

$\beta = 8*0.2245817973 = 1.796654379$.

We want to find the variance of this distribution, which we can do using the formula

$\text{Var}[M] = r\beta(1+ \beta) = 5*1.796654379*2.796654379 =$ **about 25.12310668.**

Problem S3L52-2. Similar to Question 32 from the Casualty Actuarial Society's Fall 2006 Exam 3. The number of leaf-eating ants in a garden on any given day follows a Poisson distribution with λ = 5. The number of leaves eaten by each ant follows a normal distribution with F(3400) = 0.8. Find the probability that at least one ant has eaten more than 3400 leaves today.

Solution S3L52-2. The probability (under the normal distribution - which is our severity distribution here) that any given any has eaten more than 3400 leaves is s(3400) = 1 - F(3400) = 1 - 0.8 = 0.2. Thus, to find the probability that at least one ant has eaten more than 3400 leaves today, we only consider that part of the Poisson frequency distribution which is relevant to ants that eat more than 3400 leaves. Since the Poisson distribution is infinitely divisible, the mean of the distribution we wish to consider is $\lambda = 5*0.2 = 1$. We want to find the complement of the probability that no ant has eaten more than 3400 leaves. The latter probability is equal to $e^{-\lambda} = e^{-1}$,

so our desired probability is **$1 - e^{-1}$ = about 0.6321205588**.

Problem S3L52-3. Similar to Question 35 from the Casualty Actuarial Society's [Fall 2006 Exam 3](). The annual force of interest is 0.07. A spaceship has the following probabilities of failure q_x during each age x:

Age x......q_x

0.........0.005

1.........0.0044

2.........0.0123

3.........0.0334

4.........0.0241

5.........0.064

6.........0.031

Spaceship failures are permanent once they occur and are uniformly distributed for fractional ages.

There are two kinds of warranties available of the spaceship. A normal 4-year warranty will pay the full purchase price of the spaceship if it fails within the first four years of its life. An extended 3-year warranty for the spaceship can also be purchased. It will only take effect if the spaceship survives to four years since the purchase time, and it will pay 0.6 of the spaceship's purchase price if the spaceship fails between prior to reaching age 7. What is the actuarial present value of such an extended warranty on a new spaceship, as a fraction of the spaceship's purchase price.

Solution S3L52-3. Let R be the spaceship's purchase price. The extended warranty only takes effect if the spaceship survives to age 4. The probability of this happening is

$(1 - q_0)(1 - q_1)(1 - q_2)(1 - q_3) = (1 - 0.005)(1 - 0.0044)(1 - 0.0123)(1 - 0.0334) =$ 0.9457575419.

This probability will need to be discounted for four years to take into account the time value of money:
$0.9457575419*e^{-0.07*4} = 0.7147881735$.

Moreover, the extended warranty only pays 0.6R, so our value above will need to be multiplied by this factor: 0.7147881735*0.6R = 0.4288729041R.

Now we need to consider under which circumstances payment will be made.
We can calculate $A^1_{4:3\neg}$, the actuarial present value (at time 4) of the extended warranty (analogous to a three-year term life insurance policy for a life aged 4), *if* payments are only made at the end of the year of failure.

$A^1_{4:3\neg} = e^{-0.07}q_4 + e^{-0.07*2}p_4q_5 + e^{-0.07*3}p_4p_5q_6$

$A^1_{4:3\neg} = e^{-0.07}*0.0241 + e^{-0.07*2}*0.9759*0.064 + e^{-0.07*3}*0.9759*0.936*0.031 = 0.0997218026$.

Now we can use the formula $\bar{A}^1_{x:n\neg} = (i/\delta)A^1_{x:n\neg}$ to find $\bar{A}^1_{4:3\neg}$, the actuarial present value of the extended warranty in question at time 4.

We find $i = e^{\delta} - 1 = e^{0.07} - 1 = 0.0725081813$.

Thus, $\bar{A}^1_{4:3\neg} = (0.0725081813/0.07)0.0997218026 = 0.1032949505$.

Our desired answer is therefore $0.4288729041R* \bar{A}^1_{4:3\neg} = 0.4288729041R*0.1032949505 =$

0.0443004054R. Expressed as a fraction of R, our answer is thus **0.0443004054**.

Problem S3L52-4.

This question uses the [Illustrative Life Table, which can be found here]().

Fully discrete benefit premiums of amount P are paid each year to insure a 40-year-old robot whose mortality follows the Illustrative Life Table at i = 6%. The insurance policy is will pay $60000 at the end of the year of the robot's failure, until the robot is 80 years old. Find P.

Solution S3L52-4. We use the formula $P^1_{x:n\neg} = A^1_{x:n\neg}/\ddot{a}_{x:n\neg}$.

We need to find $A^1_{40:40\neg}$. We do so using the formula $A^1_{x:n\neg} = A_x - {}_nE_x*A_{x+n}$. Thus,

$A^1_{40:40\neg} = A_{40} - {}_{40}E_{40}*A_{80} = A_{40} - {}_{20}E_{40}*{}_{20}E_{60}*A_{80} = 0.16132 - 0.27414*0.14906*0.66575=$

$A^1_{40:40\neg} = 0.1341152524$.

Now we need to find $\ddot{a}_{40:40\neg}$. We do so using the formula $\ddot{a}_{x:n\neg} = \ddot{a}_x - {}_nE_x*\ddot{a}_{x+n}$. Thus,

$\ddot{a}_{40:40\neg} = \ddot{a}_{40} - {}_{40}E_{40}*\ddot{a}_{80} = \ddot{a}_{40} - {}_{20}E_{40}*{}_{20}E_{60}*\ddot{a}_{80} = 14.8166 - 0.27414*0.14906*5.9050 =$

$\ddot{a}_{40:40\neg}$ = 14.57530216.

We want to find P = 60000*$P^1_{40:40\neg}$ = 60000*$A^1_{40:40\neg}$/ $\ddot{a}_{40:40\neg}$ = 60000*0.1341152524/14.57530216 = P = \$552.0925092 = **about \$552.09**.

Problem S3L52-5. Similar to Question 36 from the Casualty Actuarial Society's Fall 2006 Exam 3. Fully discrete benefit premiums are paid each year to insure a 40-year-old robot whose mortality follows the Illustrative Life Table at i = 6%. The insurance policy is will pay \$60000 at the end of the year of the robot's failure, until the robot is 80 years old. For the first 20 years of the policy, premiums of amount Q are paid. For the last 20 years of the policy, premiums of amount 1.5Q are paid. Find Q.

Solution S3L52-5. We use the formula $P^1_{x:n\neg}$ = $A^1_{x:n\neg}$/$\ddot{a}_{x:n\neg}$.

We need to find $A^1_{40:40\neg}$. We do so using the formula $A^1_{x:n\neg}$ = A_x - ${}_nE_x$*A_{x+n}. Thus,

$A^1_{40:40\neg}$ = A_{40} - ${}_{40}E_{40}$*A_{80} = A_{40} - ${}_{20}E_{40}$*${}_{20}E_{60}$*A_{80} = 0.16132 - 0.27414*0.14906*0.66575=

$A^1_{40:40\neg}$ = 0.1341152524.

Here, $\ddot{a}_{40:40\neg}$ is not calculated on the basis of uniform payments each period. Rather, the payments from time 20 to time 40 are 1.5 times the payments from time 0 to time 20.

Thus, $\ddot{a}_{40:40\neg}$ = $\ddot{a}_{40:20\neg}$ + 1.5*${}_{20}E_{40}$*$\ddot{a}_{60:20\neg}$.

Why was 1.5*$\ddot{a}_{60:20\neg}$ also multiplied by ${}_{20}E_{40}$? Because we need to discount 1.5*$\ddot{a}_{60:20\neg}$ by 20 years *and* only consider it in the event that the robot survives to age 60.

We know that ${}_nE_x$ = v^n*${}_np_x$, so including a factor of ${}_{20}E_{40}$ takes both of these considerations into account.

Now we find $\ddot{a}_{40:20\neg}$ = $\ddot{a}_{40}$ - ${}_{20}E_{40}$*$\ddot{a}_{60}$ = 14.8166 - 0.27414*11.1454 = 11.7612

Now we find $\ddot{a}_{60:20\neg}$ = $\ddot{a}_{60}$ - ${}_{20}E_{60}$*$\ddot{a}_{80}$ = 11.1454 - 0.14906*5.9050 = 10.2652

Thus, $\ddot{a}_{40:40\neg}$ = $\ddot{a}_{40:20\neg}$ + 1.5*${}_{20}E_{40}$*$\ddot{a}_{60:20\neg}$ = 11.7612 + 1.5*0.27414*10.2652 = 15.98235289.

Thus, Q = 60000$A^1_{40:40\neg}$/$\ddot{a}_{40:40\neg}$ = 60000*0.1341152524/15.98235289 = Q = \$503.4875152 = **about \$503.49.**

Section 53

Order Statistics, Likelihood Functions, and Maximum Likelihood Estimates

Following Larsen and Marx 2006, we let Y be a continuous random variable for which y'_1, y'_2, ..., y'_n are values of a random sample of size n such that $y'_1 < y'_2 < ... < y'_n$.

We can define the variable Y'_i, the **ith order statistic** of Y, as $Y'_i = y'_i$, where $1 \leq i \leq n$.

The ith order statistic of Y has the following probability density function (p. d. f.):

$\mathbf{f_{Y'\_i}(y) = (n!/[(i-1)!(n-i)!])*F(y)^{i-1}*(1-F(y))^{n-i}*f(y)}$, where f(y) is the p. d. f. of Y and F(y) is the cumulative distribution function (c. d. f.) of Y.

We can also find $E[Y'_i] = {}_0^\infty\int y*f_{Y'\_i}(y)*dy = {}_0^\infty\int s_{Y'\_i}(y)*dy$.

Then $E[Y'_i]$ can be used to estimate some particular value Y_k of Y.

If this is the case, then the **bias** of the ith order statistic can be determined as follows:

$\mathbf{Bias(Y'_i) = E[Y'_i] - Y_k}$**.**

The following **integration rule** offers a useful shortcut for problems involving integration by parts:

$\mathbf{{}_0^\infty\int t^n e^{-at}*dt = n!/a^{n+1}}$.

The following definition of the **likelihood function** is offered by Larsen and Marx.

"Let $k_1, k_2, ..., k_n$ be a random sample of size n from the discrete p. d. f. $p_X(k; \theta)$, where θ is an unknown parameter. The *likelihood function,* L(θ), is the product of the p. d. f. evaluated at the n k_is. That is, $\mathbf{L(\theta) = {}_{i=1}^{n}\Pi\ p_X(k_i; \theta)}$."

If the p. d. f. in question is equal to $f_Y(y_i; \theta)$ and is continuous for random sample $y_1, y_2, ..., y_n$ of size n, then $\mathbf{L(\theta) = {}_{i=1}^{n}\Pi\ f_Y(y_i; \theta)}$**.**

Now we can define the **maximum likelihood estimate,** following Larsen and Marx.

"Let $L(\theta) = {}_{i=1}^{n}\Pi\ p_X(k_i; \theta)$and $L(\theta) = {}_{i=1}^{n}\Pi\ f_Y(y_i; \theta)$ be the likelihood functions corresponding to random samples $k_1, k_2, ..., k_n$ and $y_1, y_2, ..., y_n$ drawn from the discrete p.d.f. $p_X(k_i; \theta)$ and the continuous p.d.f. $f_Y(y_i; \theta)$, respectively, where θ is an unknown parameter. In each case, let θ_e

be a value of the parameter such that $L(\theta_e) \geq L(\theta)$ for all possible values of θ. Then θ_e is called a **maximum likelihood estimate** for θ."

Some of the problems in this section were designed to be similar to problems from past versions of the Casualty Actuarial Society's Exam 3L and the Society of Actuaries' Exam MLC. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

Sources: Broverman, Sam. Actuarial Exam Solutions - CAS Exam 3 - Fall 2006.

Larsen, Richard J. and Morris L. Marx. *An Introduction to Mathematical Statistics and Its Applications.* Fourth Edition. Pearson Prentice Hall: 2006. pp. 241-243, 347.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L53-1.

Similar to part of Question 1 from the Casualty Actuarial Society's Fall 2006 Exam 3.

The 2^{nd} order statistic in a sample of 4 is used to estimate the 40^{th} percentile of an exponential distribution with mean θ. What is the actual 40^{th} percentile of this distribution, in terms of θ?

Solution S3L53-1. The 40^{th} percentile of an exponential distribution is the value y for which F(y) = 0.4 and s(y) = 1 - F(y) = 0.6. For an exponential distribution, $s(y) = e^{-y/\theta}$.

Thus, for the 40^{th} percentile, $0.6 = e^{-y/\theta}$ and $\ln(0.6) = -y/\theta$, so $y = -\ln(0.6)*\theta = \ln(5/3)\theta$.

So the actual 40^{th} percentile is **ln(5/3)θ**.

Problem S3L53-2.

Similar to part of Question 1 from the Casualty Actuarial Society's Fall 2006 Exam 3.

The 2^{nd} order statistic in a sample of 4 is used to estimate the 40^{th} percentile of an exponential distribution with mean θ. What is the estimated 40^{th} percentile of this distribution, in terms of θ, given by using the 2^{nd} order statistic?

Solution S3L53-2. We need to find $E[Y'_2] = {}_0{}^{\infty}\int y*f_{Y'\_2}(y)*dy$.

We find $f_{Y'\_2}(y)$ using the formula $f_{Y'\_i}(y) = (n!/[(i-1)!(n-i)!])*F(y)^{i-1}*(1-F(y))^{n-i}*f(y)$.

Here, n = 4, i = 2, $f(y) = (1/\theta)e^{-y/\theta}$, and $F(y) = 1 - e^{-y/\theta}$.

Thus,

$f_{Y'\_2}(y) = [4!/(1!*2!)](1 - e^{-y/\theta})(e^{-y/\theta})^2*(1/\theta)e^{-y/\theta}$

$f_{Y'\_2}(y) = 12*(1/\theta)(1 - e^{-y/\theta})(e^{-y/\theta})^3$

$f_{Y'\_2}(y) = 12*(1/\theta)(e^{-3y/\theta} - e^{-4y/\theta})$

Thus, $E[Y'_2] = {}_0{}^{\infty}\int y*12*(1/\theta)(e^{-3y/\theta} - e^{-4y/\theta})dy$

$E[Y'_2] = (12/\theta){}_0{}^{\infty}\int y*(e^{-3y/\theta} - e^{-4y/\theta})dy$

$E[Y'_2] = (12/\theta)({}_0{}^{\infty}\int y*e^{-3y/\theta}dy - {}_0{}^{\infty}\int y*e^{-4y/\theta}dy)$

Now we can use the integration shortcut ${}_0{}^{\infty}\int t^n e^{-at}*dt = n!/a^{n+1}$ to find

${}_0{}^{\infty}\int y*e^{-3y/\theta}dy = 1/(3/\theta)^2$ and ${}_0{}^{\infty}\int y*e^{-4y/\theta}dy = 1/(4/\theta)^2$.

Thus, $E[Y'_2] = (12/\theta)(1/(3/\theta)^2 - 1/(4/\theta)^2)$

$E[Y'_2] = (12/\theta)(\theta^2/9 - \theta^2/16) = 12\theta(1/9 - 1/16) =$ **$E[Y'_2] = 7\theta/12$**.

Problem S3L53-3.

Similar to part of Question 1 from the Casualty Actuarial Society's Fall 2006 Exam 3.

The 2nd order statistic in a sample of 4 is used to estimate the 40th percentile of an exponential distribution with mean θ. What is the bias resulting from estimating the 40th percentile of this distribution by using the 2nd order statistic? Give your answer in terms of θ.

Solution S3L53-3. We use the formula $\text{Bias}(Y'_i) = E[Y'_i] - Y_k$.

From Solutions S3L53-1 and S3L53-2, we know that $E[Y'_2] = 7\theta/12$ and $Y_k = \ln(5/3)\theta$.

Thus, $\text{Bias}(Y'_2) = 7\theta/12 - \ln(5/3)\theta =$ **about 0.0725077096θ**.

Problem S3L53-4.

Similar to part of Question 2 from the Casualty Actuarial Society's Fall 2006 Exam 3.

The amount of time in hours it takes Hernando to write a property deed follows the cumulative distribution function $F(x) = x^{\theta+5}$, where $0 \le x \le 1$ and $\theta > -5$ is an unknown parameter. You are aware of the following sample of six times during each of which Hernando completed one property deed:

0.24, 0.643, 0.46, 0.45, 0.34, 0.36

Find the likelihood function L(θ) based on this sample.

Solution S3L53-4. We use the formula $L(\theta) = {}_{i=1}{}^{n}\Pi\, f_Y(y_i; \theta)$.

We find $f_X(x) = F'(x) = (\theta +5)x^{\theta+5}$.

Thus,

$L(\theta) = (\theta +5)0.24^{\theta+5}*(\theta +5)0.643^{\theta+5}*(\theta +5)0.46^{\theta+5}*(\theta +5)0.45^{\theta+5}*(\theta +5)0.34^{\theta+5}* (\theta +5)0.36^{\theta+5}$

$L(\theta) = (\theta +5)^6(0.24*0.643*0.46*0.45*0.34*0.36)^{\theta+5}$

$\mathbf{L(\theta) = (\theta +5)^6(0.003909975)^{\theta+5}}$**.**

Problem S3L53-5. Similar to part of Question 2 from the Casualty Actuarial Society's Fall 2006 Exam 3.

The amount of time in hours it takes Hernando to write a property deed follows the cumulative distribution function $F(x) = x^{\theta+5}$, where $0 \le x \le 1$ and $\theta > -5$ is an unknown parameter. You are aware of the following sample of six times during each of which Hernando completed one property deed:

0.24, 0.643, 0.46, 0.45, 0.34, 0.36

Find the maximum likelihood estimate for θ using this sample.

Solution S3L53-5. We know from Solution S3L53-4 that $L(\theta) = (\theta +5)^6(0.003909975)^{\theta+5}$.

The maximum of L(θ) will occur where L'(θ) = 0.

We find $L'(\theta) = 6(\theta +5)^5(0.003909975)^{\theta+5} + (\theta +5)^6*\ln(0.003909975)(0.003909975)^{\theta+5}$

We set $0 = 6(\theta +5)^5(0.003909975)^{\theta+5} + (\theta +5)^6*\ln(0.003909975)(0.003909975)^{\theta+5}$ and find θ.

$6(\theta +5)^5(0.003909975)^{\theta+5} = -(\theta +5)^6*\ln(0.003909975)(0.003909975)^{\theta+5}$

$6(\theta +5)^5= -(\theta +5)^6*\ln(0.003909975)$

$6 = -(\theta +5)*\ln(0.003909975)$

$\theta +5 = 6/-\ln(0.003909975)$

$\theta = 6/-\ln(0.003909975) - 5 =$ **θ = about -3.917792703**.

Section 54

Exam-Style Questions on the Method of Moments, Significance Levels, and the Neyman-Pearson Lemma

Following Larsen and Marx 2006, we describe the **method of moments** as follows.

"Let y_1, y_2, ..., y_n be a random sample from the continuous p.d.f. $f_Y(y; \theta_1, \theta_2, \ldots, \theta_s)$. The *method of moments* estimates $\theta_{1e}, \theta_{2e}, \ldots, \theta_{se}$ for the model's unknown parameters are the solutions of the s simultaneous equations

$$\int_{-\infty}^{\infty} y * f_Y(y; \theta_1, \theta_2, \ldots, \theta_s)dy = (1/n)\sum_{i=1}^{n} y_i$$
$$\int_{-\infty}^{\infty} y^2 * f_Y(y; \theta_1, \theta_2, \ldots, \theta_s)dy = (1/n)\sum_{i=1}^{n} y_i^2$$
...
$$\int_{-\infty}^{\infty} y^s * f_Y(y; \theta_1, \theta_2, \ldots, \theta_s)dy = (1/n)\sum_{i=1}^{n} y_i^s$$

"If the underlying random variable is discrete with p.d.f. $p_X(k; \theta_1, \theta_2, \ldots, \theta_s)$, the method of moments estimates are the solutions of the system of equations

$\sum_{\text{all } k} k^j * p_X(k; \theta_1, \theta_2, \ldots, \theta_s) = (1/n)\sum_{\text{all } k} y_i^k$" (357-358).

Let M be the random variable representing the values in our *sample* and X be the random variable representing the values given by the *distribution* we are considering. According to the method of moments, the system of equations that can be set up to solve for our s unknown parameters can be conceived of as follows:

$E[X] = E[M]$.
$E[X^2] = E[M^2]$
...
$E[X^s] = E[M^s]$

This is a more convenient way to remember the method of moments and is especially useful when one has memorized the formulas for the moments of the distribution with which one is working.

A **single-parameter Pareto distribution** is different from the two-parameter Pareto distribution introduced in Section 47.

The single-parameter Pareto distribution has survival function

$s(x) = 1 - (\theta/x)^\alpha$, for some given value of θ and for $x > \theta$.

The single-parameter Pareto distribution only describes values of x larger than θ. For instance, θ could be some minimum claim or loss size.

The mean of a single-parameter Pareto distribution is

$E[X] = \theta\alpha/(\alpha - 1)$.

Broverman defines a **significance level** for a hypothesis test as "the probability of rejecting the hypothesis H_0, given that H_0 is true." In problems involving significance levels, you will be given some **critical region** for values within which the hypothesis H_0 must be rejected. Then you will be asked to find the conditional probability that, if H_0 is true, you will find a value within that critical region.

The **Neyman-Pearson Lemma** is used to perform a hypothesis test between two point hypotheses: H_0: $\theta = \theta_0$ and H_1: $\theta = \theta_1$.

Hypothesis H_0 is rejected in favor of H_1 when
$\Lambda(x) = L(\theta_0 \mid x)/L(\theta_1 \mid x) \leq \eta$, where $P(\Lambda(x) \leq \eta \mid H_0) = \alpha$.

In the context of Exam 3L, you will likely be asked to analyze a hypothesis test under the following conditions.

A certain **significance level** will be given, as a percentage or probability.

You will be given a multiplicity of points and asked to determine which points are within the **critical region** defined by the given significance level.

The answer to such a question is as follows. **A point is within the critical region defined by a given significance level k if $P(N = n \mid \theta = \theta_0) < k$.**

So to answer a problem involving this application of the Neyman-Pearson Lemma, it is necessary to consider the probability distribution of N, *given that* $\theta = \theta_0$. You will typically be given such a distribution, so you will simply need to examine the conditional probability for each n, given that $\theta = \theta_0$, to find whether this probability is less than k.

Source: Broverman, Sam. Actuarial Exam Solutions - CAS Exam 3 - Fall 2006.

Larsen, Richard J. and Morris L. Marx. *An Introduction to Mathematical Statistics and Its Applications*. Fourth Edition. Pearson Prentice Hall: 2006. pp. 357-358.

"Neyman-Pearson Lemma" by Zaqrfv.

"Neyman-Pearson Lemma." Wikipedia, the Free Encyclopedia.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L54-1. Similar to Question 3 from the Casualty Actuarial Society's Fall 2006 Exam 3. A single-parameter Pareto distribution with parameter α models financial loss amounts of 200 or greater. You have a random sample of losses modeled by this distribution: 430, 316, 766, 453, 243, 424, 224, 535. Use the method of moments to estimate α for this sample.

Solution S3L54-1. If a Pareto distribution only models amounts of 200 or greater, this means that θ = 200 for this distribution. For a single-parameter Pareto distribution, we know that $_{-\infty}{}^{\infty}\int y*f_Y(y)dy = {}_{\theta}{}^{\infty}\int y*f_Y(y)dy = E[Y] = \theta\alpha/(\alpha - 1)$, **which in our case is**

$200\alpha/(\alpha - 1)$. We are also given n = 8.

We use the method of moments, which in our case only requires a single equation:

$_{-\infty}{}^{\infty}\int y*f_Y(y; \theta_1, \theta_2, \ldots ,\theta_s)dy = (1/n)_{i=1}{}^{n}\sum y_i$

$_{\theta}{}^{\infty}\int y*f_Y(y)dy = (1/8)_{i=1}{}^{8}\sum y_i$

$200\alpha/(\alpha - 1) = (1/8)(430+316+766+453+243+424+224+535)$

$200\alpha/(\alpha - 1)= 423.875$

$200\alpha = 423.875\alpha - 423.875$

$423.875 = 223.875\alpha$

α = about 1.893355667.

Problem S3L54-2. Similar to Question 5 from the Casualty Actuarial Society's Fall 2007 Exam 3.

Now you are given a typical *two-parameter* Pareto distribution with parameters θ and α. A random sample from this distribution contains the following values: 2000, 13515, 400000, 16437, 4993, 12000.

Find the parameter α for this distribution.

(Hint: The first and second moments for this distribution are given in Section 51).

Solution S3L54-2. Here, we are solving for two unknown parameters, so we will need a system of two equations. Let M be the random variable representing the values in our *sample* and X be the random variable representing the values given by the *distribution* we are considering. According to the method of moments, the system of equations that can be set up to solve for θ and α can be conceived of as follows:

E[X] = E[M].
$E[X^2] = E[M^2]$.
For a two-parameter Pareto distribution,
$E[X] = \theta/(\alpha - 1)$ and $E[X^2] = 2\theta^2/[(\alpha - 1)(\alpha - 2)]$.
We can compute E[M] = (2000 + 13515 + 400000 + 16437 + 4993 + 12000)/6 = 74824.1666667
We can compute $E[M^2] = (2000^2 + 13515^2 + 400000^2 + 16437^2 + 4993^2 + 12000^2)/6$ = 26770960040.5
Thus, our system of equations appears as follows.
(i) $\theta/(\alpha - 1) = 74824.1666667$
(ii) $2\theta^2/[(\alpha - 1)(\alpha - 2)] = 26770960040.5$.
We divide (ii) by (i) to get
(iii) $2\theta/(\alpha - 2) = 357784.9408$.
We transform (i) and (iii):
(i)': $\theta = 74824.1666667(\alpha - 1)$
(iii)': $\theta = 178892.4704(\alpha - 2)$
Thus, we have the following equality:
$74824.1666667(\alpha - 1) = 178892.4704(\alpha - 2)$
$\alpha - 1 = 2.390838126(\alpha - 2)$
$\alpha - 1 = 2.390838126\alpha - 4.781676252$
$3.781676252 = 1.390838126\alpha$
α = about 2.718990931.

Problem S3L54-3. Similar to Question 4 from the Casualty Actuarial Society's Fall 2006 Exam 3. The probability distribution of the random variable N is dependent on the value of parameter θ, which can be either θ_0 or θ_1. The probability distributions that result are illustrated as follows:

n......f(n; θ_0).... f(n; θ_1)

3.......0.3..........0.12

4......0.03.........0.135

5......0.312........0.123

6......0.158........0.422

7......0.12.........0.07

8......0.08.........0.13

You are testing the following two hypotheses using the Neyman-Pearson Lemma and a single observation: H_0: $\theta = \theta_0$ and H_1: $\theta = \theta_1$.

Which of these points is in the critical region defined by the significance level of 10%? More than one correct answer is possible.

(a) n is 3

(b) n is 4

(c) n is 5

(d) n is 6

(e) n is 7

(f) n is 8

Solution S3L54-3. We use the fact that a point is within the critical region defined by a given significance level k if $P(N = n \mid \theta = \theta_0) < k$. Here, we need to find all points n for which $P(N = n \mid \theta = \theta_0) < 0.1$.

(a) n is 3 → $P(N = 3 \mid \theta = \theta_0) = 0.3 > 0.1$, so (a) is not in the critical region.

(b) n is 4 → $P(N = 4 \mid \theta = \theta_0) = 0.03 < 0.1$, so (b) is in the critical region.

(c) n is 5 → $P(N = 5 \mid \theta = \theta_0) = 0.312 > 0.1$, so (c) is not in the critical region.

(d) n is 6 → $P(N = 6 \mid \theta = \theta_0) = 0.158 > 0.1$, so (d) is not in the critical region.

(e) n is 7 → $P(N = 7 \mid \theta = \theta_0) = 0.12 > 0.1$, so (e) is not in the critical region.

(f) n is 8 → $P(N = 8 \mid \theta = \theta_0) = 0.08 < 0.1$, so (f) is in the critical region.

Thus, **(b) and (f) are the correct answers.**

Problem S3L54-4. Similar to Question 6 from the Casualty Actuarial Society's Fall 2007 Exam 3. You are using two methods to estimate the parameter θ for an exponential distribution. The estimate θ_a is the maximum likelihood estimate. The estimate θ_b is obtained via the method of moments. Assume you have a sample of size n, and let the sample mean be equal to θ_s. What is the value of $\theta_a - \theta_b$, expressed using solely numbers and/or the values θ_s and n?

Solution S3L54-4. We find, θ_b by using the fact that the method of moments system of equations in this case will be the single equation E[X] = E[M].We are given the sample mean $E[M] = \theta_s$, and we know that the parameter θ is the mean of an exponential distribution. So E[X] from the method of moments is just θ_b, and therefore, $\theta_b = \theta_s$.

Now we find $\theta_a = \max(L(\theta))$, where $L(\theta) = {}_{i=1}{}^{n}\Pi\, f_Y(y_i; \theta)$.

For every y_i under an exponential distribution, $f_Y(y_i; \theta) = (1/\theta)\exp[-y_i/\theta]$.

Thus, $L(\theta) = {}_{i=1}{}^{n}\Pi\, f_Y(y_i; \theta) = (1/\theta)^n\exp[-{}_{i=1}{}^{n}\sum y_i/\theta]$

To find the maximum of its function, we can set its derivative equal to zero.

However, the natural logarithm of this function will also have a derivative of zero wherever the original function has a derivative of zero, and the natural logarithm function is easier to work with.

We find $N(\theta) = \ln(L(\theta)) = \ln[(1/\theta)^n \exp[-_{i=1}{}^n\sum y_i/\theta]] = \ln(1/\theta)^n + \ln(\exp[-_{i=1}{}^n\sum y_i/\theta]]) =$

$n*\ln(1/\theta) + -_{i=1}{}^n\sum y_i/\theta$.

We also note that θ_s, the sample mean, is equal to $(1/n)\ _{i=1}{}^n\sum y_i$, and so $_{i=1}{}^n\sum y_i = n\theta_s$.

Thus, $N(\theta) = n*\ln(1/\theta) + n\theta_s/\theta$.
We take $N'(\theta) = n/\theta - n\theta_s/\theta^2$. We then set $N'(\theta_a) = 0$.
Thus, $0 = n/\theta_a - n\theta_s/\theta_a{}^2$.
$n/\theta_a = n\theta_s/\theta_a{}^2$.
$n\theta_a{}^2/\theta_a = n\theta_s$.
$\theta_a = \theta_s$.

We have shown that for *any* exponential distribution and *any* sample of *any* size n, $\theta_a = \theta_b = \theta_s$. That is, both the maximum likelihood estimate and the estimate using the method of moments will be equal to the sample mean. Therefore, it is always the case that $\boldsymbol{\theta_a - \theta_b = 0}$.

Problem S3L54-5. Similar to Question 7 from the Casualty Actuarial Society's Fall 2007 Exam 3. The random variable Y is a sum of random variables X_1 through X_n, each of which follows a Poisson distribution with mean Λ. Here, the sample size n is equal to 20. You are performing a test between two hypotheses: H_0: $\Lambda = 0.7$ and H_1: $\Lambda < 0.7$. The critical region for this test for rejecting H_0 is $Y \leq 2$. What is the significance level of the this hypothesis test? **Hint:** The sum of n Poisson variables, each with mean Λ, has mean $n\Lambda$. **Hint II:** The answer will be very small!

Solution S3L54-5. We recall that Broverman defines a significance levelfor a hypothesis test as "the probability of rejecting the hypothesis H_0, given that H_0 is true."
This is the probability $\Pr(Y \leq 2 \mid H_0$ is true.$) = \Pr(Y \leq 2 \mid \Lambda = 0.7)$.
If H_0 is true, then Y is the sum of 20 Poisson random variables, each with mean $\Lambda = 0.7$.
Thus, Y has a mean of $\lambda = 0.7*20 = 14$.
Thus, we want to find $\Pr(Y \leq 2 \mid \lambda = 14) = \Pr(Y = 0, Y = 1, \text{or } Y = 2 \mid \lambda = 14) =$
$e^{-\lambda} + \lambda e^{-\lambda} + (1/2)\lambda^2 e^{-\lambda} = e^{-14} + 14e^{-14} + (1/2)14^2e^{-14} =$ **about 0.00009396274526**.

Section 55

Chi-Square Goodness-of-Fit Test

The **Chi-square distribution (χ^2 distribution)** can be defined as follows:

The p.d.f. of $U = {}_{j=1}{}^{m}\sum[Z_j{}^2]$, where each Z_j is a standard normal random variable independent of all the other Z_j's, is called the **Chi-square distribution with m degrees of freedom.** (Larsen and Marx 2006, p. 474).

It is easy to remember the Chi-square distribution as the p.d.f. that results from adding the *squares* of m *standard normal random variables*, where m is the number of degrees of freedom for the Chi-square distribution.

The chi-square distribution has **critical values** associated with a given **significance level** and a number of degrees of freedom. You can look up these critical values in a Table of Critical Values for the Chi-Square Distribution.

Larsen and Marx define a **goodness of fit test** as "any procedure that seeks to determine whether a set of data could reasonably have originated from some given probability distribution, or *class* of probability distributions" (599).

The **Chi-square goodness of fit test** can be phrased as follows.

"Let $r_1, r_2, ..., r_t$ be the set of possible outcomes (or ranges of outcomes) associated with each of n independent trials, where $P(r_i) = p_i$, for i = 1, 2, ..., t. Let X_i = the number of times r_i occurs. Then the following holds.

a. The random variable $D = {}_{i=1}{}^{t}\sum[(X_i - np_i)^2/np_i]$ has approximately a χ^2 distribution with t-1 degrees of freedom. For the approximation to be adequate, the t classes should be defined so that $np_i \geq 5$ for all i.

b. Let $k_1, k_2, ..., k_t$ be the observed frequencies for the outcomes $r_1, r_2, ..., r_t$, respectively, and let $np_{1o}, np_{2o}, ..., np_{to}$ be the corresponding expected frequencies based on the null hypothesis. At the α level of significance, H_0: $f_Y(y) = f_o(y)$ (or H_0: $p_X(k) = p_o(k)$) is rejected if

$d = {}_{i=1}{}^{t}\sum[(k_i - np_{io})^2/(np_{io})] \geq \chi^2{}_{1-\alpha,t-1}$." (Larsen and Marx 2006, p. 616)

For instance, if we have a distribution of observations, and we know both the expected numbers of observations of each kind and the actual numbers of observations of this kind, then we can let k_i be the actual number of observations of the type i and np_{io} be the expected number of observations of the type i. Then we can find $d = {}_{i=1}{}^{t}\sum[(k_i - np_{io})^2/(np_{io})]$ and compare it to

$\chi^2{}_{1-\alpha,t-1}$, the chi-square statistic with t-1 degrees of freedom and significance level of 1-α.

If $d \geq \chi^2_{1-\alpha,t-1}$, then we reject the null hypothesis H_0.

Sources: Broverman, Sam. Actuarial Exam Solutions - CAS Exam 3 - Fall 2006.

Larsen, Richard J. and Morris L. Marx. *An Introduction to Mathematical Statistics and Its Applications.* Fourth Edition. Pearson Prentice Hall: 2006. pp. 474, 599, 616.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L55-1. Consider the following table of observed sightings of leopard-mice, by territory and color.
.......................White....Blue....Yellow.....Magenta........Total
Territory 1..........3........36.......63.............62..............164
Territory 2..........15......531.....351............11...............908
Territory 3..........315.....60.......14..............2..............391
Total.................333.....627......428............75............1463.

If you have a hypothesis regarding the distribution of mice in each territory and you use a Chi-square goodness-of-fit test to evaluate your hypothesis, how many degrees of freedom would the associated Chi-square distribution need to have?

Solution S3L55-1. It is useful to think of degrees of freedom as the number of values in the table that you can pick without being constrained by your previous choices to make some particular choice. For instance, in Territory 1, you can pick the number of white, blue, and yellow leopard-mice, but these three choices will determine the number of magenta leopard-mice, provided that the total number of mice in Territory 1 must remain fixed at 164. So you have 3 degrees of freedom from Territory 1. Likewise, in Territory 2, you can make 3 selections for the numbers of three kinds of mice, which will entirely determine the number of mice of the fourth kind. You have *no* degrees of freedom from Territory 3, because the totals of each kind of mice by color are fixed and cannot be changed, so your selections from Territories 1 and 2 will determine the values in Territory 3. Thus, the Chi-square distribution associated with this goodness-of-fit test will need to have **6 degrees of freedom.**

Problem S3L55-2. Consider the following table of observed sightings of leopard-mice, by territory and color.
.......................White....Blue....Yellow.....Magenta........Total
Territory 1..........3........36.......63.............62..............164
Territory 2..........15......531.....351............11...............908
Territory 3..........315.....60.......14..............2..............391
Total.................333.....627......428............75............1463.

You have a hypothesis regarding the distribution of mice in each territory and you use a Chi-square goodness-of-fit test to evaluate your hypothesis. Your hypothesis is that in each territory, the distribution of mice by color is the same. The significance level of the Chi-square statistic you are using is 10%. Find the critical value associated with the significance level for this test. Use a Table of Critical Values for the Chi-Square Distribution.

Solution S3L55-2. We want the value corresponding to a significance level of 0.1 and 6 degrees of freedom (as we found in Solution S3L55-1). We look up this value in the table and find it to be **10.645**.

Problem S3L55-3. Consider the following table of observed sightings of leopard-mice, by territory and color.
.......................White....Blue....Yellow.....Magenta........Total
Territory 1..........3........36.......63.............62..............164
Territory 2..........15......531.....351............11..............908
Territory 3..........315.....60.......14..............2..............391
Total.................333.....627......428............75............1463.

You have a hypothesis regarding the distribution of leopard-mice in each territory and you use a Chi-square goodness-of-fit test to evaluate your hypothesis. Your hypothesis is that in each territory, the distribution of mice by color is the same. The significance level of the Chi-square statistic you are using is 10%. Create a table of *expected* observations of leopard-mice by territory if you believe that the *total* distribution of mice among territories is what will be reflected in the distribution of mice by color. Round your answers to the nearest whole leopard-mouse in each case.

Solution S3L55-3. For each color of leopard-mouse, we would expect 164/1463 = 0.1120984279 of the mice to be in Territory 1, 908/1463 = 0.6206425154 of the mice to be in Territory 2, and 391/1463 = 0.2672590567 of the mice to be in Territory 3, if our null hypothesis is indeed correct. To fit these probabilities in our table while doing computations, we can let A = 0.1120984279, B = 0.6206425154, and C = 0.6206425154. (If you have a TI-83 Plus calculator, you can actually program it to associate these letters with the probabilities given here.)

Thus, we get the following expected distribution of observations:
.......................White....Blue....Yellow.....Magenta........Total
Territory 1..........333A....627A ...428A........75A...............164
Territory 2..........333B... 627B.....428B........75B...............908
Territory 3..........333C....627C.... 428C........75C...............391
Total.................333......627......428..........75............1463.

This is the same table as the following, with all values rounded to the nearest whole number.
.......................White....Blue....Yellow.....Magenta........Total
Territory 1..........37.........70.......48...........8...............164
Territory 2..........207..... 389......266...........47...............908
Territory 3..........89.......168......114..........20..............391
Total.................333......627......428..........75............1463.

Problem S3L55-4. Similar to Question 5 from the Casualty Actuarial Society's Fall 2006 Exam 3.

Consider the following table of observed sightings of leopard-mice, by territory and color.
.......................White....Blue....Yellow.....Magenta........Total
Territory 1..........3........36.......63.............62..............164
Territory 2..........15......531.....351............11...............908
Territory 3..........315.....60.......14..............2..............391
Total.................333.....627......428............75............1463.

You have a hypothesis regarding the distribution of leopard-mice in each territory and you use a Chi-square goodness-of-fit test to evaluate your hypothesis. Your hypothesis is that in each territory, the distribution of mice by color is the same. The significance level of the Chi-square statistic you are using is 10%. Find the *test statistic* d for this Chi-square goodness of fit test. For your table of expected values, use the answer to Problem S3L55-3 (i.e., use the table with rounded values. The work will proceed much faster if you do.). Also decide whether this test would require you to reject or fail to reject the null hypothesis.

Solution S3L55-4. We have the following to tables to consider.

Table of actual observations (Table for values of k_i):
.......................White....Blue....Yellow.....Magenta........Total
Territory 1..........3........36.......63.............62..............164
Territory 2..........15......531.....351............11...............908
Territory 3..........315.....60.......14..............2..............391
Total.................333.....627......428............75............1463.

Table of expected observations (Table for values of np_{io}):
.......................White....Blue....Yellow.....Magenta........Total
Territory 1..........37.........70.......48...........8..............164
Territory 2..........207..... 389......266..........47...............908
Territory 3..........89.......168......114..........20..............391
Total.................333......627......428..........75............1463.

We use the formula $d = {}_{i=1}{}^{t}\sum[(k_i - np_{io})^2/(np_{io})] =$
$(3 - 37)^2/37 + (36 - 70)^2/70 + (63 - 48)^2/48 + (62 - 8)^2/8 + (15 - 207)^2/207 + (531 - 389)^2/389 +$
$(351 - 266)^2/266 + (11 - 47)^2/47 + (315 - 89)^2/89 + (60 - 168)^2/168 + (14 - 114)^2/114.$

Thus, **d = about 1363.212281**. We recall from Solution S3L55-2 that our critical value for this test is 10.645. Since d is substantially greater than 10.645, **we reject the null hypothesis.**

Problem S3L55-5. Similar to Question 5 from the Casualty Actuarial Society's Fall 2006 Exam 3. You have the following table of expected observations for the number of managers in a given large firm who are completely idle, distributed by their positions in the management hierarchy.

Upper ....Upper-Mid-Level ....Mid-Mid-Level......Lower-Mid-Level............Total
20.............70..................................150...................560.................................800

Your null hypothesis is that the distribution of idle managers corresponds to the above table.

Here is what you actually observe with regard to managerial idleness:

Upper ....Upper-Mid-Level ....Mid-Mid-Level......Lower-Mid-Level............Total
16.............96..................................175...................513......,,,,,,......................800

You are performing a Chi-square goodness-of-fit test on your hypothesis. Your significance level for this test is 5%. Find the difference between the test statistic d and the critical value c for this test. Use a Table of Critical Values for the Chi-Square Distribution as necessary. Would you need to reject the null hypothesis?

Solution S3L55-5. The Chi-square distribution associated with this test will have 3 degrees of freedom, because once we have picked any 3 values in the table, the fourth value will be determined by our prior choices. We look up the critical value c, the value associated with 3 degrees of freedom and a significance level of 5%. This value is 7.815.

Now we find d = ${}_{i=1}{}^{t}\sum[(k_i - np_{io})^2/(np_{io})] = (16 - 20)^2/20 + (96 - 70)^2/70 + (175 - 150)^2/150 +$

$(513 - 560)^2/560$ = d = 18.56845238.

Now we find d - c = 18.56845238 - 7.815 = **10.75345238.** Since d - c > 0, **the null hypothesis needs to be rejected.**

Section 56

Hypothesis Testing: Type I and Type II Errors, P-Values, and the Student-t Distribution

A **Type I error** occurs when one rejects the null hypothesis H_0, even though H_0 is true. Thus, the probability of a Type I error occurring is **Pr(Type I error) = Pr(reject H_0 │ H_0 is true)**.

The **level of significance** of a hypothesis test is precisely the probability of committing a Type I error. The symbol **α** is often used to denote the level of significance.

A **Type II error** occurs when one fails to reject the null hypothesis H_0, even though H_0 is false. Thus, the probability of a Type I error occurring is **β =Pr(Type II error) = Pr(Do not reject H_0 │ H_0 is false)**.

The complement of β, **1- β**, is the probability of rejection of H_0 when H_1 is true. This value is known as the **power** of the test. It is often intuitively easier to find 1- β first and then determine β from it.

According to Larsen and Marx, "The **p-value** associated with an observed test statistic is the probability of getting a value for that test statistic as extreme or more extreme than what was actually observed (relative to H_1) *given that H_0 is true*" (437). If the p-value of a given test is less than or equal to α, then we can reject the null hypothesis at the significance level of α. Another way to define the p-value is as "the smallest α at which we can reject H_0" (437).

Now we will discuss the **Student t distribution.**

Following Larsen and Marx, "Let Z be a standard normal random variable and let U be a chi square random variable independent of Z with n degrees of freedom. The *Student t ratio with n degrees of freedom* is denoted T_n, where $T_n = Z/\sqrt{(U/n)}$" (476).

In this section, we will be focusing on hypothesis testing using the Student t distribution and more particularly, the following property.

Following Larsen and Marx, "Let $Y_1, Y_2, ..., Y_n$ be a random sample from a normal distribution with mean μ and standard deviation σ. Then

$T_{n-1} = (\hat{Y} - \mu)/(S/\sqrt{(n)})$ has a Student t distribution with n - 1 degrees of freedom." (Ordinarily, when circumstances permit it, the "Y" has a *straight line* over the top.)

Reminder: Here, S is the *sample* standard deviation, and σ is the standard deviation of the given normal distribution. These may or may not be the same!

In any particular test involving the Student t distribution, you will have some random sample of size n, drawn from a normal distribution. You can find the *test statistic* t, where

$t = (\mu_{actual} - \mu_{hypothesized})/(S/\sqrt{(n)})$. Here, μ_{actual} is the observed sample mean and $\mu_{hypothesized}$ is the mean value predicted via the null hypothesis H_0.

Once you find the value of the test statistic t, you can find the p-value associated with the hypothesis test. The p-value of t is equal to Pr(|T| > t) or Pr(T > t) or Pr(T < t), depending on the nature of the null hypothesis.

If for some constant k, H_0: μ = k, then p = Pr(|T| > t).

If for some constant k, H_0: μ ≤ k, then p = Pr(T > t).

If for some constant k, H_0: μ ≥ k, then p = Pr(T < t).

The p-value of t can be found by examining the Table of Percentage Points of the t Distribution. Look in the row with the appropriate number of degrees of freedom (n - 1) and then find the value that corresponds to the observed test statistic t. Then look at the heading of the column to see which *tail probabilities* t corresponds to. If you seek to find Pr(|T| > t), then your desired p-value is the probability associated with *two tails* of the distribution. If you seek to find Pr(T > t), then your desired p-value is the probability associated with *one tail* of the distribution.

Source: Broverman, Sam. Actuarial Exam Solutions - CAS Exam 3 - Fall 2006.

Larsen, Richard J. and Morris L. Marx. *An Introduction to Mathematical Statistics and Its Applications.* Fourth Edition. Pearson Prentice Hall: 2006. pp. 436-437, 447-449, 476-478.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L56-1. Similar to Question 6 from the Casualty Actuarial Society's Fall 2006 Exam 3. You are using a 7% level of significance for a hypothesis test where H_0 states that the random variable X is uniformly distributed on the interval [5, 46], and H_1 states that the random variable X is uniformly distributed on the interval [36,46]. Your hypothesis test uses only a single observation. Find the probability of a Type II error associated with this test.

Solution S3L56-1. We are given the level of significance, which is the same as Pr(Type I error) = Pr(reject H_0 | H_0 is true) = 0.07. If H_0 is true, then X is uniformly distributed on the interval

[5, 46]. We note that H_1 states that X can assume values within the upper tail of the distribution of X according to H_0. If we are to reject H_0, should do so if the values of x we get are within the highest 7% of the values in the interval [5, 46]. The minimum value of x for which we reject H_0 is (46-5)(1-0.07) + 5 = 43.13.

Now we can calculate 1- β = Pr(Reject H_0 | H_1 is true). If H_1 is true, then X is uniformly distributed on [36,46]. Thus, 1- β = Pr(x $\geq$ 43.13 | X is uniformly distributed on [36,46]) =

(46-43.13)/(46-36) = 0.287. Thus, our desired probability is β = 1 - 0.287 = **β = 0.713**.

Problem S3L56-2. Similar to part of Question 7 from the Casualty Actuarial Society's Fall 2006 Exam 3. You have collected a sample of 37 values from a normal distribution. The sample mean is μ_X = 5. You also know that $_{i=1}{}^{37}\sum(X_i - \mu_X)/36$ = 12.98644507.

You are testing the following two hypotheses.

H_0: μ = 4

H_1: $\mu \neq 4$

What is the test statistic t for this test?

Solution S3L56-2. Since we have a random sample drawn from a normal distribution, we need to use the test statistic t, which can be found via the following formula:

t = (μ_{actual} - $\mu_{hypothesized}$)/(S/$\sqrt{}$(n)). We are given μ_{actual} = 5, $\mu_{hypothesized}$ = 4, and n = 37. The expression $_{i=1}{}^{37}\sum(X_i - \mu_X)/36$ is equivalent to the *sample variance*, or S^2. Thus, we know that S^2 = 12.98644507 and so S = 3.603532305.

Now we can find t = (5 - 4)/(3.603532305/$\sqrt{}$(37)) = **t = 1.688**.

Problem S3L56-3. Similar to part of Question 7 from the Casualty Actuarial Society's Fall 2006 Exam 3. You have collected a sample of 37 values from a normal distribution. The sample mean is μ_X = 5. You also know that $_{i=1}{}^{37}\sum(X_i - \mu_X)/36$ = 12.98644507.

You are testing the following two hypotheses.

H_0: μ = 4

H_1: $\mu \neq 4$

What is the p-value for this test? Use the Table of Percentage Points of the t Distribution.

Solution S3L56-3. In Solution S3L56-2, we found our test statistic t to be 1.688. The probability we seek to find as our p-value is Pr(|T| > t). Why do we use the absolute value of T rather than just T? We do this because the null hypothesis is a statement of *equality* with respect to the mean. If the actual mean gets too far from 4 in *either* direction, we will reject the null hypothesis. Thus, in our t-Distribution table, we will need to consider the *two-tail* probabilities.

We look in the row corresponding to 37-1 = 36 degrees of freedom. Conveniently enough (you knew this was going to happen), we find our t value of 1.688 in the column corresponding to a two-tail probability of 0.10. Thus, for this test, **p = 0.10.**

Problem S3L56-4. You have collected a sample of 20 values from a normal distribution. The sample mean is $\mu_X = 19$. You also know that ${}_{i=1}{}^{20}\sum(X_i - \mu_X)/19 = 18.26213422$.

You are testing the following two hypotheses.

H_0: $\mu \leq 17$

H_1: $\mu > 17$

What is the test statistic t for this test?

Solution S3L56-4. Since we have a random sample drawn from a normal distribution, we need to use the test statistic t, which can be found via the following formula:

$t = (\mu_{actual} - \mu_{hypothesized})/(S/\sqrt{(n)})$. We are given $\mu_{actual} = 19$, $\mu_{hypothesized} = 17$, and n = 20. The expression ${}_{i=1}{}^{20}\sum(X_i - \mu_X)/19$ is equivalent to the *sample variance*, or S^2. Thus, we know that $S^2 = 18.26213422$ and so S = 4.273421839.

Now we can find $t = (19 - 17)/(4.273421839/\sqrt{(20)})$ = **t = 2.093**.

Problem S3L56-5. You have collected a sample of 20 values from a normal distribution. The sample mean is $\mu_X = 19$. You also know that ${}_{i=1}{}^{20}\sum(X_i - \mu_X)/19 = 18.26213422$.

You are testing the following two hypotheses.

H_0: $\mu \leq 17$

H_1: $\mu > 17$

What is the p-value for this test? Use the Table of Percentage Points of the t Distribution.

Solution S3L56-5. In Solution S3L56-4, we found our test statistic t to be 2.093. The probability we seek to find as our p-value is Pr(T > t). Why do we use just T instead of the absolute value of T? We do this because the null hypothesis is a statement of *inequality* with respect to the mean. We only reject the null hypothesis if our observed sample mean becomes *too large*.

Thus, in our t-Distribution table, we will need to consider the *one-tail* probabilities.

We look in the row corresponding to 20-1 = 19 degrees of freedom. Conveniently enough (you knew this was going to happen), we find our t value of 2.093 in the column corresponding to a one-tail probability of 0.025. Thus, for this test, **p = 0.025.**

Section 57

Least-Squares Regression and Assorted Exam-Style Questions

Here, we will discuss the method of **least-squares regression.** According to Larsen and Marx, "Given n points $(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)$, the straight line y = a + bx minimizing

$L = {}_{i=1}{}^{n}\sum[y_i - (a + bx_i)]^2$ has slope

$\mathbf{b = [n*{}_{i=1}{}^{n}\sum(x_iy_i) - {}_{i=1}{}^{n}\sum(x_i)\, {}_{i=1}{}^{n}\sum(y_i)]/[n*{}_{i=1}{}^{n}\sum(x_i^2) - {}_{i=1}{}^{n}\sum(x_i)^2]}$

and y-intercept

$\mathbf{a = [{}_{i=1}{}^{n}\sum(y_i) - b*{}_{i=1}{}^{n}\sum(x_i)]/n}$." (647-648)

Alternatively, $\mathbf{b = {}_{i=1}{}^{n}\sum[(x_i - \mu_x)(y_i - \mu_y)]/{}_{i=1}{}^{n}\sum(x_i - \mu_x)^2}$ **and** $\mathbf{a = \mu_y - b*\mu_x}$, where μ_y is the mean of the values y_1 through y_n and μ_x is the mean of the values x_1 through x_n.

Source: Broverman, Sam. Actuarial Exam Solutions - CAS Exam 3 - Fall 2006.

Larsen, Richard J. and Morris L. Marx. *An Introduction to Mathematical Statistics and Its Applications.* Fourth Edition. Pearson Prentice Hall: 2006. pp. 647-648.

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Problem S3L57-1. Similar to Question 8 from the Casualty Actuarial Society's Fall 2006 Exam 3. The number y_i of complimentary ordinary widgets that you get with the purchase of x_i superwidgets, follows the relationship $y_i = a + bx_i$, where a and b are found using the method of least-squares regression. Today, you purchase 54 superwidgets and are given the following additional information. ${}_{i=1}{}^{17}\sum[(x_i - \mu_x)(y_i - \mu_y)] = 450$ and ${}_{i=1}{}^{17}\sum(x_i - \mu_x)^2 = 50$. Moreover, the mean μ_x is equal to 60 and the mean μ_y is equal to 194. Find how many complimentary ordinary widgets you will be getting. Non-whole numbers of superwidgets are acceptable.

Solution S3L57-1. We first find b using the formula $b = {}_{i=1}{}^{n}\sum[(x_i - \mu_x)(y_i - \mu_y)]/{}_{i=1}{}^{n}\sum(x_i - \mu_x)^2 =$ 450/50 = b = 9. Now we find $a = \mu_y - b*\mu_x$ = 194 - 9*60 = -346.

Then, for $x_i = 54$, we have $y_i = a + bx_i$ = -346 + 9*54 = **140 ordinary widgets.**

Problem S3L57-2. You are given the following points (4, 5), (6, 7), (10, 23). Using the method of least-squares regression, where y = a + bx, find b.

Solution S3L57-2. We use the formula

$b = [n*{}_{i=1}{}^{n}\sum(x_iy_i) - {}_{i=1}{}^{n}\sum(x_i)\ {}_{i=1}{}^{n}\sum(y_i)]/[n*{}_{i=1}{}^{n}\sum(x_i{}^2) - {}_{i=1}{}^{n}\sum(x_i)^2]$, where n = 3.

We have ${}_{i=1}{}^{n}\sum(x_iy_i) = 4*5 + 6*7 + 10*23 = 292$.

Moreover, ${}_{i=1}{}^{n}\sum(x_i) = 4 + 6 + 10 = 20$ and ${}_{i=1}{}^{n}\sum(y_i) = 5 + 7 + 23 = 35$.

Moreover, ${}_{i=1}{}^{n}\sum(x_i{}^2) = 4^2 + 6^2 + 10^2 = 152$ and ${}_{i=1}{}^{n}\sum(x_i)^2 = 20^2 = 400$.

Thus, b = (3*292 - 20*35)/(3*400 - 152) = **b = about 0.1679389313**.

Problem S3L57-3. You are given the following points of the form (x, y): (4, 5), (6, 7), (10, 23). Using the method of least-squares regression, where y = a + bx, find a.

Solution S3L57-3. We use the formula $a = [{}_{i=1}{}^{n}\sum(y_i) - b*{}_{i=1}{}^{n}\sum(x_i)]/n$, where n = 3 and b = 0.1679389313 (from Solution S3L57-2). We also know that

${}_{i=1}{}^{n}\sum(x_i) = 4 + 6 + 10 = 20$ and ${}_{i=1}{}^{n}\sum(y_i) = 5 + 7 + 23 = 35$.

Thus, a = (35 - 20*0.1679389313)/3 = **a = about 10.54707379**.

Problem S3L57-4. Similar to Question 9 from the Casualty Actuarial Society's Fall 2006 Exam 3. This question is also an excellent review of the concepts in Section 53.

You are taking a sample of 30 random values from an exponential distribution whose mean θ is equal to 100.

The ith order statistic of Y has the following probability density function (p. d. f.):
$f_{Y'\_i}(y) = (n!/[(i-1)!(n-i)!])*F(y)^{i-1}*(1-F(y))^{n-i}*f(y)$, where f(y) is the p. d. f. of Y and F(y) is the cumulative distribution function (c. d. f.) of Y.

Find the probability that the *second* order statistic of Y is between 40 and 80. The answer will be very small!

Solution S3L57-4. We are given that Y follows an exponential distribution with mean 100. Thus, $f(y) = 0.01e^{-0.01y}$, $F(y) = 1 - e^{-0.01y}$, and $1 - F(y) = e^{-0.01y}$. Here, n = 30 and i = 2, so

$f_{Y'\_2}(y) = (30!/[(1)!(28)!])*(1 - e^{-0.01y})*(e^{-0.01y})^{28}*0.01e^{-0.01y} =$

$f_{Y'\_2}(y) = 8.7(1 - e^{-0.01y})(e^{-0.29y})$

$f_{Y'\_2}(y) = 8.7(e^{-0.29y} - e^{-0.3y})$

Thus, $\Pr(40 < Y_2 < 80) = {}_{40}{}^{80}\int 8.7(e^{-0.29y} - e^{-0.3y})dy =$

$(8.7/-0.29)(e^{-0.29y}) - (8.7/-0.3)(e^{-0.3y}) \Big|_{40}^{80} =$

$[(8.7/-0.29)(e^{-0.29*80}) - (8.7/-0.3)(e^{-0.3*80})] - [(8.7/-0.29)(e^{-0.29*40}) - (8.7/-0.3)(e^{-0.3*40})] =$

$Pr(40 < Y_2 < 80)$ = about 0.00009679904811.

Problem S3L57-5. Similar to Question 5 from the Casualty Actuarial Society's Spring 2008 Exam 3L. This question is also an excellent review of the concepts in Section 56.

X is a normally distributed random variable, whose standard deviation σ is 2 and whose mean μ is either 50 or 55. You perform a hypothesis test between these two hypotheses:
H_0: σ = 2, μ = 50.

H_1: σ = 2, μ = 55.

You observe one random value x of X and will reject the null hypothesis if x > k, for some specific k. The probability of a Type I error is 0.1469. What is the probability of Type II error?

Use the **Table of Values for the Standard Normal Distribution**.

Solution S3L57-5. Pr(Type I error) = Pr(reject H_0 | H_0 is true) = Pr(x > k | μ = 50) = 0.1469.

For the standard normal random variable Z and observations z of Z, we know that

Pr(z > k | μ = 50) = 0.1469, so 1 - Φ(z) = 0.1469. so Φ(z) = 0.8531.

Using the **Table of Values for the Standard Normal Distribution**, we find that z = 1.05, which means that x = 50 + 1.05*2 = x = 52.1.

Thus, if x > 52.1, we accept hypothesis H_1.

Now we can find Pr(Type II error) = Pr(Do not reject H_0 | H_0 is false) = Pr(x < 52.1 | μ = 55).

We find the z value corresponding to 52.1, if σ = 2 and μ = 55.

This value is z = (52.1 - 55)/2 = -1.45.

Thus, Pr(x < 52.1 | μ = 55) = Φ(-1.45) = 1 - Φ(1.45) = 1 - 0.9265 = **Pr(Type II error) =0.0735.**

Section 58

Fully Discrete Benefit Reserves

Let us suppose we have a life insurance policy that pays some benefits and is paid for via some benefit premiums.

Then the **benefit reserve** at time t is "the conditional expectation of the difference between the present value of future benefits and the present value of future benefit premiums, the conditional event being survivorship of the insured to time t." (Bowers et. al. 1997, p. 205).

Fully discrete benefit reserves occur in conjunction with fully discrete benefit premiums (see Section 52) and insurance policies paying benefits at the end of the year of death (see Section 34).

For a fully discrete benefit reserve on a whole life insurance policy on life (x) on the condition of that life's survival for k additional years, the value of the benefit reserve is denoted $_kV_x$ and can be found via the following formula.

$\mathbf{_kV_x = A_{x+k} - P_x*ä_{x+k}}$

We recall that $P_x = A_x/ä_x$, where A_x is the actuarial present value of a fully discrete whole life insurance policy and $ä_x$ is the actuarial present value of a fully discrete whole life annuity-due. We recall also from Section 41 that $\mathbf{ä_x = {}_{k=0}{}^{\infty}\Sigma v^k*{}_kp_x}$.

Alternatively, $\mathbf{_kV_x = 1 - ä_{x+k}/ä_x}$.

Each of the above formulas may be useful in different circumstances.

For a fully discrete n-year endowment insurance policy with actuarial present value $A_{x:n¬}$, the benefit reserve at time k < n can be denoted as $\mathbf{_kV_{x:n¬}}$ and can be found as follows:

$\mathbf{_kV_{x:n¬}} = \mathbf{1 - ä_{x+k:n-k¬}/ä_{x:n¬}}$.

Prospective formula: $\mathbf{_kV_{x:n¬}} = \mathbf{A_{x+k:n-k¬} - P_{x:n¬}*ä_{x+k:n-k¬}}$

Retrospective formula: $\mathbf{_kV_{x:n¬} = (1/_kE_x)(P_{x:n¬}*\, ä_{x:k¬} - A^1{}_{x:k¬})}$

Recall: $ä_{x:n¬} = {}_{k=0}{}^{n-1}\Sigma v^k*{}_kp_x$, $P_{x:n¬} = A_{x:n¬}/ä_{x:n¬}$, $_kE_x = v^n*{}_np_x$, and $A^1{}_{x:n¬} = {}_{k=0}{}^{n-1}\sum v^{k+1}*{}_kp_x*q_{x+k}$.

Some of the problems in this section were designed to be similar to problems from past versions of the Casualty Actuarial Society's Exam 3L and the Society of Actuaries' Exam MLC. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1997. Second Edition. Society of Actuaries: Itasca, Illinois. pp. 205, 215-218.

Broverman, Sam. [Actuarial Exam Solutions - CAS Exam 3 - Fall 2006].

Broverman, Sam. [Actuarial Exam Solutions - CAS Exam 3 - Fall 2007].

Broverman, Sam. [Actuarial Exam Solutions - CAS Exam 3L - Spring 2008].

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Problem S3L58-1. Iago owns a whole life insurance policy which pays a benefit of 1 whose benefits are payable at the end of the year of death. Iago is currently 35 years old and can purchase a discrete whole life annuity-due that pays a benefit of 1 at the beginning of each year for \$65. If Iago survived for another 17 years, he would be able to purchase such a whole life annuity-due for \$56. Find the fully discrete benefit reserve on Iago's insurance policy 17 years from now.

Solution S3L58-1. We want to find ${}_{17}V_{35}$, which we will do using the formula ${}_kV_x = 1 - \ddot{a}_{x+k}/\ddot{a}_x$.

We are given x = 35, k = 17, $\ddot{a}_{35} = 65$, and $\ddot{a}_{52} = 56$. Thus, ${}_{17}V_{35} = 1 - 56/65 =$ **${}_{17}V_{35}$ = about 0.1384615385**.

Problem S3L58-2. Roland is 95 years old and owns a whole life insurance policy that pays a benefit of 1 at the end of the year of death and whose benefit premium 0.19. In 29 years, the actuarial present value of Roland's policy will be 0.89, and Roland will be able to purchase can purchase a discrete whole life annuity-due that pays a benefit of 1 at the beginning of each year for \$5. Find the fully discrete benefit reserve on Roland's policy 29 years from now.

Solution S3L58-2. We want to find ${}_{29}V_{95}$. We use the formula ${}_kV_x = A_{x+k} - P_x*\ddot{a}_{x+k}$. We are given $A_{124} = 0.89$, $P_{95} = 0.19$, and $\ddot{a}_{124} = 5$. Thus, ${}_{29}V_{95} = 0.89 - 0.19*5 =$ **${}_{29}V_{95}$ = -0.06**.

(Note: Nothing theoretically precludes benefit reserves from being negative. This problem is one example where a negative benefit reserve indeed occurs. A negative benefit premium simply means that the present value of future benefit premiums is greater than the present value of future benefits, under the condition that the insured life survives during the specified time period.)

Problem S3L58-3. Similar to Question 37 from the Casualty Actuarial Society's [Fall 2006 Exam 3]. The following probabilities are associated with the lives of rabbit-bears aged 5.

${}_{k|}q_5 = (k + 1)/15$ for k = 0, 1, 2, 3, 4. The annual effective interest rate is 0.09. Reabtibbar the Rabbit-Bear is 5 years old has a whole life insurance policy that pays a benefit of 1 at the end of the year of death. Find the fully discrete benefit reserve on Reabtibbar's policy 3 years from now.

Solution S3L58-3. We use the formula ${}_kV_x = 1 - ä_{x+k}/ä_x$.

The above probabilities of death in each of the years 0, 1, 2, 3, 4 add to 1, indicating that Reabtibbar will not survive past 5 years from now. Thus, to find $ä_5$, we only need to consider payments in years 0 through 4. Thus, we use the formula $ä_x = {}_{k=0}{}^{\infty}\Sigma v^k * {}_kp_x$, which, applied to our case, is as follows.

$ä_5 = 1 + v*p_5 + v^2*{}_2p_5 + v^3*{}_3p_5 + v^4*{}_4p_5$

$ä_5 = 1 + (1/1.09)(1 - 1/15) + (1/1.09)^2(1 - 3/15) + (1/1.09)^3(1 - 6/15) + (1/1.09)^4(1 - 10/15)$

$ä_5 = 3.229064933$

To find $ä_{5+3} = ä_8$ we only need to consider payments in years 3 and 4.

$ä_8 = 1 + v*p_8$. We find $p_8 = (1 - 10/15)/(1 - 6/15) = (5/9)$.

Thus, $ä_8 = 1 + (1/1.09)(5/9) = 1.509683996$.

Thus, ${}_3V_5 = 1 - ä_8/ä_5 = 1 - 1.509683996/3.229064933 =$ **${}_3V_5$ = about 0.5324702267**.

Problem S3L58-4. Similar to Question 24 from the Casualty Actuarial Society's [Spring 2008 Exam 3L]. Arachne is currently 78 years old and owns a 7-year fully discrete endowment insurance policy that pays a benefit of 1. You know the following: $ä_{78:7¬} = 5.9$. Find the ${}_6V_{78:7¬}$, the fully discrete benefit reserve for this policy six years from now.

Solution S3L58-4. We use the formula ${}_kV_{x:n¬} = 1 - ä_{x+k:n-k¬}/ä_{x:n¬}$. In our case, ${}_6V_{78:7¬} = 1 - ä_{84:1¬}/ä_{78:7¬}$. We only need to consider what the value of $ä_{84:1¬}$ might be. We use the formula $ä_{x:n¬} = {}_{k=0}{}^{n-1}\Sigma v^k*{}_kp_x$, so

$ä_{84:1¬} = {}_{k=0}{}^{1-1}\Sigma v^k*{}_kp_{84} = {}_{k=0}{}^{0}\Sigma v^k*{}_kp_{84} = v^0*{}_0p_{84} = 1*1 = 1$.

Thus, ${}_6V_{78:7¬} = 1 - 1/5.9 =$ **${}_6V_{78:7¬}$ = about 0.8305084746**.

Note: Exam Question 24 from the Spring 2008 Exam 3L is slightly defective. It only makes sense if the value asked for is ${}_4V_{35:5¬}$, the *fully discrete* benefit reserve, rather than the fully continuous benefit reserve.

Problem S3L58-5. Similar to Question 39 from the Casualty Actuarial Society's [Fall 2007 Exam 3].

A given 30-year endowment insurance policy on the life of a 40-year-old white panther pays a benefit of $10000 and has a benefit premium of $20. You know that the one-year discount factor v is 0.95 and that $p_{40} = 0.88$ for white panthers. Find the fully discrete benefit reserve one year from now (${}_1V_{40:30¬}$) for this policy.

Solution S3L58-5. We use the retrospective formula ${}_kV_{x:n¬} = (1/{}_kE_x)(P_{x:n¬} * ä_{x:k¬} - A^1{}_{x:k¬})$

In our case, ${}_1V_{40:30¬} = (1/{}_1E_{40})(P_{40:30¬} * ä_{40:1¬} - A^1{}_{40:1¬})$

We use the formula ${}_kE_x = v^n * {}_np_x$ to find ${}_1E_{40} = v*p_{40} = 0.95*0.88 = {}_1E_{40} = 0.836$.

Since the formulas we use assume a benefit of 1 for the insurance policy, we need to scale our value of the benefit and the benefit premium by (1/10000) to make them fit the conditions of the formulas.

Thus, $P_{40:30¬} = 20/10000 = P_{40:30¬} = 0.002$.

Moreover, we use the formula $A^1{}_{x:n¬} = {}_{k=0}{}^{n-1}\sum v^{k+1} * {}_kp_x * q_{x+k}$ to find

$A^1{}_{40:1¬} = {}_{k=0}{}^{0}\sum v^{k+1} * {}_kp_{40} * q_{40+k} = v * {}_0p_{40} * q_{40} = 0.95*1*(1 - p_{40}) = 0.95(1 - 0.88) = A^1{}_{40:1¬} = 0.114$.

We use the formula $ä_{x:n¬} = {}_{k=0}{}^{n-1}\Sigma v^k * {}_kp_x$ to find

$ä_{40:1¬} = {}_{k=0}{}^{1-1}\Sigma v^k * {}_kp_{40} = {}_{k=0}{}^{0}\Sigma v^k * {}_kp_{40} = v^0 * {}_0p_{40} = 1*1 = 1$.

Thus, we have ${}_1V_{40:30¬} = (1/0.836)(0.002*1 - 0.114) = -0.1339712919$.

But we need to scale our value of ${}_1V_{40:30¬}$ by 10000 to have it match the size of the policy's benefit.
Thus, our final answer is -0.1339712919*1000 = **about -$1339.71.**

Section 59

The Central Limit Theorem and the Continuity Correction for Normal Approximations

Some problems in this section will involve the **Central Limit Theorem,** whose statement is provided by Larsen and Marx:

"Let W_1, W_2,... be an infinite sequence of independent random variables, each with the same distribution. Suppose that the mean μ and the variance σ^2 of $f_W(w)$ are both finite. Then for any numbers a and b, $\lim_{n\to\infty}\Pr(a \le (W_1 + ... + W_n - n\mu)/(\sqrt{(n)}*\sigma) \le b) = \Phi(b) - \Phi(a)$." (302)

Here, $\Phi(x) = \Pr(Z \le x)$, where Z is the standard normal random variable.

You can find Φ(x) in MS Excel by using the function "=NormSDist(x)".

In other words, when you have a sum of n identically distributed independent random variables, each with mean μ and variance σ^2, you can use the Central Limit Theorem to approximate the probability that this sum is between any two values by using the probabilities associated with the standard normal distribution. Let S_n be the sum $(W_1 + ... + W_n)$. Then

$\Pr(a \le (S_n - n\mu)/(\sqrt{(n)}*\sigma) \le b) \approx \Phi(b) - \Phi(a)$.

However, when making approximations using the normal distribution, it is often necessary to use a **continuity correction.** This needs to happen when you use a continuous distribution such as the normal distribution to approximate a distribution where only discrete outcomes (such as integer outcomes) are possible. For instance, using the normal distribution in approximating a binomial distribution with a large number of possible discrete outcomes requires the continuity correction.

When you have an integer g, then, by the continuity correction,

Pr(More than g) = Pr(At least g + 1) = Pr(g + 1 or more) = $1 - \Phi((g + 0.5 - \mu)/\sigma)$.

When you have an integer g, then, by the continuity correction,

Pr(Less than g) = Pr(At most g - 1) = Pr(g - 1 or less) = $\Phi((g - 0.5 - \mu)/\sigma)$.

Some of the problems in this section were designed to be similar to problems from past versions of the Casualty Actuarial Society's Exam 3L and the Society of Actuaries' Exam MLC. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

Source: Broverman, Sam. Actuarial Exam Solutions - CAS Exam 3L - Spring 2008.

Larsen, Richard J. and Morris L. Marx. *An Introduction to Mathematical Statistics and Its Applications.* Fourth Edition. Pearson Prentice Hall: 2006. p. 302.

Mahler, Howard. "Sample Pages for Stochastic Models (CAS 3L / SOA MLC)."

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L59-1. You are examining a binomially distributed random variable G, with mean 67 and variance 54. Find the probability that $G \leq 55$. Use a normal approximation with a continuity correction.

Solution S3L59-1. We want to find the probability that G is *at most* 55, which, by the formula

Pr(At most g - 1) = $\Phi((g - 0.5 - \mu)/\sigma)$ (for g = 56), is = $\Phi((56 - 0.5 - 67)/\sqrt{(54)}) = \Phi(-1.56495178)$, which we find in MS Excel using the function "=NormSDist(-1.56495178)" = **about 0.058797116**.

Problem S3L59-2. You are examining a binomially distributed random variable G, with mean 127 and variance 23. Find the probability that $G \geq 129$. Use a normal approximation with a continuity correction.

Solution S3L59-2. We want to find the probability that G is *at least* 129, which, by the formula

Pr(At least g + 1) = $1 - \Phi((g + 0.5 - \mu)/\sigma)$ (for g = 128) is $1 - \Phi((128 + 0.5 - 127)/\sqrt{(23)}) =$

$1 - \Phi(0.3127716211)$, which we find in MS Excel using the function "=1 - NormSDist(0.3127716211)" = **about 0.377227154**.

Problem S3L59-3. You are examining a binomially distributed random variable G, with mean 25 and variance 10. Find the probability that $G < 28$. Use a normal approximation with a continuity correction.

Solution S3L59-3. We want to find the probability that G is less than 28, which, by the formula

Pr(Less than g) = $\Phi((g - 0.5 - \mu)/\sigma)$ (for g = 28) = $\Phi((28 - 0.5 - 25)/\sqrt{(10)}) = \Phi(0.790569415)$, which we find in MS Excel using the function "=NormSDist(0.790569415)" = **about 0.785402416**.

Problem S3L59-4. The variable S is a sum of 45 independent, identically distributed, and *continuous* random variables for each of which the mean is 930 and the variance is 3451. Find the approximate probability that $41650 \leq S \leq 42150$ using the Central Limit Theorem.

Solution S3L59-4. Since the random variables being added are each continuous, no continuity correction needs to be applied when using the normal approximation. We use the formula

$\Pr(a \leq (S_n - n\mu)/(\sqrt{(n)}*\sigma) \leq b) \approx \Phi(b) - \Phi(a)$. We need to find a and b.

We find $n\mu = 45*930 = 41850$. We also find $\sqrt{(n)}*\sigma = \sqrt{(45*3451)} = 399.1804103$.

Thus, $b = (42150 - 41850)/399.1804103 =$ about 0.7515398858 and

$a = (41650 - 41850)/399.1804103 =$ about -0.5010265905.

Thus, we want to find $\Phi(0.7515398858) - \Phi(-0.5010265905)$, which we do in MS Excel using the function "=NormSDist(0.7515398858) - NormSDist(-0.5010265905)" = **about 0.465659972**.

Problem S3L59-5. Similar to Question 1 from the Casualty Actuarial Society's Spring 2008 Exam 3L. Nrocinu is a missionary for the World Church of the Pink Unicorn. Nrocinu's job is to hand out pamphlets to people on the street, convincing them to believe in the Pink Unicorn. His probability of convincing any given person on the street whom he approaches is 0.03. Nrocinu will be given a promotion to Grand Unicorn if he manages to convince at least 340 people to believe in the Pink Unicorn during the course of 10 days. During each of these days, Nrocinu will hand out pamphlets to 1000 people. What is Nrocinu's probability of getting the promotion?

Solution S3L59-5. Because each person whom Nrocinu approaches has a 0.03 probability of being convinced, and Nrocinu can only convince whole numbers of people, the number of people he convinces (N) follows a binomial distribution with $n = 10*1000 = 10000$ and $p = 0.03$. Thus, $\mu = np = 10000*0.03 = \mu = 300$, and $\sigma^2 = np(1-p) = 300*0.97 = \sigma^2 = 291$. We want to find $\Pr(N \geq 340)$. In approximating this probability using the normal distribution, we apply the continuity correction: Pr(At least 339 + 1) = $1 - \Phi((339 + 0.5 - \mu)/\sigma) = 1 - \Phi((339 + 0.5 - 300)/\sqrt{(291)}) = 1 - \Phi(2.315531008)$), which we find in MS Excel using the function "=1 - NormSDist(2.315531008)" = **about 0.010291919**.

Section 60

Characteristics of Estimators, Mean Square Error, and Assorted Exam Style Questions

When you use an estimate $\hat{W}$ for a value W, then the **mean square error** associated with the estimate is defined as **MSE=** $\mathbf{E[(\hat{W} - W)^2]}$.

We now define what it means for an estimator to be **consistent,** following Larsen and Marx.

"An estimator $\hat{Y}_n = h(W_1, W_2, ..., W_n)$ is said to be *consistent* for Y if it converges in probability to Y - that is, if for all $\varepsilon > 0$, $\lim_{n\to\infty}\Pr(|\hat{Y}_n - Y| < \varepsilon) = 1$" (406).

An estimator $\hat{Y}$ of Y is an **unbiased** estimator if $\hat{Y}$ does not "systematically err in any particular direction" (Larsen and Marx 2006, p. 381). More formally, an estimator $\hat{Y}$ for Y is unbiased if $\mathbf{E[\hat{Y}] = Y}$ for all Y.

If you are given that values $x_1,...., x_n$ follow an **inverse exponential distribution** with parameter θ, then this is the same as saying that the values $1/x_1,...., 1/x_n$ follow an exponential distribution with parameter 1/θ. Therefore, you should convert all relevant values to the values pertaining to an exponential distribution and apply known techniques to them.

Reminder: Both the maximum likelihood estimator and the method of moments estimator for the mean of an exponential distribution are equal to the mean of the sample drawn from that distribution.

Source: Broverman, Sam. Actuarial Exam Solutions - CAS Exam 3L - Spring 2008.

Larsen, Richard J. and Morris L. Marx. *An Introduction to Mathematical Statistics and Its Applications.* Fourth Edition. Pearson Prentice Hall: 2006. pp. 381, 406.

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Problem S3L60-1. Similar to Question 2 from the Casualty Actuarial Society's Spring 2008 Exam 3L. The true value of a parameter Y is 3600. An estimator $\hat{Y}_n$ of Y, for any positive integer sample size n, is as follows: $\hat{Y}_n = 3600*n/(n+2)$. Which of the following statements about $\hat{Y}_n$ are correct? More than one answer may be possible.

(a) $\hat{Y}_n$ is an unbiased estimator of Y.
(b) $\hat{Y}_n$ is a consistent estimator of Y.
(c) The mean square error of $\hat{Y}_{19}$ is less than 100000.

Solution S3L60-1. We examine statement (a): $\hat{Y}_n$ can be an unbiased estimator of Y only if $E[\hat{Y}_n] = Y$ for all Y. In this case, each value of n corresponds to a single value of $\hat{Y}_n$, and for any value of n, 3600*n/(n+2) < 3600, so $\hat{Y}_n$ *cannot be equal* to Y and thus cannot be an unbiased estimator of Y. Thus, **(a) is false.**

We examine statement (b): $\hat{Y}_n$ can be a consistent estimator of Y if and only if for all ε > 0, $\lim_{n\to\infty}\Pr(|\hat{Y}_n - Y| < \varepsilon) = 1$.

$|\hat{Y}_n - Y| = |3600*n/(n+2) - 3600|$. $\lim_{n\to\infty}3600*n/(n+2) = 3600$ by a single application of L'Hôpital's rule, so $\lim_{n\to\infty}\Pr(|\hat{Y}_n - Y| < \varepsilon) = \Pr(|3600 - 3600| < \varepsilon) = \Pr(0 < \varepsilon) = 1$, since ε > 0. Thus, the condition for consistency is met and $\hat{Y}_n$ is a consistent estimator of Y. So **(b) is true.**

We examine statement (c): We use the formula $E[(\hat{W} - W)^2] = E[(\hat{Y}_{19} - Y)^2] = (\hat{Y}_{19} - Y)^2 = (3600*19/21 - 3600)^2$ = MSE = 117551.0204 > 100000, so **(c) is false.**

Thus, **only (b) is true.**

Problem S3L60-2. Similar to Question 3 from the Casualty Actuarial Society's [Spring 2008 Exam 3L]. The random variable X follows an inverse exponential distribution with parameter θ. A sample of 5 values is drawn from this distribution: 45, 513, 95, 675, 134. What is the maximum likelihood estimator for θ?

Solution S3L60-2. Since X follows an inverse exponential distribution with parameter θ, it follows that 1/X follows an exponential distribution with parameter (mean) 1/θ. We can restate our sample values as values of the random variable 1/X: 1/45, 1/513, 1/95, 1/675, 1/134. Because 1/X follows an exponential distribution, the maximum likelihood estimator of 1/θ is just the sample mean: (1/45 + 1/513 + 1/95 + 1/675 + 1/134)/5 = about 0.0087284048. The maximum likelihood estimator for θ is just the inverse of the above result, or 1/0.0087284048 = **about 114.5684724**.

Problem S3L60-3. Similar to Question 4 from the Casualty Actuarial Society's [Spring 2008 Exam 3L]. This question is a review of the concepts in Section 55. It is hypothesized that the number of rabid rhinoceros attacks experienced by a member of a particular tribe in a year follows a Poisson distribution with mean 0.2. For 1000 tribe members, this is the distribution of rabid rhinoceros attacks during the year.

Number of members suffering 0 attacks:879
Number of members suffering 1 attack: 98
Number of members suffering 2 or more attacks: 23.
Find the chi-square statistic associated with the hypothesis above.

Solution S3L60-3. We use the formula $D = {}_{i=1}{}^{t}\sum[(X_i - np_i)^2/np_i]$, where n = 1000, the X_i's are the given sample values, and the p_i's are calculated using probabilities of the Poisson distribution. We find $p_0 = e^{-0.2}$, so $np_0 = 1000e^{-0.2} = 818.7307531$.
We find $p_1 = 0.2e^{-0.2}$, so $np_1 = 200e^{-0.2} = 163.7461506$.
We find $p_{2\text{ or more}} = 1 - p_0 - p_1 = 1 - 1.2e^{-0.2}$, so $np_{2\text{ or more}} = 1000(1 - 1.2e^{-0.2}) = 17.52309631$.

Thus, we can find D = $(879 - 818.7307531)^2/818.7307531 + (98 - 163.7461506)^2/163.7461506 + (23 - 17.52309631)^2/17.52309631$ = **D = about 32.40033295**.

Problem S3L60-4. Similar to Question 6 from the Casualty Actuarial Society's Spring 2008 Exam 3L. This question is a review of concepts in Section 56. You have a sample of 51 values. The sample mean is 94500 and the sample standard deviation is 20000. You perform a two-sided hypothesis test for the mean of the distribution in question, with H_0: $\mu = 98000$ and H_1: $\mu \neq 98000$. Which of the following decisions would you legitimately make with regard to accepting or rejecting the null hypothesis H_0 at significance levels α.

(a) Do not reject at $\alpha = 0.1$.
(b) Do not reject at $\alpha = 0.05$, but reject at $\alpha = 0.1$.
(c) Do not reject at $\alpha = 0.02$, but reject at $\alpha = 0.05$.
(d) Do not reject at $\alpha = 0.01$, but reject at $\alpha = 0.02$.
(e) Reject at $\alpha = 0.01$.

Use the Table of Percentage Points of the t Distribution where appropriate.

Solution S3L60-4. Since we have sample size n = 51, sample standard deviation 20000, and sample mean μ_{actual} = 94500, we can use the Student t distribution with test statistic

$t = (\mu_{actual} - \mu_{hypothesized})/(S/\sqrt{(n)})$. The t distribution used here will have 51 - 1 = 50 degrees of freedom. We find $t = (94500 - 98000)/(20000/\sqrt{(50)})$ = about -1.237436867. Since we have a two-sided test, it does not really matter whether our t value is positive or negative. We do to a t value of -1.237436867 precisely what we would do to a t value of 1.237436867.

We use the table linked above to find the row of values of t associated with 50 degrees of freedom. We refer to the *two-tail* probabilities associated with each entry in that row. We find that for 50 degrees of freedom, the t-statistic associated with $\alpha = 0.1$ is 1.299 and 1.237436867 < 1.299. Thus, our t statistic is small enough for us to not reject the null hypothesis at the 0.1 significance level. Therefore, **(a) is true.**

Problem S3L60-5. Similar to Question 7 from the Casualty Actuarial Society's Spring 2008 Exam 3L. You have a random sample of four values x, and you know that the values are uniformly distributed over the interval 1000 < x < 2400. Find the probability that no more than three of these values will be greater than 1900. (Hint: This is the kind of problem that you can solve without any special knowledge, except basic knowledge of what the uniform distribution is. Think about how you might be able to do this.)

Solution S3L60-5. The only possible way that more than three of these values can be greater than 1900 is if *all four* of the values are greater than 1900. The probability of any given value being greater than 1900 is (2400 - 1900)/(2400 - 1000) = 5/14. So the probability of all four values being greater than 1900 is $(5/14)^4$. We desire the complement of that probability or $1 - (5/14)^4$ = **about 0.9837307372.**

Section 61

Residuals in Linear Regression and Assorted Exam-Style Questions

In least-squares linear regression, the **ith residual** is the difference between an observed value y_i when $x = x_i$ and the value of y_i predicted by the least squares line. The greater the ith residual, the greater the failure of the least squares line to accurately model the location of the point

(x_i, y_i). The ith residuals for all values of i can be graphed to produce a **residual plot.**

Some of the problems in this section were designed to be similar to problems from past versions of the Casualty Actuarial Society's Exam 3L and the Society of Actuaries' Exam MLC. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

Sources: Broverman, Sam. Actuarial Exam Solutions - CAS Exam 3L - Spring 2008.

Larsen, Richard J. and Morris L. Marx. *An Introduction to Mathematical Statistics and Its Applications.* Fourth Edition. Pearson Prentice Hall: 2006. pp. 650.

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Problem S3L61-1. Similar to Question 8 from the Casualty Actuarial Society's Spring 2008 Exam 3L. This question is also an excellent review of the concepts in Section 53.

You are taking a sample of 2 random values from an exponential distribution whose mean θ is equal to 700.

The ith order statistic of Y has the following probability density function (p. d. f.):
$f_{Y'\_i}(y) = (n!/((i-1)!(n-i)!))*F(y)^{i-1}*(1-F(y))^{n-i}*f(y)$, where f(y) is the p. d. f. of Y and F(y) is the cumulative distribution function (c. d. f.) of Y.

Find the mathematical expectation of the larger of the two values drawn from the sample.

Solution S3L61-1. We are asked to find $E(Y'_2)$. We are given that Y follows an exponential distribution with mean 700. Thus, $f(y) = (1/700)e^{-y/700}$, $F(y) = 1 - e^{-y/700}$, and $1 - F(y) = e^{-y/700}$.
Here, 2 = 30 and i = 2, so
$f_{Y'\_2}(y) = (2!/[(1)!(0)!])*(1 - e^{-y/700})^1*(e^{-y/700})^0*(1/700)e^{-y/700} =$
$f_{Y'\_2}(y) = (2/700)(1 - e^{-y/700})e^{-y/700}$
$f_{Y'\_2}(y) = (2/700)(e^{-y/700} - e^{-y/350})$

$f_{Y'\_2}(y) = 2*(1/700)(e^{-y/700}) - (1/350)e^{-y/350}$
We find $E(Y'_2) = {}_0{}^{\infty}\int y(2*(1/700)(e^{-y/700}) - (1/350)e^{-y/350})dy =$
$2*{}_0{}^{\infty}\int(y/700)(e^{-y/700})dy - {}_0{}^{\infty}\int(y/350)e^{-y/350})dy$. But ${}_0{}^{\infty}\int(y/700)(e^{-y/700})dy$ is 700 - the expected value for an exponential distribution with mean 700. Likewise, ${}_0{}^{\infty}\int(y/350)e^{-y/350})dy$ is 350 - the expected value for an exponential distribution with mean 350. So $E(Y'_2) = 2*700 - 350 =$ **$E(Y'_2)$ = 1050**.

Problem S3L61-2. Similar to part of Question 9 from the Casualty Actuarial Society's Spring 2008 Exam 3L. This question is also an excellent review of the concepts in Section 57.

You are given the following six points in a data sample.
(a): (6, 7)
(b): (18, 25)
(c): (34, 49)
(d): (45, 56)
(e): (67, 99)
(f): (100, 150)

You model this data with a line of the form $y_i = a + bx_i$, where a and b are found using the method of least-squares regression. Find the value of b.

Solution S3L61-2.
We use the formula $b = (n*{}_{i=1}{}^{n}\sum(x_iy_i) - {}_{i=1}{}^{n}\sum(x_i)\ {}_{i=1}{}^{n}\sum(y_i))/(n*{}_{i=1}{}^{n}\sum(x_i^2) - {}_{i=1}{}^{n}\sum(x_i)^2)$.
Here, n = 6.
${}_{i=1}{}^{6}\sum(x_iy_i) = 6*7 + 18*25 + 34*49 + 45*56 + 67*99 + 100*150 = {}_{i=1}{}^{6}\sum(x_iy_i) = 26311$.
${}_{i=1}{}^{n}\sum(x_i) = 6 + 18 + 34 + 45 + 67 + 100 = {}_{i=1}{}^{n}\sum(x_i) = 270$.
${}_{i=1}{}^{n}\sum(y_i) = 7 + 25 + 49 + 56 + 99 + 150 = {}_{i=1}{}^{n}\sum(y_i) = 386$
${}_{i=1}{}^{n}\sum(x_i^2) = 6^2 + 18^2 + 34^2 + 45^2 + 67^2 + 100^2 = 18030$
Thus, $b = (6*26311 - 270*386)/(6*18030 - 270^2) =$ **b = about 1.520578231**.

Problem S3L61-3. Similar to part of Question 9 from the Casualty Actuarial Society's Spring 2008 Exam 3L. This question is also an excellent review of the concepts in Section 57.

You are given the following six points in a data sample.
(a): (6, 7)
(b): (18, 25)
(c): (34, 49)
(d): (45, 56)
(e): (67, 99)
(f): (100, 150)

You model this data with a line of the form $y_i = a + bx_i$, where a and b are found using the method of least-squares regression. Find the value of a.

Solution S3L61-3. We use the formula a = $({}_{i=1}{}^{n}\sum(y_i) - b*{}_{i=1}{}^{n}\sum(x_i))/n$, where, from Solution S3L61-2, we know that n = 6, b = 1.520578231, ${}_{i=1}{}^{n}\sum(y_i)$ = 386, and ${}_{i=1}{}^{n}\sum(x_i)$ = 270. Thus, a = (386 - 1.520578231*270)/6 = **a = about -4.092687075**.

Problem S3L61-4. You are given the following six points in a data sample.
(a): (6, 7)
(b): (18, 25)
(c): (34, 49)
(d): (45, 56)
(e): (67, 99)
(f): (100, 150)

You model this data with a line of the form $y_i = a + bx_i$, where a and b are found using the method of least-squares regression. Find the residual at the point (67, 99).

Solution S3L61-4. The residual at point (67, 99) is the actual value of y minus the estimated value of y. The actual value of y is 99. The estimated value of y is a + b*67, where, from Solutions S3L61-2 and S3L61-3, we know that b = 1.520578231 and a = -4.092687075.

Thus, the estimated value of y is -4.092687075 + 1.520578231*67 = 97.78605442, and our desired residual is 99 - 97.78605442 = **about 1.213945578.**

Problem S3L61-5. Similar to Question 10 from the Casualty Actuarial Society's Spring 2008 Exam 3L. This question is also an excellent review of the concepts in Section 50. The number of rabid wild ostrich attacks near a remote desert village follows a Poisson distribution with a mean of 9 per day on Mondays, Tuesdays, and Thursdays, a mean of 5 per day on Wednesdays and Saturdays, and a mean of 7 per day on Fridays and Sundays. Find the probability that on a given week, there will be exactly 53 rabid wild ostrich attacks.

Solution S3L61-5. The mean of the sums of homogeneous Poisson random variables is the sum of their means. Thus, the mean for the number of rabid wild ostrich attacks in a week is 9*3 + 5*2 + 7*2 = 51. We recall the formula Pr[M = k] = $e^{-\Lambda}\Lambda^k/k!$. Here, k = 53 and Λ = 51. Thus, Pr[M = 53] = $e^{-51}51^{53}/53!$ = **about 0.0526352263**.

Section 62

Assorted Exam-Style Questions on Compound Poisson Processes, Normal Approximations, Force of Mortality, and Multiple Forces of Decrement

When determining the variance of the sum of independent Poisson random variables, you need to simply add the variances of each Poisson random variable. Note that the Poisson random variable X has the same variance as the Poisson random variable -X.

When doing some of the problems in this section, recall that the Poisson distribution is a discrete distribution and so when it is approximated by the continuous normal distribution, a continuity correction is required.

From Section 59, when you have an integer g, then, by the continuity correction,

Pr(More than g) = Pr(At least g + 1) = Pr(g + 1 or more) = 1 - $\Phi((g + 0.5 - \mu)/\sigma)$.

When you have an integer g, then, by the continuity correction,

Pr(Less than g) = Pr(At most g - 1) = Pr(g - 1 or less) = $\Phi((g - 0.5 - \mu)/\sigma)$.

A **compound Poisson process** S(t) occurs when some Poisson process N(t) accompanies some random variable X. For instance, N(t) can describe the frequency of losses of a particular kind, and X can describe the severity of each loss.

The expectation and variance of S(t) can be calculated as follows.

E(S(t)) = E(X)*E(N(t)) and **Var(S(t)) = E(N(t))*Var(X) + Var(N(t))*E(X)^2 = E(N(t))*E(X^2)**.

A useful expression relating survival probabilities to the force of mortality is

$\mathbf{{}_np_x = \exp(-\int_x^{x+n}\mu(t)*dt)}$. In many situations, this formula gives an easy way of calculating ${}_np_x$ given an expression for $\mu(t)$.

Sources: Broverman, Sam. Actuarial Exam Solutions - CAS Exam 3L - Spring 2008.

Daniel, James W. 2008. Poisson Processes (and Mixture Distributions).

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L62-1. Similar to Question 11 from the Casualty Actuarial Society's Spring 2008 Exam 3L. This question is also an excellent review of the concepts in Section 50.

The number of irrationally motivated financial bailouts in each subsequent year is expected to follow a Poisson distribution with the following means for each year:

2009: 43
2010: 24
2011: 53
2012: 45
2013: 55
2014: 54

Use a normal approximation to find the probability that the number of irrationally motivated financial bailouts during the years 2009-2011 will exceed the number of irrationally motivated financial bailouts during the years 2012-2014.

Solution S3L62-1. Let X be the random variable representing the number of irrationally motivated financial bailouts during the years 2009-2011. Let Y be the random variable representing the number of irrationally motivated financial bailouts during the years 2012-2014. We want to find Pr(X - Y > 0).

X has mean 43 + 23 + 53 = 119 and variance 119.

Y has mean 45 + 55 + 54 = 154 and variance 154.

X - Y has mean 119 - 154 = -35 and variance 154 + 119 = 273. Thus, X - Y has standard deviation $\sqrt{(273)}$ = 16.52271164.

Using the normal distribution and the continuity correction, we recall that

Pr(More than g) = 1 - $\Phi((g + 0.5 - \mu)/\sigma)$, so Pr(More than 0) =

1 - $\Phi((0 + 0.5 + 35)/16.52271164)$ = 1 - $\Phi(2.148557741)$.

In MS Excel, we use the input "=1 - NormSDist(2.148557741)" to find our desired probability, **0.015834737**.

Problem S3L62-2. Similar to Question 12 from the Casualty Actuarial Society's Spring 2008 Exam 3L. The number of highway robberies to which a particular person falls victim in a year follows a Poisson distribution with $\lambda = 3$. The amount of money lost to every highway robbery follows an exponential distribution with mean 5000. An insurance company insures this individual against highway robberies, paying the full cost of damages whenever a robbery occurs. The *risk* load of this policy is calculated as 15% of the sum of the standard deviation of the yearly loss and the expected loss per year. Find the risk load for this policy.

Solution S3L62-2. Let N(t) denote the frequency of claims and X denote the severity of claims. The expected loss per year is E(S(t)) = E(X)*E(N(t)) = 5000*3 = E(S(t)) =15000.

The variance of the yearly loss is Var(S(t)) = E(N(t))*Var(X) + Var(N(t))*E(X)^2.

We recall that for an exponential distribution, the variance is equal to the square of the mean.

Thus, Var(S(t)) = 3*5000^2 + 3*5000^2= 6*5000^2= 150000000.

SD(S(t)) = √(150000000) = 12247.44871. Thus, E(S(t)) + SD(S(t)) = 15000 + 12247.44871 = 27247.44871 and the risk load is 0.15*27247.44871 = about **4087l.117307.**

Problem S3L62-3. Similar to Question 17 from the Casualty Actuarial Society's [Spring 2008 Exam 3L](#). The lifetimes of two white frogs are independent and each follow an identical Weibull distribution for which the hazard rate function (force of mortality) is $\lambda_x = kx^3$. Find k if $p_{5:5}$, the probability that *both* of the white frogs will survive to age 6 if both of them have survived to age 5, is equal to 0.3.

Solution S3L62-3. For each white frog, we use the formula ${}_np_x = \exp(-\int_x^{x+n}\mu(t)*dt)$. Thus,

$p_5 = \exp(-\int_5^6\mu(t)*dt)$, where $\mu(t) = \lambda_t = kt^3$. Thus, $p_5 = \exp(-\int_5^6 kt^3*dt) = \exp(-(kt^4/4\,|_5^6)) =$

exp(-167.75k). Because the two lives are independent, $p_{5:5} = p_5{}^2$ = exp(-167.75k)^2.

We know that $p_{5:5}$ = 0.3 = exp(-167.75k)^2. Thus, √(0.3) = exp(-167.75k).

Thus, ln(√(0.3)) = -167.75k. k = ln(√(0.3))/-167.75 = **k = about 0.0035885926**.

Problem S3L62-4. Similar to Question 18 from the Casualty Actuarial Society's [Spring 2008 Exam 3L](#). The lifetimes A, B, and C of three independent lives each follow the probability density function (p. d. f.) f(z) = 1/(1+z)^3. Find the joint survival function $s_{A,B,C}$(a, b, c) for these three lives.

Solution S3L62-4. For each of the lives, f(z) = 1/(1+z)^3 implies that F'(z) = f(z). We also need to consider that F(0) = 0 and F(∞) = 1. Which function has a derivative of 1/(1+z)^3 and a value of 0 at z = 0? If we simply take the antiderivative of 1/(1+z)^3, we get -1/(1+z)^2 + C. We need to find C. We note that -1/(1+0)^2 + C = C - 1, and we need to set C - 1 = 0. Thus, C = 1 and so F(z) = 1 - 1/(1+z)^2. Thus, s(z) = 1 - F(z) = 1/(1+z)^2. To get $s_{A,B,C}$(a, b, c), we simply multiply the survival functions of A, B, and C, since these three variables are independent. Thus,
$s_{A,B,C}$(a, b, c) = 1/((1+a)^2(1+b)^2(1+c)^2).

Problem S3L62-5. Similar to Question 19 from the Casualty Actuarial Society's Spring 2008 Exam 3L. This question is an excellent review of the concepts in Section 46.

Under a hypothetical oppressive state, there are five possible causes of decrement for a given life (x), each with an associated force of mortality.

Death by waiting in line for government health care: $\mu_x^{1}(t) = t^4/500$.

Death by military conscription: $\mu_x^{2}(t) = 4t^3/500$.

Death by freezing due to energy-use restrictions: $\mu_x^{3}(t) = 6t^2/500$.

Death by starvation due to hyperinflation: $\mu_x^{4}(t) = 4t/500$.

Death by random police/military violence: $\mu_x^{5}(t) = 1/500$.

Find $q_x^{(\tau)}$, the probability of death this year due to any of these four causes.

Solution S3L62-5. To find $\mu_x^{(\tau)}(t)$, we simply add up the individual forces of mortality:

$t^4/500 + 4t^3/500 + 6t^2/500 + 4t/500 + 1/500 = \mu_x^{(\tau)}(t) = (t + 1)^4/500$.

Now we use the formula $p_x = \exp(-{}_x{}^{x+1}\int\mu(t)*dt)$. Since we know $\mu_x^{(\tau)}(t)$, and we are *starting* with a life aged x, we can let x = 0. Thus, $p_x^{(\tau)} = \exp(-{}_0{}^1\int((t + 1)^4/500)dt) =$

$\exp(-(t + 1)^5/2500)\,\Big|_0^1) = \exp(-0.0124)$. Thus, $q_x^{(\tau)} = 1 - p_x^{(\tau)} = 1 - \exp(-0.0124) =$

$\mathbf{q_x^{(\tau)} = 0.0123234368}$.

Section 63

Exam-Style Questions on Markov Chains and Percentile Calculations

Here is the general form of the kind of problem you need to solve in order to perform a **percentile calculation.** For $0 \le n \le 100$, when you are looking for the *nth percentile* of the distribution of any random variable X, you are trying to find the value of c such that

Pr(X ≤ c) = n. The rest is algebra and is context-specific.

Some of the problems in this section were designed to be similar to problems from past versions of the Casualty Actuarial Society's Exam 3L and the Society of Actuaries' Exam MLC. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

Sources: Broverman, Sam. Actuarial Exam Solutions - CAS Exam 3 - Spring 2007.

Broverman, Sam. Actuarial Exam Solutions - CAS Exam 3 - Fall 2007.

Broverman, Sam. Actuarial Exam Solutions - CAS Exam 3L - Spring 2008.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L63-1. Similar to Question 20 from the Casualty Actuarial Society's Spring 2008 Exam 3L. This question is also an excellent review of the concepts in Section 49.

You are given the following homogeneous Markov Chain with states X, Y, and Z and probabilities of transition among the states in each time period given by the following transition matrix.

...X......Y.....Z
X(0.1...0.5...0.4)
Y(0.4...0.3...0.3)
Z(0.1...0.7...0.2)

Find the probability that an entity in state Z at time 0 will be in state Z at time 2.

Solution S3L63-1. There are three ways in which an entity in state Z at time 0 will be in state Z at time 2. The movement among states from time 0 to time 1 to time 2 can be represented by the following rather intuitive diagrams: Z → Z → Z; Z → X → Z; Z → Y → Z.

We find Pr(Z → Z → Z) = 0.2*0.2 = 0.04.
We find Pr(Z → X → Z) = 0.1*0.4 = 0.04.
We find Pr(Z → Y → Z) = 0.7*0.3 = 0.21.
Thus, our answer is Pr(Z → Z → Z) + Pr(Z → X → Z) + Pr(Z → Y → Z) = 0.04 + 0.04 + 0.21 = **0.29.**

Problem S3L63-2. Similar to Question 25 from the Casualty Actuarial Society's Spring 2008 Exam 3L. This question is also an excellent review of the concepts in Section 49.

An entity moves between states X and Y each year according to the following transition matrix. All movements occur at the end of each year.

...X.....Y
X(0.3...0.7)
Y(0.2...0.8)

Cash flows to the entity occur during the midpoint of each year according to the following matrix:

...Cash
X(0)
Y(100).

The entity starts at state X at time t = 0. The annual effective interest rate is 0.1.

What is the actuarial present value of the entity's earnings by time t = 3.

Solution S3L63-2. By time t = 3, the entity will have been in state X from time 0 to time 1, in X or Y from time 1 to time 2, and in X or Y from time 2 to time 3. There are four ways in which this can happen: X → X → X; X → X → Y; X → Y → X; X → Y → Y.

X → X → X results in no cash flows, so we can simply disregard it in our calculation of actuarial present value.

X → X → Y results in a cash flow of 100 at time 2.5, whose present value is $100/1.1^{2.5}$ = 78.79856109.

Pr(X → X → Y) = 0.3*0.7 = 0.21, so the actuarial present value of this outcome is 0.21*78.79856109 = 16.54769783.

X → Y → X results in a cash flow of 100 at time 1.5, whose present value is $100/1.1^{1.5}$ = 86.6784172.

Pr(X → Y → X) = 0.7*0.2 = 0.14, so the actuarial present value of this outcome is 0.14*86.6784172 = 12.13497841.

X → Y → Y results in a cash flow of 100 at time 1.5, whose present value is $100/1.1^{1.5}$ = 86.6784172 and a cash flow of 100 at time 2.5, whose present value is $100/1.1^{2.5}$ = 78.79856109.

Pr(X → Y → Y) = 0.7*0.8 = 0.56, so the actuarial present value of this outcome is 92.66710784.

Thus, the actuarial present value of the entity's earnings by time t = 3 is 16.54769783 + 12.13497841 + 92.66710784 = **about 121.3497841**.

Problem S3L63-3. Similar to Question 23 from the Casualty Actuarial Society's Spring 2008 Exam 3L. The annual force of interest is 0.1. The lives of red rhinoceroses exhibit a constant force of mortality of 0.08. A red rhinoceros aged x has a whole life insurance policy that pays a benefit of 1 immediately upon death. Find the 75th percentile of the distribution of the *present value* of the benefit premium.

Solution S3L63-3. Let T be the random variable denoting the future lifetime of the red rhinoceros. Then the present value of the benefit premium is $1*e^{-0.1T} = e^{-0.1T}$.
We want to find c such that $\Pr(e^{-0.1T} \leq c) = 0.75$.
$\Pr(e^{-0.1T} \leq c) = \Pr(-0.1T \leq \ln(c)) = \Pr(T \geq \ln(c)/-0.1)$.
Let a = ln(c)/-0.1. Then $\Pr(T \geq \ln(c)/-0.1) = \Pr(T \geq a) = {}_ap_x$ by definition of ${}_ap_x$.

We have a constant force of mortality, so we know that T is exponentially distributed and so ${}_ap_x = e^{-\mu a} = e^{-0.08a}$. So we need to find a such that $0.75 = e^{-0.08a}$. This value of a is a = ln(0.75)/-0.08 = a = 3.596025906. Then we have 3.596025906 = ln(c)/-0.1, so c = exp(3.596025906*-0.1) = **c = 0.6979536443**.

Problem S3L63-4. Similar to Question 40 from the Casualty Actuarial Society's Fall 2007 Exam 3. You are given the following homogeneous Markov Chain with states X, Y, and Z and probabilities of transition among the states in each year given by the following transition matrix.

...X......Y.....Z
X(1......0.....0)
Y(0.4...0.3...0.3)
Z(0......0.7...0.3)

You are also given a matrix of cash flows associated with each transition.

...X......Y.....Z
X(0......0.....0)
Y(300...0...20)
Z(0......60.....0)

Cash flows occur at the end of each year, along with each transition among states, and the annual effective interest rate is 0.05. An entity starts in state Z at time 0 and makes its first transition at time 1. Find the actuarial present value of the cash flows accruing to that entity over the course of two transitions.

Solution S3L63-4. The only transitions that generate positive cash flows are Z → Y, Y → Z, and Y → X. We also know that transitions Z → X, X → Y, and X → Z are impossible.

Thus, the only two-period transition sequences that have nonzero cash flows associated with them are as follows. Z → Z → Y; Z → Y → Y; Z → Y → Z; Z → Y → X.

Pr(Z → Z → Y) = 0.3*0.7 = 0.21. The cash flow associated with this transition is 60 at time 2, whose present value is $60/1.05^2$ = 54.42176871. Thus, the actuarial present value (APV) of this cash flow is 0.21*54.42176871 = 11.42857143.

Pr(Z → Y → Y) = 0.7*0.3 = 0.21. The cash flow associated with this transition is 60 at time 1, whose present value is 60/1.05 = 57.14285714. Thus, the actuarial present value (APV) of this cash flow is 0.21*57.14285714 = 12.

Pr(Z → Y → Z) = 0.7*0.3 = 0.21. The cash flows associated with this transition are 60 at time 1, whose present value is 57.14285714, and 20 at time 2, whose present value is $20/1.05^2$ = 18.14058957.

The APV of these cash flows is thus 0.21*(57.14285714 + 18.14058957) = 15.80952381.

Pr(Z → Y → X) = 0.7*0.4 = 0.28. The cash flows associated with this transition are 60 at time 1, whose present value is 57.14285714, and 300 at time 2, whose present value is $300/1.05^2$ = 272.1088435. The APV of these cash flows is thus 0.28*(57.14285714 + 272.1088435) = 92.19047619.
The APV of all of these possible cash flows is 11.42857143 + 12 + 18.14058957 + 92.19047619 = **about 131.4285714.**

Problem S3L63-5. Similar to Question 40 from the Casualty Actuarial Society's Spring 2007 Exam 3. Consider the following non-homogeneous Markov chains where the transitions that occur at times t = 1 and t =2 between states X and Y are described by transition matrices Q_1 and Q_2.

Q_1:

...X.....Y
X(0.3...0.7)
Y(0.2...0.8)

Q_2:

...X.....Y
X(0.6...0.4)
Y(0.1...0.9)

The annual effective interest rate is 0.1. The cost of transitioning from state X to state Y is 30, and the cost of transitioning from state Y to state X is 60. There is no cost for remaining in either state. An entity starts in state X at time t = 0. Find the actuarial present value of the costs this entity can be expected to incur up to and including time t = 2.

Solution S3L63-5. There are four transition paths that can happen: X → X → X; X → X → Y; X → Y → X; X → Y → Y. X → X → X entails no transitions between states and so involves no costs.
X → X → Y involves a cost of 30 at t = 2. The probability of this path happening is 0.3*0.4 = 0.12, so the APV of this path is $0.12*(30/1.1^2) = 2.975206612$.

X → Y → Y involves a cost of 30 at t = 1. The probability of this path happening is 0.7*0.9 = 0.63, so the APV of this path is 0.63*(30/1.1) = 17.18181818.

X → Y → X involves a cost of 30 at t = 1 and a cost of 60 at t = 2. The probability of this path happening is 0.7*0.1 = 0.07, so the APV of this path is $0.07*(30/1.1 + 60/1.1^2) = 5.380165289$.
The APV of the costs incurred up to and including time t = 2 is the sum of the APVs of the above three paths, which is equal to 2.975206612 + 17.18181818 + 5.380165289 = **about 25.53719008.**

Section 64

Poisson Processes and the Gamma Distribution

Property 64.1. If the *time* between events follows an exponential distribution with mean θ, then the *number* of events that occur within a given unit of time follows a Poisson process with a rate of $\lambda = 1/\theta$.

The **Gamma function** is defined as $\mathbf{\Gamma(r) = {}_0{}^{\infty}\!\int y^{r-1} * e^{-y} * dy}$, for any real number r. Mostly, we will be working with cases where r is an integer. For these values of r, a more convenient expression for the Gamma function exists. Namely, **where r is an integer,** $\mathbf{\Gamma(r) = (r - 1)!}$**.**

If events are occurring according to a Poisson process with a constant rate λ for every unit of time, then we can express the probability density function (p. d. f.) of the random variable Y, the waiting time for the rth event, as $\mathbf{f_Y(y) = (\lambda^r/\Gamma(r))\, y^{r-1} * e^{-\lambda y}}$, for y > 0. Then Y is said to follow the **Gamma distribution.** For integer values of r, $\mathbf{f_Y(y) = (\lambda^r/(r-1)!)\, y^{r-1} * e^{-\lambda y}}$.

The mean and variance of a Gamma distribution with parameters r and λ are easy to calculate. If Y follows a Gamma distribution, then $\mathbf{E(Y) = r/\lambda}$ and $\mathbf{Var(Y) = r/\lambda^2}$.

While the above is an intuitive way to think of the Gamma distribution with relation to Poisson processes, the exam will use different notation for the Gamma distribution, which we will also discuss here. This *alpha-theta* notation is also used by James Daniel (Daniel 2008, p. 11).

Let Y be a Gamma random variable. Then, for some c > 0, α > 0, and θ > 0,

$\mathbf{f_Y(y) = cy^{\alpha-1} * e^{-y/\theta}}$. Furthermore, $\mathbf{E(Y) = \alpha\theta}$ and $\mathbf{Var(Y) = \alpha\theta^2}$. From comparing these expressions with those obtained using the r-lambda notation, it is clear that $\mathbf{r = \alpha}$ and $\mathbf{\lambda = 1/\theta}$**.**

Now we consider that when α = 1, a gamma distribution becomes just a friendly exponential distribution with mean θ. When we look back to Property 64.1, we recognize that λ = 1/θ in that case as well.So Property 64.1 is just a specific case of the general relationship described below.

Property 64.2. If the *number* of events that occur within a given unit of time follows a Poisson process with a rate of λ = 1/θ, then the time until the αth eventfollows a Gamma distribution with parameters α and θ.

Some additional properties of the Gamma distribution are useful.

Property 64.3. If Y is a Gamma random variable with parameters α and θ, then for any constant k, kY is a Gamma random variable with parameters α and kθ.

Property 64.4. The sum of α independent exponential random variables, each with parameter θ, is a Gamma random variable with parameters α and θ.

Some of the problems in this section were designed to be similar to problems from past versions of the Casualty Actuarial Society's Exam 3L and the Society of Actuaries' Exam MLC. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

Sources: Broverman, Sam. Actuarial Exam Solutions - CAS Exam 3 - Fall 2007.

Daniel, James W. 2008. Poisson Processes (and Mixture Distributions).

Larsen, Richard J. and Morris L. Marx. *An Introduction to Mathematical Statistics and Its Applications.* Fourth Edition. Pearson Prentice Hall: 2006. p. 327-330.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L64-1. Similar to Question 1 from the Casualty Actuarial Society's Fall 2007 Exam 3. The time between major stock market crashes in the year 2009 follows an exponential distribution and remains constant throughout the year. The probability that no major stock market crash will occur within an 80-day period is 0.8. How many stock market crashes can one expect to occur during the first 180 days of 2009?

Solution S3L64-1. We are given s(80) = 0.8. We know that s(80) = $e^{-80/\theta}$ = 0.8, so -80/θ = ln(0.8) and ln(0.8)/-80 = 1/θ = 0.0027892944. This is the value of λ for a Poisson process that describes the number of major stock market crashes that we can expect every *day*. To get the number of major stock market crashes that we can expect in 180 days, we multiply 180*0.0027892944 = **about 0.5020729905.**

Problem S3L64-2. Similar to Question 2 from the Casualty Actuarial Society's Fall 2007 Exam 3. The number of American currency devaluations in the years 2010-2011 follows a Poisson distribution with a mean of 5 per year. The number of American currency devaluations in the years 2012-2016 follows a Poisson distribution with a mean of 2 per year. It is currently January 1, 2015, and 17 currency devaluations have happened since the year 2010. At what time would you expect the next currency devaluation to occur? (Assume that each month is exactly 1/12 of one year).

Solution S3L64-2. Non-overlapping Poisson processes are completely independent of each other, and so it does not matter what happened prior to the year 2015 in determining when we expect the next devaluation to occur. In 2015, devaluations happen with a mean of 2 per year, which means that we should expect one devaluation to occur per half-year, and so the next devaluation can be expected to occur 6 months from now on **July 1, 2015.**

Problem S3L64-3. Similar to Question 3 from the Casualty Actuarial Society's Fall 2007 Exam 3. This question is an excellent review of the concepts in Section 62.

The number of U. S. government defaults on its debt in the years 2009-2015 follows a Poisson distribution with a mean of 3 per year. The amount in dollars which the U. S. government fails to pay during those years follows a Pareto distribution with θ = 2,000,000,000 and α = 3. The U. S. government employs actuaries to calculate risk loads pertaining to the defaults. The risk load is set at 20% of the standard deviation of the total amount of debt which the government fails to repay. Find the risk load for the years 2009-2015.

Solution S3L64-3. The expected number of defaults over the course of 7 years (2009 through 2015) is 7*3 = E(N(t)) = Var(N(t)) = 21, since we are working with a Poisson process.
$E(X^2)$, the second moment of the sum not repaid per default is $E(X^2) = 2\theta^2/((\alpha - 1)(\alpha - 2)) = 2(2000000000)^2/(2*1) = 4*10^{18}$.
The total amount of debt which the government fails to repay is a compound Poisson process S(t) for which $Var(S(t)) = E(N(t))*E(X^2) = 21*4*10^{18} = 8.4*10^{19}$.
Thus, $SD(S(t)) = \sqrt{(8.4*10^{19})}$ = 9,165,151,390 and the risk load is 0.2*9,165,151,390 = **about $1,833,030,278.**

Problem S3L64-4. Similar to Question 4 from the Casualty Actuarial Society's Fall 2007 Exam 3. This question is an excellent review of the concepts in Section 60.

The random variable Z is equal to $Z = X_1 + X_2 + ... + X_{50}$, where each of the random variables X_1 through X_{50} follows a Gamma distribution with α = 10 and an unknown value of θ that is the same for all of these variables. You are trying to estimate the value of θ by using the random variable cZ as your estimator, for some value of c. Find c such that cZ is an unbiased estimator.

Solution S3L64-4. For cZ to be an unbiased estimator of θ, it must be the case that $E(cZ) = \theta$. We know that for every one of the random variables X, $E(X) = \alpha\theta = 10\theta$. Moreover, Z is the sum of 50 identical Gamma random variables, so, by Property 64.3, $E(Z) = 50*\alpha\theta = 500\theta$. Since Z follows a Gamma distribution with parameters α = 10 and 50θ, it follows that cZ follows a Gamma distribution with parameters α = 10 and $c*50\theta$. Thus, $E(cZ) = 10*c*50\theta = 500c\theta$. For cZ to be an unbiased estimator of θ, it must be the case that $500c\theta = \theta$ and so c = 1/500 = **c = 0.002**.

Problem S3L64-5. The time between major stock market crashes in the years 2009-2015 follows an exponential distribution and remains constant throughout the year. The probability that no major stock market crash will occur within an 80-day period is 0.8. How many days can one expect to pass until the 4^{th} stock market crash?

Solution S3L64-5. For the exponential distribution, we are given s(80) = 0.8. We know that $s(80) = e^{-80/\theta} = 0.8$, so $-80/\theta = \ln(0.8)$ and $\ln(0.8)/-80 = 1/\theta = 0.0027892944$, so $\theta = 358.5136094$. By Property 64.4, the time T until the fourth stock market crash will follow a Gamma distribution with α = 4 and θ = 358.5136094, and so $E(T) = \alpha*\theta$ = 4*358.5136094 = **about 1434.054438 days**.

Section 65

The F Distribution and Hypothesis Testing of Variances

Property 65.1. If we have a sample of size n drawn from a normal distribution with mean μ and standard deviation σ, and if the unbiased sample variance is equal to S^2, then $(n - 1)^2S^2/\sigma^2$ follows a Chi-square distribution with (n - 1) degrees of freedom. To find percentiles of this distribution, you can use a Table of Critical Values for the Chi-Square Distribution.

The **F distribution** is used to perform hypothesis tests regarding the comparison of variances of different samples. The following Table of Upper Critical Values for the F Distribution is given for the 0.05 significance level.

The following tests can be performed using the F distribution, according to Larsen and Marx 2006, p. 569.

Let $x_1, x_2, ..., x_n$ and $y_1, y_2, ..., y_m$ be independent random samples from normal distributions with means μ_X and μ_Y and standard deviations σ_X and σ_Y, respectively. Our **test statistic** using the F distribution will be the ratio of the *sample* variances: $\mathbf{s_Y^2/s_X^2}$.

Then the following properties hold.

Property 65.2. To test $H_0: \sigma_X^2 = \sigma_Y^2$ versus $H_1: \sigma_X^2 > \sigma_Y^2$ at the α level of significance, reject H_0 if

$s_Y^2/s_X^2 \leq F_{\alpha,m-1,n-1}$. To find $F_{\alpha,m-1,n-1}$, use the Table of Upper Critical Values for the F Distribution.

Property 65.3. To test $H_0: \sigma_X^2 = \sigma_Y^2$ versus $H_1: \sigma_X^2 < \sigma_Y^2$ at the α level of significance, reject H_0 if

$s_Y^2/s_X^2 \geq F_{1-\alpha,m-1,n-1}$. To find $F_{1-\alpha,m-1,n-1}$, use the methodology described in "Critical Values of the F Distribution".

Property 65.4. To test $H_0: \sigma_X^2 = \sigma_Y^2$ versus $H_1: \sigma_X^2 \neq \sigma_Y^2$ at the α level of significance, reject H_0 if

$s_Y^2/s_X^2$is their (1) $\leq F_{\alpha/2,m-1,n-1}$ or (2) $\geq F_{1-\alpha/2,m-1,n-1}$.

Note: The value (m - 1) can be called the **numerator degrees of freedom** and the value (n - 1) can be called the **denominator degrees of freedom.**

We can also use a convenient property pertaining to **order statistics.**

Property 65.5. In a sample of size n drawn from the distribution of a random variable Y, the cumulative distribution function (c. d. f.) $F_{Y\_n}(t)$ of the nth order statistic is equal to the (c. d. f.) of Y taken to the nth power. That is, $\mathbf{F_{Y\_n}(t) = (F_Y(t))^n}$.

Source: Broverman, Sam. Actuarial Exam Solutions - CAS Exam 3 - Fall 2007.

Larsen, Richard J. and Morris L. Marx. *An Introduction to Mathematical Statistics and Its Applications.* Fourth Edition. Pearson Prentice Hall: 2006. p. 569, 683.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L65-1. Similar to Question 9 from the Casualty Actuarial Society's Fall 2007 Exam 3. This question is an excellent review of the concepts in Section 55. In any group of beneficiaries from a financial bailout, the hypothesized distribution of beneficiaries among various groups is as follows.

Politicians: 35%

Reckless speculators: 25%

Reckless lenders: 20%

Reckless borrowers: 15%

Socialist demagogues: 5%

In a sample of 1000 bailout beneficiaries, you find that 378 are politicians, 245 are reckless speculators, 201 are reckless lenders, 160 are reckless borrowers, and 16 are socialist demagogues. Find the chi-square statistic that would be used in testing whether the hypothesized distribution above is correct.

Solution S3L65-1. We use the formula $d = {}_{i=1}{}^{n}\sum[(k_i - np_{io})^2/(np_{io})]$, where n = 5 and the k_i are the observed values, while the np_{io} are the expected values (1000 times the expected percentage for each category). Thus, our chi-square statistic d is equal to

$(378 - 350)^2/350 + (245 - 250)^2/250 + (201 - 200)^2/200 + (160 - 150)^2/150 + (16 - 50)^2/50 =$

d = about 26.131666666667.

Problem S3L65-2. Similar to Question 10 from the Casualty Actuarial Society's Fall 2007 Exam 3. This question is an excellent review of the concepts in Section 55. You are analyzing a sample of 48 values drawn from a normal distribution with mean 4 and variance 10. S^2 is the unbiased sample variance. Find the critical value c such that $\Pr(S^2 \le c) = 0.9$. If it is necessary, use a Table of Critical Values for the Chi-Square Distribution.

Solution S3L65-2. By Property 65.1, $(n - 1)^2S^2/\sigma^2$ follows a Chi-square distribution with (n - 1) degrees of freedom. Here, n = 48 and $\sigma^2 = 10$. Thus, $47^2S^2/10$follows a Chi-square distribution with 47 degrees of freedom, and therefore $220.9S^2$ follows a Chi-square distribution with 47 degrees of freedom. We examine the Table of Critical Values for the Chi-Square Distribution to find the entry in the row corresponding to 47 degrees of freedom and a 0.10 probability of exceeding the critical value, which is the same as a 0.90 probability of being *within* the critical value. This value is 59.774. Thus,

$\Pr(220.9S^2 \leq 59.774) = 0.9$ and so $\Pr(S^2 \leq 0.2705930285) = 0.9$ and therefore **c = about 0.2705930285**.

Problem S3L65-3. Similar to Question 11 from the Casualty Actuarial Society's Fall 2007 Exam 3.

You have a sample of values $x_1, x_2, ..., x_n$ drawn from the Normal distribution of the random variable X. You know that n = 5, the distribution mean is μ_X and the distribution standard deviation is σ_X.

You also have a sample of values $y_1, y_2, ..., y_m$ drawn from the Normal distribution of the random variable Y. You know that m = 15, the distribution mean is μ_Y and the distribution standard deviation is σ_Y.

You test H_0: $\sigma_X^2 = \sigma_Y^2$ versus H_1: $\sigma_X^2 > \sigma_Y^2$ at the 0.05 level of significance. Your test statistic is s_Y^2/s_X^2. Express the region of rejection of the null hypothesis as an inequality with the test statistic on one side and some numerical value on the other. Use the Table of Upper Critical Values for the F Distribution.

Solution S3L65-3. We recall Property 65.2:To test H_0: $\sigma_X^2 = \sigma_Y^2$ versus H_1: $\sigma_X^2 > \sigma_Y^2$ at the α level of significance, reject H_0 if $s_Y^2/s_X^2 \leq F_{\alpha,m-1,n-1}$. Thus, in our table, we seek to find $F_{0.05,15-1,5-1}$ = $F_{0.05,14,4}$ = 5.873. (14 is the number of our *numerator* degrees of freedom, and 4 is the number of our *denominator* degrees of freedom.) Thus, **we reject H_0 if $s_Y^2/s_X^2 \leq 5.873$**.

Problem S3L65-4. Similar to Question 11 from the Casualty Actuarial Society's Fall 2007 Exam 3.

You have a sample of values $x_1, x_2, ..., x_n$ drawn from the Normal distribution of the random variable X. You know that n = 19, the distribution mean is μ_X and the distribution standard deviation is σ_X.

You also have a sample of values $y_1, y_2, ..., y_m$ drawn from the Normal distribution of the random variable Y. You know that m = 14, the distribution mean is μ_Y and the distribution standard deviation is σ_Y.

You test H_0: $k*\sigma_X^2 = \sigma_Y^2$ versus H_1: $k*\sigma_X^2 < \sigma_Y^2$ at the 0.05 level of significance, for some constant value of k. Your test statistic is s_Y^2/s_X^2.

You know that the critical region for your null hypothesis has a value of 6.93. That is, you reject H_0 if $s_Y^2/s_X^2 \geq 6.93$. Find the value of k in this hypothesis test. Use the table and methodology presented in "Critical Values of the F Distribution" in the course of your work.

Solution S3L65-4. We recall Property 65.3:To test H_0: $\sigma_X^2 = \sigma_Y^2$ versus H_1: $\sigma_X^2 < \sigma_Y^2$ at the α level of significance, reject H_0 if $s_Y^2/s_X^2 \geq F_{1-\alpha,m-1,n-1}$. Here, m = 14 and n = 19. If we were testing H_0: $\sigma_X^2 = \sigma_Y^2$ versus H_1: $\sigma_X^2 < \sigma_Y^2$, then we would reject H_0 if $s_Y^2/s_X^2 \geq F_{0.95,13,18} = 2.31$ according to the table and methodology presented in "Critical Values of the F Distribution". But the hypotheses we are comparing are H_0: $k*\sigma_X^2 = \sigma_Y^2$ and H_1: $k*\sigma_X^2 < \sigma_Y^2$, and we know that we would reject H_0 if $s_Y^2/s_X^2 \geq 6.93$. It makes sense that if we accept the claim that $\sigma_X^2 < \sigma_Y^2$ for all values greater than 2.31 and we accept the claim that $k*\sigma_X^2 < \sigma_Y^2$ for all values greater than 6.93, then $k*\sigma_X^2$ must be 6.93/2.31 = 3 times greater than σ_X^2. Thus, $k*\sigma_X^2/\sigma_X^2$ = **k = 3**.

Problem S3L65-5. Similar to Question 12 from the Casualty Actuarial Society's Fall 2007 Exam 3.

A sample of seven random values is drawn from a Pareto distribution with $\theta = 17000$ and $\alpha = 3.4$.

Find the probability that the seventh order statistic of Y is greater than 30000.

Solution S3L65-5. From Section 47, we know that for a Pareto distribution, $F(y) = 1 - \theta^\alpha/(y + \theta)^\alpha$

and $f(y) = \alpha\theta^\alpha/(y + \theta)^{\alpha+1}$. By Property 65.5, since Y_7 is the largest order statistic of Y, $F_{Y\_7}(y) = (F_Y(y))^7$,

$F_{Y\_7}(y) = (1 - \theta^\alpha/(y + \theta)^\alpha)^7$, and $F_{Y\_7}(30000) = (1 - \theta^\alpha/(y + \theta)^\alpha)^7 = (1 - 17000^{3.4}/(30000 + 17000)^{3.4})^7 =$

$F_{Y\_7}(30000) = \Pr(Y_7 \leq 30000) = 0.7992423032$. Our desired probability, $\Pr(Y_7 > 30000)$, is 1 - 0.7992423032 = **about 0.2007576968.**

Section 66

Joint-Life Status and Last-Survivor Status

For lives (x) and (y), a status that continues so long as both lives survive and terminates whenever either life fails is called a **joint-life status.** For lives (x) and (y), the joint-live status is denoted as (xy). This status his life table functions associated with it, defined analogously to the life table functions in Section 1 and Section 2. In some cases, especially for independent lives, it is possible to define the life table functions pertaining to (xy) in terms of the life table functions pertaining to (x) and (y). For instance, if lives (x) and (y) are independent, then we have $\mathbf{{}_tp_{xy} = ({}_tp_x)({}_tp_y)}$.

Another important status to consider is the **last-survivor status.** For lives (x) and (y), the last-survivor status continues so long as *either* of the lives remains alive. It is denoted as $(\overline{xy})$, with a straight line drawn *directly over* the "xy" whenever possible.

Both the joint-life status and the last-survivor status can be applied to indefinitely many lives. The joint-life status prevails so long as *all* of the lives survive. The last-survivor status prevails so long as *any* life survives. The following expressions hold with regard to the joint-life and last-survivor statuses:

$$\mathbf{{}_tp_{xy} + {}_tp_{\overline{xy}} = {}_tp_x + {}_tp_y}$$
$$\mathbf{{}_tq_{xy} + {}_tq_{\overline{xy}} = {}_tq_x + {}_tq_y}$$

We can also find the expectation of life $\mathbf{\dot{e}_{xy}}$ or $\mathbf{\dot{e}_{\overline{xy}}}$ and the curate expectation of life $\mathbf{e_{xy}}$ or $\mathbf{e_{\overline{xy}}}$ for the joint-life and last survivor statuses.

$$\mathbf{\dot{e}_{xy} = \int_0^{\infty} {}_tp_{xy} * dt}$$
$$\mathbf{\dot{e}_{\overline{xy}} = \int_0^{\infty} {}_tp_{\overline{xy}} * dt}$$
$$\mathbf{\dot{e}_{\overline{xy}} = \dot{e}_x + \dot{e}_y - \dot{e}_{xy}.}$$
$$\mathbf{e_{xy} = \sum_1^{\infty} {}_kp_{xy}.}$$
$$\mathbf{e_{\overline{xy}} = \sum_1^{\infty} {}_kp_{\overline{xy}}.}$$
$$\mathbf{e_{\overline{xy}} = e_x + e_y - e_{xy}.}$$

Source: Bowers, Gerber, et. al. *Actuarial Mathematics.* 1997. Second Edition. Society of Actuaries: Itasca, Illinois. pp. 263-273.

Broverman, Sam. Actuarial Exam Solutions - CAS Exam 3 - Fall 2007.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L66-1. Life (x) is a 38-year-old pink pterodactyl with survival function $s(x) = e^{-0.01x}$. Life (y) is a 54-year-old white crow with survival function s(y) = 1 - y/70. The two lives are independent. Find ${}_2p_{xy}$ for these two lives.

Solution S3L66-1. We use the formula ${}_tp_{xy} = ({}_tp_x)({}_tp_y)$. We find ${}_2p_x = {}_2p_{38} = s(40)/s(38) = e^{-0.01*40}/e^{-0.01*38} = e^{-0.02}$. We find ${}_2p_y = {}_2p_{54} = s(56)/s(54) = (14/70)/(16/70) = 7/8$. Thus, ${}_2p_{xy} = ({}_2p_x)({}_2p_y) = $ **$(7/8)e^{-0.02} = 0.8576738391$**.

Problem S3L66-2. Life (x) is a 38-year-old pink pterodactyl with survival function $s(x) = e^{-0.01x}$. Life (y) is a 54-year-old white crow with survival function $s(y) = 1 - y/70$. The two lives are independent. Find ${}_2p_{\overline{xy}}$ for these two lives.

Solution S3L66-2. We use the formula ${}_tp_{xy} + {}_tp_{\overline{xy}} = {}_tp_x + {}_tp_y$, which we rearrange as follows:

${}_tp_{\overline{xy}} = {}_tp_x + {}_tp_y - {}_tp_{xy}$. We know from Solution S3L66-1 that ${}_2p_x = e^{-0.02}$, ${}_2p_y = 7/8$, and ${}_2p_{xy} = (7/8)e^{-0.02}$. Thus, ${}_2p_{\overline{xy}} = e^{-0.02} + 7/8 - (7/8)e^{-0.02} = $ **${}_2p_{\overline{xy}}$ = about 0.9975248342**.

Problem S3L66-3. Life (x) is a 38-year-old pink pterodactyl with survival function $s(x) = e^{-0.01x}$. Life (y) is a 54-year-old white crow with survival function $s(y) = 1 - y/70$. The two lives are independent. Find $\mathring{e}_{xy}$ for these two lives. Set up the appropriate integral and then use any calculator to evaluate it.

Solution S3L66-3. We use the formula $\mathring{e}_{xy} = \int_0^\infty {}_tp_{xy}*dt$. In order to apply this formula, we need to find a general expression for ${}_tp_{xy}$. Since the lifetime of (x) is exponentially distributed with force of mortality 0.01, we know that ${}_tp_x = e^{-0.01t}$. Since the lifetime of (y) is uniformly distributed, and the future lifetime of (y) has p. d. f. $f(y) = 1/(70 - 54) = 1/16$, we know that ${}_tp_y = (1 - t/16)$. The upper bound on our integral will be 16 and not infinity, because life (y) cannot survive past 16 years from now.

Thus, ${}_tp_{xy} = ({}_tp_x)({}_tp_y) = (1 - t/16)e^{-0.01t}$, and so **$\mathring{e}_{xy} = \int_0^{16}(1 - t/16)e^{-0.01t}*dt$ = about 7.589868104**.

Problem S3L66-4. Similar to Question 33 from the Casualty Actuarial Society's Fall 2007 Exam 3. You are given the following information about two independent lives (x) and (y):

${}_{k|}q_x = 0.25$ for k = 0, 1, 2, 3.

$q_{y+k} = 0.5^{k+1}$ for k = 0, 1, 2, 3.

Find $e_{\overline{xy}:\overline{2|}}$, the *temporary* curate expectation of life pertaining to the last-survivor status.

Hint: The formula for performing this calculation is the same as that for performing the calculation for the complete curate expectation of life pertaining to the last-survivor status, except that the upper bound of the summation is different in a rather intuitive way.

Solution S3L66-4. If $e_{\overline{xy}} = \sum_1^\infty {}_kp_{\overline{xy}}$, then $e_{\overline{xy}:\overline{2|}} = \sum_1^2 {}_kp_{\overline{xy}} = p_{\overline{xy}} + {}_2p_{\overline{xy}}$.

We can find $p_x = 1 - {}_{0|}q_x = 1 - 0.25 = p_x = 0.75$.

Moreover, ${}_2p_x = 1 - {}_{0|}q_x - {}_{1|}q_x = 1 - 0.25 - 0.25 = {}_2p_x = 0.5$.

Moreover, $p_y = 1 - q_y = 1 - 0.5^{0+1} = p_y = 0.5$.

Moreover, ${}_2p_y = (1 - q_y)(1 - q_{y+1}) = 0.5*(1 - 0.5^2) = {}_2p_y = 0.375$.

Since the two lives are independent, ${}_tp_{xy} = ({}_tp_x)({}_tp_y)$ and so $p_{xy} = p_xp_y = 0.75*0.5 = p_{xy} = 0.375$.

Likewise, ${}_2p_{xy} = ({}_2p_x)({}_2p_y) = 0.5*0.375 = {}_2p_{xy} = 0.1875$.

We find $p_{\overline{xy}} = p_x + p_y - p_{xy} = 0.75 + 0.5 - 0.375 = p_{\overline{xy}} = 0.875$.

We find ${}_2p_{\overline{xy}} = {}_2p_x + {}_2p_y - {}_2p_{xy} = 0.5 + 0.375 - 0.1875 = {}_2p_{\overline{xy}} = 0.6875$.
Thus, $e_{\overline{xy}:\overline{2}|} = p_{\overline{xy}} + {}_2p_{\overline{xy}} = 0.875 + 0.6875 =$ $\mathbf{e_{\overline{xy}:\overline{2}|} = 1.5625}$.

Problem S3L66-5. Similar to Question 34 from the Casualty Actuarial Society's Fall 2007 Exam 3.

Two independent superwidgets are key components to the functionality of a system. If either widget fails, so does the system. The operational lifetime in years of each superwidget can be modeled by the life table probability function ${}_tp_0 = 1 - t/400$, $t < 400$ and 0 otherwise. A modification has been made to the system such that the system can continue to work if at least one superwidget remains operational. How many years has this modification added to the expected operational lifetime of the system?

Solution S3L66-5. We are asked to find the difference between $\mathring{e}_{\overline{xy}}$ (the system's expectation-of-life after the modification) and $\mathring{e}_{xy}$ (the system's expectation-of-life before the modification).

We first find $\mathring{e}_{xy} = {}_0^\infty\int {}_tp_{xy}*dt$. Since the two lifetimes are independent, ${}_tp_{xy} = ({}_tp_x)({}_tp_y) = (1 - t/400)^2$ and
$\mathring{e}_{xy} = {}_0^{400}\int(1 - t/400)^2dt = (-400(1 - t/400)/3) \Big|_0^{400} = 400/3$.
We find ${}_tp_{\overline{xy}} = {}_tp_x + {}_tp_y - {}_tp_{xy} = 2(1 - t/400) - (1 - t/400)^2$.
Thus, $\mathring{e}_{\overline{xy}} = {}_0^\infty\int {}_tp_{\overline{xy}}*dt = {}_0^{400}\int(2(1 - t/400) - (1 - t/400)^2)dt = {}_0^{400}\int(2(1 - t/400)dt - 400/3 =$
$-400(1 - t/400)^2 \Big|_0^{400} - 400/3 = 400 - 400/3 = \mathring{e}_{\overline{xy}} = 800/3$.
Our desired answer is $\mathring{e}_{\overline{xy}} - \mathring{e}_{xy} = 800/3 - 400/3 = 400/3 =$ **about 133.333333 years.**

Section 67

Assorted Exam-Style Review Questions for Actuarial Exam 3L

This section will be devoted to reviewing the materials covered in an assortment of previous sections of this study guide, so as to ensure students' familiarity with these ideas when they are encountered in a variety of contexts.

Some of the problems in this section were designed to be similar to problems from past versions of the Casualty Actuarial Society's Exam 3L and the Society of Actuaries' Exam MLC. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

Sources:

Broverman, Sam. Actuarial Exam Solutions - CAS Exam 3 - Spring 2007.

Broverman, Sam. Actuarial Exam Solutions - CAS Exam 3 - Fall 2007.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L67-1. Similar to Question 35 from the Casualty Actuarial Society's Fall 2007 Exam 3. This question is an excellent review of Section 46.

Red bluefish can suffer from two possible causes of decrement, 1 and 2. You are studying a group of 1400 red bluefish, each of age 3 years. You know the following life table probability functions:

$q_3^{(1)} = 0.04$, $q_3^{(2)} = 0.03$, $q_4^{(1)} = 0.09$, $q_4^{(2)} = 0.014$, $q_5^{(1)} = 0.05$, $q_5^{(2)} = 0.3$. Find $l_6^{(\tau)}$, the number of red bluefish you expect to remain alive at age 6 years. Fractional red bluefish are possible.

Solution S3L67-1. Here, we are given $l_3^{(\tau)} = 1400$. We know that $l_6^{(\tau)} = l_3^{(\tau)}*{}_3p_3^{(\tau)} = l_3^{(\tau)}*p_3^{(\tau)}*p_4^{(\tau)}*p_5^{(\tau)} = l_3^{(\tau)}*(1 - q_3^{(\tau)})*(1 - q_4^{(\tau)})*(1 - q_5^{(\tau)}) = l_3^{(\tau)}*(1 - q_3^{(1)} - q_3^{(2)})*(1 - q_4^{(1)} - q_4^{(2)})*(1 - q_5^{(1)} - q_5^{(2)}) =$

$1400(1 - 0.04 - 0.03)(1 - 0.09 - 0.014)(1 - 0.05 - 0.3) =$ **758.2848 red bluefish.**

Problem S3L67-2. Similar to Question 36 from the Casualty Actuarial Society's Fall 2007 Exam 3. This question is an excellent review of Section 49. A certain kind of chameleon has the option of changing its color only once a day and can only have two colors - orange and magenta. The transition of chameleons between these two colors can be modeled by a homogeneous Markov process. An orange chameleon has a 0.7 probability of remaining orange the next day. A magenta chameleon has a 0.5 probability of being magenta two days from now. Find the probability that a magenta chameleon today will become an orange chameleon tomorrow.

Solution S3L67-2. Since we are working with a homogeneous Markov process, the transition matrix for each time period is the same. Let A be the probability that a magenta chameleon today will become an orange chameleon tomorrow. Then we have the following transition matrix for each time period.

...O......M
O(0.7...0.3)
M(A....1-A)

There are two paths that a magenta chameleon can follow in order to be magenta two days from now: M → O → M and M → M → M. Pr(M → O → M) = A*0.3 = 0.3A, and

Pr(M → M → M) = $(1 - A)^2$. Thus, the probability that a magenta chameleon today is magenta two days from now is 0.5 = 0.3A + $(1 - A)^2$. This can be rearranged into the following quadratic equation:

$0.5 = 0.3A + 1 - 2A + A^2 \rightarrow A^2 - 1.7A + 0.5 = 0 \rightarrow A = 1.321699057$ or $A = 0.3783009434$. Since probabilities must be between 0 and 1, the only sensible answer is **A = 0.3783009434**.

Problem S3L67-3. Similar to Question 37 from the Casualty Actuarial Society's [Fall 2007 Exam 3]. This question is an excellent review of Section 33. For a life aged 65, you are given three values: ${}_2E_{65} = 0.77$, $p_{65} = 0.98$, and ${}_{1|}q_{65} = 0.07$. Find $A^1_{65:2\neg}$, the present value of the benefit payment of a discrete two-year term life insurance pertaining to this life.

Solution S3L67-3. This insurance only pays benefits if the insured life dies within the next two years, and, if it pays this benefit, it will only do so either 1 year or 2 years from now. We use the formula

$A^1_{x:n\neg} = {}_{k=0}{}^{n-1}\sum v^{k+1} * {}_kp_x * q_{x+k} = {}_{k=0}{}^{n-1}\sum v^{k+1}{}_{k|}q_x$. In this case, $A^1_{65:2\neg} = v*q_{65} + v^2*{}_{1|}q_{65}$. We know that ${}_{1|}q_{65} = 0.07$ and $q_{65} = 1 - p_{65} = 0.02$. We need to find v. We use the formula ${}_nE_x = v^n*{}_np_x$.

Here, ${}_2E_{65} = v^2*{}_2p_{65}$. We find ${}_2p_{65} = 1 - q_{65} - {}_{1|}q_{65} = 1 - 0.02 - 0.07 = {}_2p_{65} = 0.91$.

Thus, $0.77 = 0.91v^2$ and $v = \sqrt{(77/91)}$. Hence, $A^1_{65:2\neg} = \sqrt{(77/91)}*0.02 + (77/91)*0.07 =$ **$A^1_{65:2\neg}$ = about 0.0776280935**.

Problem S3L67-4. Similar to Question 1 from the Casualty Actuarial Society's [Spring 2007 Exam 3]. This question is an excellent review of Section 64. The number of character assassination attempts engaged in by politicians during an election cycle follows a Poisson distribution with a mean of 20 attempts per 30 days. Find the probability that more than 4 days will pass between the 15^{th} and the 16^{th} character assassination attempt.

Solution S3L67-4. If the mean of character assassination attempts per 30 days is 20, then the mean of character assassination attempts per day is 2/3. It follows that the waiting time between any two character assassination attempts follows an exponential distribution with mean 3/2. The probability that more than 4 days will pass between *any* two attempts is $s(4) = e^{-4/(3/2)} = \mathbf{e^{-8/3}}$ **= about 0.0694834512**.

Problem S3L67-5. Similar to Question 2 from the Casualty Actuarial Society's Spring 2007 Exam 3. This question is an excellent review of Section 62. The number of defective financial institutions that collapse on any given day follows a Poisson distribution with a mean of 5. The government then bails out each defective financial institution by giving it an amount of money that is normally distributed with mean \$1,000,000,000 and coefficient of variation (σ/μ) of 0.4. Find the standard deviation of the amount of taxpayer money the government gives to save defective financial institutions in a 30-day period.

Solution S3L67-5. The formula for the variance of a compound Poisson process is

$Var(S(t)) = E(N(t))*Var(X) + Var(N(t))*E(X)^2 = E(N(t))*E(X^2)$.

This is a compound Poisson process, with N(t) following a Poisson distribution such that E(N(t)) (over 30 days) is 5*30 = 150. Furthermore, $Var(X) = \sigma^2$. Since $(\sigma/\mu) = 0.4$ and $\mu = 1,000,000,000$, it follows that $\sigma = 400,000,000$ and $\sigma^2 = 1.6*10^{17}$. Since $Var(X) = E(X^2) - E(X)^2$, it follows that

$E(X^2) = Var(X) + E(X)^2 = 1.6*10^{17} + 1,000,000,000^2 = 1.16*10^{18}$.

Thus, $Var(S(t)) = E(N(t))*E(X^2) = 150(1.16*10^{18}) = 1.74*10^{20}$ and so $SD(S(t)) = \sqrt{(1.74*10^{20})}$ = **about \$13,190,905,958**.

Section 68

Challenging Exam-Style Review Questions for Actuarial Exam 3L

This section will be devoted to offering some more challenging exam-style questions that address the material already covered in the study guide in unusual ways. The best ways to learn how to do these problems is to simply try to reason through them. If you are having difficulty, take a look at either my answer keys or those provided by Dr. Broverman to the corresponding exam questions.

Some of the problems in this section were designed to be similar to problems from past versions of the Casualty Actuarial Society's Exam 3L and the Society of Actuaries' Exam MLC. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values and calculations in the problems here are the original work of Mr. Stolyarov.

Sources:

Broverman, Sam. Actuarial Exam Solutions - CAS Exam 3 - Spring 2007.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L68-1. Similar to Question 7 from the Casualty Actuarial Society's Spring 2007 Exam 3. Two independent lives, (x) and (y), have a joint life annuity-due that pays $1 per year with certainty for the first 7 years. Thereafter, the annuity pays $1 per year for every year that *both* individuals are alive and $1/2 per year for every year that *either* individual is alive. These two individuals have joint-life status denoted (xy). You also know the following about discrete life annuities-due for these indivdiuals:

The annual effective interest rate $i = 0.09$
$\ddot{a}_x = 20.34$
$\ddot{a}_{x:\overline{7|}} = 4.901$
$\ddot{a}_y = 18.75$
$\ddot{a}_{y:\overline{7|}} = 3.998$
$\ddot{a}_{xy} = 15.431$
$\ddot{a}_{xy:\overline{7|}} = 3.368.$

Find the actuarial present value of the annuity in question.

Solution S3L68-1. APV(Annuity) = PV(certain payments for first 7 years) + APV(payments when both individuals are alive after 7 years) + APV(payments when only (x) is alive after 7 years) + APV(payments when only (y) is alive after 7 years).

PV(certain payments for first 7 years) = $\ddot{a}_{7\neg}$ = $1.09(1 - 1.09^{-7})/0.09$ = 5.48591859.

APV(payments when both individuals are alive after 7 years) = ($\ddot{a}_{xy}$ - $\ddot{a}_{xy:7\neg}$) = 15.431 - 3.368 = 12.063.

APV(payments when only (x) is alive after 7 years) = (1/2)(($\ddot{a}_x$ - $\ddot{a}_{x:7\neg}$) - ($\ddot{a}_{xy}$ - $\ddot{a}_{xy:7\neg}$)) =

(1/2)((20.34 - 4.901) - 12.063) = 1.688.

APV(payments when only (y) is alive after 7 years) = (1/2)(($\ddot{a}_y$ - $\ddot{a}_{y:7\neg}$) - ($\ddot{a}_{xy}$ - $\ddot{a}_{xy:7\neg}$)) =

(1/2)((18.75 - 3.998) - 12.063) = 1.3445.

So APV(Annuity) = 5.48591859 + 12.063 + 1.688 + 1.3445 = **about $20.58141859.**

Problem S3L68-2. Similar to Question 9 from the Casualty Actuarial Society's Spring 2007 Exam 3. This question uses the Illustrative Life Table, which can be found here. This question also uses the Normal Distribution Table, which is found in the same PDF file as the Illustrative Life Table.

200 independent lives, each of age 40, is insured by a fund that pays a benefit of $1000 at the end of the year of death of each life. Mortality follows the Illustrative Life Table, and the annual effective rate of interest is 0.06. Use a normal approximation in order to find out how much money the fund needs to have initially in order to have a 67% probability of paying all claims.

Solution S3L68-2. Let K = $Z_1 + Z_2 + ... + Z_{200}$, where each subscripted Z is a random variable denoting the present value of the death benefit paid to one of the lives. K is then equal to the present value of the total amount that will be paid to each of the insured lives. Each of the subscripted Z's is independent of every other and has the exact same expected value, the APV of a discrete whole life insurance policy on a life (40), paying a benefit of 1 at the end of the year of death. We want to find X, where $\Pr(K \leq X) = 0.67$.

K has a mean of 200*1000*E(Z), where E(Z) = A_{40} = 0.16132, from the Illustrative Life Table.

Thus, E(K) = 200*1000*0.16132 = E(K) = 32264.

We can also find Var(K) = $200*1000^2*\text{Var}(Z) = 200*1000^2*({}^2A_{40} - (A_{40})^2)$.

From the Illustrative Life Table, ${}^2A_{40}$ = 0.04863. Thus,

Var(K) = $200*1000^2*(0.04863- (0.16132)^2)$ = Var(K) = 4521171.52, which implies that

$SD(K) = 4521171.52^{0.5} = 2126.304663$.

Thus, we can use a normal approximation where the normal distribution has mean 32264 and standard deviation 2126.304663.

Thus, $\Pr(K \leq X) = \Pr((K - 32264)/2126.304663 \leq (X - 32264)/2126.304663) = 0.67$, so

$\Phi((X - 32264)/2126.304663)) = 0.67$ and, looking up the corresponding value in the Normal Distribution Table, $(X - 32264)/2126.304663 = 0.44$. Thus $X = 0.44*2126.304663 + 32264 =$ **about $33199.57405.**

Problem S3L68-3. Similar to Question 10 from the Casualty Actuarial Society's Spring 2007 Exam 3. This question is a review of Section 53. You have the following sample drawn from a distribution with p.d.f. $f(x) = e^{-\theta+x}$ for any value of θ and any x > θ: 25, 2, 34, 35, 56, 67, 3, 5.

Which of these is a maximum likelihood estimate for θ?

(a) 3348660000
(b) 2
(c) 67
(d) 227
(e) 10669

Solution S3L68-3. For any sample value X_i, it must be the case $\theta < X_i$. If any estimate θ_e other than 2 is chosen for θ from the above possibilities, then there will be values X_i such that $\theta_e > X_i$, and this estimate will therefore have a likelihood of zero of being the true estimate. Therefore, the maximum likelihood estimate for θ is **(b): 2.**

Problem S3L68-4. Similar to Question 11 from the Casualty Actuarial Society's Spring 2007 Exam 3. This question is a review of Section 53. You are given that the proportion of GDP spent annually by governments throughout the world follows the distribution described by

$f(x) = (\theta +1)x^{\theta}$. You take a sample of 6 governments and find the following data about spending as a proportion of GDP: 0.36, 0.54, 0.23, 0.65, 0.24, 0.44.

Find the maximum likelihood estimate of θ.

Solution S3L68-4. We first find $L(\theta) = {}_{i=1}{}^{n}\Pi\ p_X(k_i; \theta) =$

$(\theta +1)0.36^{\theta}*(\theta +1)0.54^{\theta}*(\theta +1)0.23^{\theta}*(\theta +1)0.65^{\theta}*(\theta +1)0.24^{\theta}*(\theta +1)0.44^{\theta} =$

$L(\theta) = (\theta +1)^6*0.0030690317^{\theta}$.

Then $L'(\theta) = 6(\theta +1)^5*0.0030690317^{\theta} + \ln(0.0030690317)(\theta +1)^6*0.0030690317^{\theta}$.

To find the maximum likelihood estimate of θ, we set L'(θ) = 0. Thus,

$0 = 6(\theta +1)^5*0.0030690317^\theta + \ln(0.0030690317)(\theta +1)^6*0.0030690317^\theta$.

$6(\theta +1)^5*0.0030690317^\theta = -\ln(0.0030690317)(\theta +1)^6*0.0030690317^\theta$.

$6 = -\ln(0.0030690317)(\theta +1)$

$\theta_e = 6/-\ln(0.0030690317) - 1 =$ **$\theta_e = 0.0369153655$**.

Problem S3L68-5. Similar to Question 8 from the Casualty Actuarial Society's Spring 2007 Exam 3. This question uses the Illustrative Life Table, which can be found here. Mortality follows the Illustrative Life Table, and the annual interest rate is i = 6%. You are analyzing a fully discrete whole life insurance policy on a life aged 63, paying a benefit of 1000 at the end of the year of death. The annual premium payable for this policy is π, and the loss-at-issue random variable is denoted L(π). Find the largest value of π such that the standard deviation of L(π) is less than or equal to 300.

Solution S3L68-5. Let Z be the present-value random variable for the whole life insurance policy on (63) that pays a benefit of 1. Then the present-value random variable for the whole life insurance policy in question is 1000Z. Moreover, the present value of all the benefit premiums associated with the policy can be denoted as π*Y, where Y is the discrete whole life-annuity random variable. The loss-at-issue is the difference between the present value of the insurance policy and the present value of the benefit premiums, i.e., L(π) = 1000Z - π*Y.

But Y can be expressed as Y = (1+i)(1 - PV(insurance policy))/i = (1+i)(1 - Z)/i.
Thus, L(π) = 1000Z - π(1+i)(1 - Z)/i, where i = 0.06. Thus,
L(π) = 1000Z - π(53/3)(1 - Z) = L(π) = (1000 +(53/3)π)Z - (53/3)π.

For any random variable X and constants k and c, $Var(kX + c) = k^2Var(X)$, so
$Var(L(\pi)) = (1000 +(53/3)\pi)^2Var(Z)$, and $Var(Z) = E(Z^2) - E(Z)^2 = {}^2A_{63} - (A_{63})^2$.

From the Illustrative Life Table, ${}^2A_{63} = 0.21113$ and $A_{63} = 0.41085$.
Thus, $Var(Z) = 0.21113 - 0.41085^2 = 0.0423322775$.
So $Var(L(\pi)) = (1000 +(53/3)\pi)^2(0.0423322775)$ and
SD(L(π)) = (1000 +(53/3)π)(0.2057480923).

We want to find the largest π such that (1000 +(53/3)π)(0.2057480923) ≤ 300.
This implies that 1000 +(53/3)π ≤ 1458.093713 and (53/3)π ≤ 458.093713, so
π ≤ 25.92983284, which means that the largest value of π meeting these conditions is **π = 25.92983284**.

Section 69

Exam-Style Questions for Actuarial Exam 3L – Part 3

We recall from Section 56 that the complement of probability of committing a Type II error, **1- β**, is the probability of rejection of H_0 when H_1 is true. This value is known as the **power** of the test.

To expand upon the discussion of the joint-life status in Section 66, we know that E(T(xy)) = $\dot{e}_{xy} = {}_0{}^{\infty}\!\int {}_tp_{xy}*dt$. The second moment of (xy), $E(T(xy)^2)$, can be found via the following formula: $\mathbf{E((xy)^2) = {}_0{}^{\infty}\!\int 2t * {}_tp_{xy} * dt}$.

From these two formulas, it is possible to find $Var(T(xy)) = E(T(xy)^2) - E(T(xy))^2$.

Source: Broverman, Sam. Actuarial Exam Solutions - CAS Exam 3 - Spring 2007.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L69-1. Similar to Question 18 from the Casualty Actuarial Society's Spring 2007 Exam 3. This question is a review of Section 55. You are testing a null hypothesis which states that the number of disastrous pieces of legislation passed is randomly distributed among each of the years 2009-2015. Here is what you actually observe regarding the number of disastrous pieces of legislation passed each year:

2009: 130
2010: 156
2011: 132
2012: 154
2013: 160
2014: 120
2015: 163
Total: 1015

Within which range is the p value that results from testing the null hypothesis above?

(a) $p < 0.001$
(b) $0.001 < p < 0.01$
(c) $0.01 < p < 0.025$
(d) $0.025 < p < 0.05$
(e) $0.05 < p < 0.1$

Use a **Table of Critical Values for the Chi-Square Distribution**.

Solution S3L69-1. The null hypothesis states that expected number of disastrous pieces of legislation should be the same for each year. We have 7 years to consider, so the mean per year is hypothesized as 1015/7 = 145. Thus, our Chi-square statistic is d = $_{i=1}{}^{t}\sum[(k_i - np_{io})^2/(np_{io})]$ =

$(130 - 145)^2/145 + (156 - 145)^2/145 + (132 - 145)^2/145 + (154 - 145)^2/145 + (160 - 145)^2/145 + (120 - 145)^2/145 + (163 - 145)^2/145$ = d = 12.20689655. This Chi-square distribution has 7 - 1 = 6 degrees of freedom. We look in the corresponding row of the **Table of Critical Values for the Chi-Square Distribution** to find 12.20689655 to be between 10.645 (the value associated with p = 0.1) and 12.592 (the value associated with p = 0.05).

Thus, the correct answer is **(e): 0.05 < p < 0.1**.

Problem S3L69-2. Similar to Question 19 from the Casualty Actuarial Society's Spring 2007 Exam 3. This question is a review of Section 55. You are testing a hypothesis that states that the distribution of spending per government is independent of the government's type.

You are investigating five systems of government:
Democracy, Democratic Republic, Republican Democracy, Social Democracy, and Democratic Socialism.

You are also investigating four types of spending:
Welfare, Miscellaneous, Auxiliary, and Miscellaneous Auxiliary.

Your hypothesis test has a Chi-square statistic associated with it. How many degrees of freedom are associated with the Chi-square statistic?

Solution S3L69-2. You can arrange the five types of government and four types of spending in a table with five rows and four columns. The entries in the last row are determined by the entries in the first four rows. Likewise, the entries in the last column are determined by the entries in the first three columns. Thus, we are only truly free to pick values in four rows and three columns, which gives us 3*4 = **12 degrees of freedom.**

Problem S3L69-3. Similar to Question 22 from the Casualty Actuarial Society's Spring 2007 Exam 3. This question is a review of Section 56. K is a random variable that can assume values of either 0 or 1, with Pr(K = 0) = p. You draw a sample of 1400 values of K in order to test the following two hypotheses:

H_0: p = 0.48; H_1: p = 0.5.

You know that the probability of a Type II error is 0.2005. Find the significance level of this test.

Use the **Table of Values for the Standard Normal Distribution**.

Solution S3L69-3. The level of significanceof a hypothesis test is precisely the probability of committing a Type I error. We know that Pr(Type II error) = Pr(Do not reject H_0 │ H_0 is false) = 0.2005 = Pr(Do not reject H_0 │ H_1 is true) = Pr(Do not reject H_0 │ p = 0.5). Since H_0 posits a lower value of p than H_1, this means that H_0 is likelier to be accepted when the incidence of 0's is lowest. Thus, H_0 will be wrongly accepted if the sample mean is within the leftmost 0.2005 of the distribution of sample means that would have been the case if H_1 were true.

If H_1 is true, then we can assume that sample means are normally distributed with mean

1400*0.5 = 700 and standard deviation √(700*(1 - 0.5)) = √(350) (for a binomial distribution).

The threshold for committing a Type II error is r standard deviations from the mean, where 0.2005 = Φ(r). Using the **Table of Values for the Standard Normal Distribution**, we find that 0.7995 = Φ(0.84) and so 0.2005 = 1 - 0.7995 = 1 - Φ(0.84) = Φ(-0.84). Thus, r = -0.84.

Thus, the threshold for committing a Type II error is 700 -0.84√(350) = 684.285039. This is the threshold for rejecting H_0 *and* the threshold for accepting H_1.

The significance level of the test is Pr(Type I error) = Pr(reject H_0 | H_0 is true).

The threshold for committing a Type I error *remains* at 684.285039, but the mean of the distribution with which we are working is different, because we now assume that p = 0.4, so the mean is 0.48*1400 = 672 and the standard deviation is √(1400*0.48*(1-0.48)) = √(349.44) (for a binomial distribution).

Thus, 684.285039 is q standard deviations away from 672, where

q = (684.285039 - 672)/√(349.44) = 0.6571889176.

Our desired significance level is 1 - Φ(0.6571889176), which we find in MS Excel using the input "=1 - NormSDist(0.6571889176)" = **0.255529725**.

Problem S3L69-4. Similar to Question 23 from the Casualty Actuarial Society's Spring 2007 Exam 3. This question is a review of Section 56. You are examining a normal distribution of the random variable X, with mean μ and variance 625. You pick a sample of n values from the distribution to test the following two hypotheses: H_0: μ = 500; H_1: μ = 525.

The significance level of this test is 0.1587, and the power of this test is 50%. Find the number of additional observations in the sample needed to raise the power of the test to 97.72%.

Solution S3L69-4. The significance level of this test is Pr(reject H_0 | H_0 is true) = 0.1587.

Using the **Table of Values for the Standard Normal Distribution**, we find that 1 - 0.1587 = 0.8413 = Φ(1). Thus, all sample means to the right of one standard deviation away from 500 will result in a rejection of H_0 and an acceptance of H_1. But the sample standard deviation is not the same as the distribution standard deviation σ = √(625) = 25. Rather, the sample standard deviation is S = σ/√(n). Thus, the threshold for accepting H_1 is located at 500 + 25/√(n).

The power of the test is Pr(reject H_0 | H_1 is true) = 0.5. We know that 0.5 = Φ(0), so we know that the threshold for accepting H_1 is *exactly* at the mean under H_1 for this test, i.e., at 525. Thus,

we know that 500 + 25/√(n) = 525 and so 25/√(n) = 25 and n = 1.

Now, in order for this test to have a power of 0.9772 (with the same significance level) , we need to add some k values to our sample such that our threshold for accepting H_1 will be 500 + 25/√(n + k) and Pr(X > 500 + 25/√(n + k) | μ = 525) = 0.9772. It must thus be the case that

500 + 25/√(n + k) is -r standard deviations away from 525, where Φ(r) = 0.9772. Using the **Table of Values for the Standard Normal Distribution**, we find that r = 2.

Thus, 525 - 2*25/√(n + k) = 500 + 25/√(n + k) and so 25 = 75/√(n + k) and thus √(n + k) = 3 and so n +k = 9. Since n = 1, it must be that **k = 8 additional observations.**

Problem S3L69-5. Similar to Question 25 from the Casualty Actuarial Society's Spring 2007 Exam 3. This question is a review of Section 66.

You are given the following information about two independent lives (x) and (y): ${}_tp_x = e^{-0.03t}$, and

${}_tp_y = 1 - 0.04t$ for $0 \leq t \leq 25$. Find Var(T(xy)), the variance of the future lifetime pertaining to the joint-life status of these two lives. Use any calculator to evaluate any integrals you encounter.

Solution S3L69-5. We know that Var(T(xy)) = E(T(xy)^2) - E(T(xy))^2 and that

E(T(xy)) = $\mathring{e}_{xy}$= ${}_0{}^{\infty}\int {}_tp_{xy}$*dt. Since (x) and (y) are independent, ${}_tp_{xy} = {}_tp_x * {}_tp_y = e^{-0.03t}(1 - 0.04t)$. The upper bounds of all our integrals will be 25, since life (y) cannot survive beyond 25 years, and the joint-life status cannot survive beyond life (y). Thus, E(T(xy)) = ${}_0{}^{25}\int e^{-0.03t}(1 - 0.04t)$*dt = 9.8829579.

Now we use the formula E((xy)^2) = ${}_0{}^{\infty}\int 2t * {}_tp_{xy}$*dt = ${}_0{}^{25}\int 2t * e^{-0.03t}(1 - 0.04t)$*dt = 145.2089483.

Thus, Var(T(xy)) = 145.2089483 - 9.8829579^2 = **Var(T(xy)) = 47.53609141**.

Section 70

Pooled Sample Variances and Covariances for Multiple-Life Statuses

The following expressions regarding the **covariance** of two random variables X and Y are useful:
$\text{Cov}(X, Y) = 0.5(\text{Var}(X + Y) - \text{Var}(X) - \text{Var}(Y))$.
$\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$

For the joint-life status (xy) and the last-survivor status ($\overline{xy}$) on independent lives (x) and (y), the following is true:
$E(T(xy)*T(\overline{xy})) = E(T(x)*T(y)) = E(T(x))*E(T(y))$.

The equality $E(T(xy)*T(\overline{xy})) = E(T(x)*T(y))$ holds because the lifetime of the joint-life status T(xy) is equal to either T(x) or T(y) (whichever life dies first), and $T(\overline{xy})$ is equal to whichever of the T(x) or T(y) that exceeds the other.

When taking any two samples X of size n_X and Y of size n_Y from a normal distribution, the **pooled sample variance,** denoted S_p^2, can be found as follows:
$S_p^2 = ((n_X - 1)S_X^2 + (n_X - 1)S_Y^2)/(n_X + n_Y - 2)$.

To perform a hypothesis test comparing the means of X and Y, μ_X and μ_Y, the following test statistic is used:
$t = (\mu_{Y\_observed} - \mu_{X\_observed})/\sqrt{(S_p^2(1/n_X + 1/n_Y))}$

The p-value of t can be found by examining the Table of Percentage Points of the t Distribution. Look in the row with the appropriate number of degrees of freedom **$(n_X + n_Y - 2)$** and then find the value that corresponds to the observed test statistic t. Then look at the heading of the column to see which *tail probabilities* t corresponds to. If you seek to find Pr(|T| > t), then your desired p-value is the probability associated with *two tails* of the distribution.

If you seek to find Pr(T > t), then your desired p-value is the probability associated with *one tail* of the distribution.

Source: Broverman, Sam. Actuarial Exam Solutions - CAS Exam 3 - Spring 2007.

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Problem S3L70-1. Similar to Question 26 from the Casualty Actuarial Society's Spring 2007 Exam 3. This question is a review of Section 66. You are analyzing two independent lives, (x) and (y), for which ${}_tp_x = e^{-0.04t}$ and ${}_tp_y = e^{-0.02t}$. Find $\text{Cov}(T(xy), T(\overline{xy}))$.

Solution S3L70-1. We use the formula Cov(X, Y) = E(XY) - E(X)E(Y), which in this case translates to $Cov(T(xy), T(\overline{xy})) = E(T(xy)T(\overline{xy})) - E(T(xy))E(T(\overline{xy}))$.

We know that $E(T(xy)*T(\overline{xy})) = E(T(x)*T(y)) = E(T(x))*E(T(y))$.

Since T(x) and T(y) are exponentially distributed, we know that $E(T(x)) = 1/\mu_x = 1/0.04 = 25$.

Likewise, $E(T(y)) = 1/\mu_y = 1/0.02 = 50$. So $E(T(xy)*T(\overline{xy})) = 25*50 = 1250$.

Since the two lives are independent, ${}_tp_{xy} = ({}_tp_x)({}_tp_y) = e^{-0.04t} * e^{-0.02t} = e^{-0.06t}$.

$E(T(xy)) = \mathring{e}_{xy} = {}_0^\infty\int {}_tp_{xy}*dt = {}_0^\infty\int e^{-0.06t}*dt = 1/0.06 = 16.66666666667$

Moreover, ${}_tp_{\overline{xy}} = {}_tp_x + {}_tp_y - {}_tp_{xy} = e^{-0.04t} + e^{-0.02t} - e^{-0.06t}$, so

$E(T(\overline{xy})) = {}_0^\infty\int {}_tp_{\overline{xy}}*dt = {}_0^\infty\int(e^{-0.04t} + e^{-0.02t} - e^{-0.06t})*dt = 1/0.04 + 1/0.02 - 1/0.06 = 58.3333333$.

Thus, $E(T(xy))E(T(\overline{xy})) = 16.66666666667*58.3333333 = 972.22222222$, and so

$Cov(T(xy), T(\overline{xy})) = 1250 - 972.22222222$ = about **277.777777778**.

Problem S3L70-2. Similar to Question 27 from the Casualty Actuarial Society's Spring 2007 Exam 3. This question is a review of Section 46. Black doves are subject to two causes of decrement, (1): turning white (after which they are no longer black doves) and (2): spontaneous invisibility disorder (SID). You are given the following double decrement table for black doves aged x.

Age: x........$q_x^{(1)}$.......... $q_x^{(2)}$
12.........0.32............0.11
13.........0.24............0.42
14.........0.17............0.67

You are examining a group of 10000 12-year-old black doves. Find ${}_{15}d_{12}^{(2)}$, the number of black doves that are expected to cease being black due to SID between ages 12 and 15. Fractional black doves are possible (because it is possible for SID only afflict a part of a black dove).

Solution S3L70-2. 0.11*10000 = 1100 12-year-old black doves will become invisible between years 12 and 13.

At time t =13, there will be 10000(1 - 0.11 - 0.32) = 5700 black doves. Of these, 0.42*5700 = 2394 will become invisible between years 13 and 14.
At time t = 14, there will be 5700(1 - 0.24 - 0.42) = 1938 black doves. Of these, 0.67*1938 = 1298.46 will become invisible between years 14 and 15.

Thus, ${}_{15}d_{12}^{(2)} = 1100 + 2394 + 1298.46 =$ **4792.46 black doves**.

Problem S3L70-3. Similar to Question 28 from the Casualty Actuarial Society's Spring 2007 Exam 3. This question is a review of Section 46. Consider the following non-homogeneous Markov process, where transitions among states X, Y, and Z occur at times 1, 2, and 3, and the following transition matrices Q_1, Q_2, and Q_3 are associated with each respective transition.

Q_1:
...X......Y.....Z
X(0.1...0.5...0.4)
Y(0.4...0.3...0.3)
Z(0.1...0.7...0.2)

Q_2:
...X......Y.....Z
X(0.3...0.2...0.5)
Y(0.2...0.1...0.7)
Z(0.25..0.35..0.4)

Q_3:
...X......Y........Z
X(0.15..0.25..0.6)
Y(0.8...0.1...0.1)
Z(0.4...0.5..0.1)

Find the probability that an entity starting in State X at time 0 will transition from state Y to state Z at time 3.

Solution S3L70-3. We want to find the probability of the following transition path:

X → □ → Y → Z, where □ can be either X, Y, or Z.

We find Pr(X → X → Y → Z) = 0.1*0.2*0.1 = 0.002.

We find Pr(X → Y → Y → Z) = 0.5*0.1*0.1 = 0.005.

We find Pr(X → Z → Y → Z) = 0.4*0.35*0.1 = 0.014.

Our desired probability is thus 0.002 + 0.005 + 0.014 = **0.021.**

Problem S3L70-4. Similar to part of Question 29 from the Casualty Actuarial Society's Spring 2007 Exam 3. This question is a review of Section 56.

You are taking two samples, each from a normal distribution.

Sample X has 15 values in it and has a mean of 6 and a variance of 8.

Sample Y has 23 values in it and has a mean of 8 and a variance of 15.

Find the pooled sample variance, S_p^2 .

Solution S3L70-4. We use the formula $S_p^2 = ((n_X - 1)S_X^2 + (n_X - 1)S_Y^2)/(n_X + n_Y - 2)$, where

$n_X = 15$, $n_Y = 23$, $S_X^2 = 8$, and $S_Y^2 = 15$. Thus,

$S_p^2 = (14*8 + 22*15)/(15 + 23 - 2) =$ **$S_p^2 = 12.2777777778$**.

Problem S3L70-5. Similar to part of Question 29 from the Casualty Actuarial Society's Spring 2007 Exam 3. This question is a review of Section 56.

You are taking two samples, each from a normal distribution.

Sample X has 15 values in it and has a mean of 6 and a variance of 8.

Sample Y has 23 values in it and has a mean of 8 and a variance of 15.

You are performing the following hypothesis test:

H_0: $\mu_Y = \mu_X$; H_1: $\mu_Y > \mu_X$.

Find $|t - T|$, the absolute value of the difference between the test statistic t and the critical value of the t-distribution at the significance level $\alpha = 0.01$.

Use the Table of Percentage Points of the t Distribution.

Solution S3L70-5. We use the formula $t = (\mu_{Y\_observed} - \mu_{X\_observed})/\sqrt{(S_p^2(1/n_X + 1/n_Y))}$.

From Solution S3L70-4, $S_p^2 = 12.2777777778$. Moreover, $n_X = 15$, $n_Y = 23$, $\mu_{Y\_observed} = 8$, and $\mu_{X\_observed} = 6$. Thus, we have $t = (8 - 6)/\sqrt{(12.2777777778(1/15 + 1/23))} = t = 1.719839268$.

To find T, we use the Table of Percentage Points of the t Distribution for $n_X + n_Y - 2 = 36$ degrees of freedom and a *one-tail* probability of 0.01, since our H_1 is $\mu_Y > \mu_X$. Thus, we have T = 2.434. Thus, $|1.719839268 - 2.434|$ = **about 0.7141607324**.

Section 71

The Coefficient of Determination in Linear Regression, Fully Continuous Benefit Reserves Under DeMoivre's Law, and Assorted Exam-Style Questions

When performing least-squares linear regression, the **coefficient of determination,** or $\mathbf{R^2}$**,** is a useful measure of the accuracy of one's estimate. The closer R^2 is to 1, the more accurate the estimate. The closer R^2 is to 0, the less accurate the estimate. For a sample of size n, R^2 can be found as follows.

$$\mathbf{R^2 = 1 - {}_{i=1}{}^{n}\sum(y_i - \hat{y}_i)^2/{}_{i=1}{}^{n}\sum(y_i - \mu_y)^2}$$

Here, the y_i are the actual (given) values of y, the $\hat{y}_i$ are the values of y estimated by the linear regression, and μ_y is the mean of the actual values of y (y_i). The quantity $\mathbf{{}_{i=1}{}^{n}\sum(y_i - \mu_y)^2}$ is known as **TSS** or the **total sum of squares.** The quantity $\mathbf{{}_{i=1}{}^{n}\sum(y_i - \hat{y}_i)^2}$ is known as **ESS** or the **error sum of squares.** Thus, a more convenient formula to memorize is

$$\mathbf{R^2 = 1 - ESS/TSS.}$$

When future lifetimes follow **DeMoivre's law** (i.e., the uniform distribution), with some limiting value ω (the maximum possible lifetime), then shortcuts exist for finding expressions for continuous whole life insurance policies. Given force of interest $\delta = \ln(1 + i)$, we have

$$\mathbf{\bar{a}_{\overline{\omega - x}|} = (1 - v^{\omega - x})/\delta},$$

$$\mathbf{\bar{A}_x = \bar{a}_{\overline{\omega - x}|}/(\omega - x)}, \text{ and}$$

$\mathbf{{}_t\bar{\bar{V}}(\bar{A}_x) = (\bar{A}_{x+t} - \bar{A}_x)/(1 - \bar{A}_x)}$, where $\mathbf{{}_t\bar{\bar{V}}(\bar{A}_x)}$ is the **fully continuous benefit reserve at time t.**

Source: Broverman, Sam. Actuarial Exam Solutions - CAS Exam 3 - Spring 2007.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L71-1. Similar to Question 30 from the Casualty Actuarial Society's Spring 2007 Exam 3. This question is a review of Section 65.

You have two samples, X and Y, each drawn from some normal distribution. Sample X contains 10 values and sample Y contains 5 values. The mean and variance of each distribution are unknown. You are aware of the sample variances: $s_X{}^2 = 25$ and $s_Y{}^2 = 36$. You are testing between the following two hypotheses:

H_0: $\sigma_Y = \sigma_X$; H_1: $\sigma_Y < \sigma_X$.

Calculate $|f - F|$, the absolute value of the difference between the test statistic f and the critical value of the F-distribution at the significance level $\alpha = 0.05$. Where necessary, use the Table of Upper Critical Values for the F Distribution.

Solution S3L71-1. The test statistic f is $s_Y{}^2/s_X{}^2 = 36/25$.

We use the Table of Upper Critical Values for the F Distribution to find F for $\alpha = 0.05$ and numerator degrees of freedom 5 - 1 = 4 and denominator degrees of freedom 10 - 1 = 9. This value of F is 3.633. Thus, $|f - F| = |36/25 - 3.633| =$ **about 2.193**.

Problem S3L71-2. Similar to Question 31 from the Casualty Actuarial Society's Spring 2007 Exam 3. This question is a review of Section 57.

You are given data for variables x and y, and you hypothesize that there is a linear relationship between x and y, such that your estimate of y, ŷ, is equal to 2x + 5. You have the following data for x, y, and ŷ:

x......y...... ŷ
5.....17......15
8.....22......21
10....24......25
15....38......35
16....39......37
17....39......39

From this information, findR^2, the coefficient of determination associated with the estimate ŷ = 2x + 5.

Solution S3L71-2. We use the formula $R^2 = 1 - ESS/TSS$.
$ESS = {}_{i=1}{}^{n}\sum(y_i - \hat{y}_i)^2 = (17 - 15)^2 + (22 - 21)^2 + (24 - 25)^2 + (38 - 35)^2 + (39 - 37)^2 + (39 - 39)^2 = ESS = 19$.
We also find $\mu_y = (17 + 22 + 24 + 38 + 39 + 39)/6 = 29.83333333333$.
$TSS = {}_{i=1}{}^{n}\sum(y_i - \mu_y)^2 = (17 - 29.83333333333)^2 + (22 - 29.83333333333)^2 + (24 - 29.83333333333)^2 + (38 - 29.83333333333)^2 + (39 - 29.83333333333)^2 + (39 - 29.83333333333)^2 = 494.83333333333$.
Thus, $R^2 = 1 - 19/494.83333333333 =$ $\mathbf{R^2 = 0.9616032334}$.

Problem S3L71-3. Similar to Question 38 from the Casualty Actuarial Society's Spring 2007 Exam 3. This question is a review of Section 58. You are given the following information pertaining to a 30-year endowment insurance policy on life (30) paying a benefit of 1000:

$_{10}E_{30} = 0.778$

$_{30}E_{30} = 0.318$

$\ddot{a}_{30:10\neg} = 8.465$

$\ddot{a}_{30:30\neg} = 24.436$

$A^1_{30:10\neg} = 0.0982$

$A^1_{30:30\neg} = 0.246$

Find $1000{*}_{10}V_{30:30\neg}$, the benefit reserve at time 10 associated with this policy.

Solution S3L71-3. We use the retrospective formula: $_kV_{x:n\neg} = (1/_kE_x)(P_{x:n\neg}{*}\ddot{a}_{x:k\neg} - A^1_{x:k\neg})$.
Here, x = 30, n = 30, and k = 10. Thus, $_kE_x = {}_{10}E_{30} = 0.778$.
We are also given $\ddot{a}_{x:k\neg} = \ddot{a}_{30:10\neg} = 8.465$ and $A^1_{x:k\neg} = A^1_{30:10\neg} = 0.0982$.
We need to find $P_{x:n\neg}$. From Section 52, $P_{x:n\neg} = A_{x:n\neg}/\ddot{a}_{x:n\neg}$, and $A_{x:n\neg} = A^1_{x:n\neg} + {}_nE_x$.
Here, $A_{30:30\neg} = A^1_{30:30\neg} + {}_{30}E_{30} = 0.246 + 0.318 = 0.564$.
Moreover, $\ddot{a}_{30:30\neg} = 24.436$. Thus, $P_{30:30\neg} = 0.564/24.436 = 0.0230807006$.
Thus,
$1000{*}_{10}V_{30:30\neg} = 1000(1/0.778)(0.0230807006{*}8.465 - 0.0982) =$ **about 124.9076229.**

Problem S3L71-4. Similar to Question 39 from the Casualty Actuarial Society's [Spring 2007 Exam 3](). This question is a review of Section 58. The annual effective rate of interest is i = 0.08%. Among a group of minocentaurs, $l_x = 55 - x$. You are considering $\bar{A}_{13}$, the present value of a fully continuous whole life insurance policy for a minocentaur, paying a benefit of 1 upon death. Find $_{18}\bar{V}(\bar{A}_{13})$, the fully continuous benefit reserve when the minocentaur is 31 years old.

Solution S3L71-4. We note that the survival of minocentaurs follows DeMoivre's law with ω = 55.

For x = 13, we first find $\bar{a}_{\omega - x\neg} = (1 - v^{\omega - x})/\delta = \bar{a}_{55 - 13\neg} = \bar{a}_{42\neg} = (1 - 1.08^{-42})/\ln(1.08) = \bar{a}_{42\neg} =$ 12.48080691.

Now we find $\bar{A}_x = \bar{a}_{\omega - x\neg}/(\omega - x) = \bar{A}_{13} = \bar{a}_{42\neg}/42 = 12.48080691/42 = \bar{A}_{13} = 0.2971620693$.

For x = 31, we first find $\bar{a}_{\omega - x\neg} = (1 - v^{\omega - x})/\delta = \bar{a}_{55 - 31\neg} = \bar{a}_{24\neg} = (1 - 1.08^{-24})/\ln(1.08) = \bar{a}_{24\neg} =$ 10.94450712.

Now we find $\bar{A}_x = \bar{a}_{\omega - x\neg}/(\omega - x) = \bar{A}_{31} = \bar{a}_{24\neg}/24 = 0.45602113$.

Now we can use the formula $_t\bar{V}(\bar{A}_x) = (\bar{A}_{x+t} - \bar{A}_x)/(1 - \bar{A}_x)$.

Thus, $_{18}\bar{V}(\bar{A}_{13}) = (\bar{A}_{31} - \bar{A}_{13})/(1 - \bar{A}_{13}) = (0.45602113 - 0.2971620693)/(1 - 0.2971620693) =$

$_{18}{}^{--}V(\bar{A}_{13})$ = 0.2260251671.

Problem S3L71-5. Similar to Question 1 from the Casualty Actuarial Society's [Spring 2006 Exam 3]. This question is a review of Section 54. The number of solvency crises per year experienced by government-subsidized banking institutions follows a negative binomial distribution with parameters r and β. A random sample of 16 government-subsidized banking institutions showed the following with regard to the number of solvency crises each institution underwent.

Number of Solvency Crises.........Institutions Experiencing This Number of Crises
0...0
1...2
2...5
3...1
4...2
5...3
6...1
7...1
8...1

Use the method of moments to estimate β for this negative binomial distribution.

Solution S3L71-5. We need to estimate two parameters here, so we use a system of two equations:
$E(X) = E(M)$
$E(X^2) = E(M^2)$
We know that $E(X) = r\beta$ and $Var(X) = E(X^2) - E(X)^2 = r\beta(1 + \beta)$, so $E(X^2) = Var(X) + E(X)^2 = r\beta(1 + \beta) + (r\beta)^2$.
We now find $E(M) = (1*2 + 2*5 + 3*1 + 4*2 + 5*3 + 6*1 + 7*1 + 8*1)/16 = 3.6875$.
We also find $E(M^2) = (1^2*2 + 2^2*5 + 3^2*1 + 4^2*2 + 5^2*3 + 6^2*1 + 7^2*1 + 8^2*1)/16 = 17.9375$.
Thus, we need to solve the following system of equations:
(i) $r\beta = 3.6875$.
(ii) $r\beta(1 + \beta) + (r\beta)^2 = 17.9375$.
We substitute 3.6875 for rβ into (ii):
$3.6875(1 + \beta) + 3.6875^2 = 17.9375$
$3.6875(1 + \beta) = 4.33984375$
$1 + \beta = 1.17690678$
β = about 0.17690678.

Section 72

Chi-Square Test for the Variance and Assorted Exam-Style Questions

Property 72.1. For a binomial distribution with probability of success p and a fixed number of trials m, the maximum likelihood estimate for p is the same as the method of moments estimate for p.

The **Chi-square test for the variance** is described as follows by Larsen and Marx (504).

"Let s^2 denote the sample variance calculated from a random sample of n observations drawn from a normal distribution with mean μ and variance σ^2. Let $\chi^2 = (n - 1)s^2/\sigma_0^2 = {}_{i=1}{}^{n}\sum(X_i - \mu_{X\_observed})^2/\sigma_0^2$."

Let σ_0^2 be the value for σ^2 posited by the null hypothesis.

Property 72.2. "To test H_0: $\sigma^2 = \sigma_0^2$ versus H_1: $\sigma^2 > \sigma_0^2$ at the α level of significance, reject H_0 if $\chi^2 \geq \chi^2_{1-\alpha,n-1}$." Use the Table of Lower Critical Values for the Chi-Square Distribution to find $\chi^2_{1-\alpha,n-1}$. (Scroll down on the page linked above to find this table).

Property 72.3. "To test H_0: $\sigma^2 = \sigma_0^2$ versus H_1: $\sigma^2 < \sigma_0^2$ at the α level of significance, reject H_0 if $\chi^2 \leq \chi^2_{\alpha,n-1}$." Use the Table of Upper Critical Values for the Chi-Square Distribution to find $\chi^2_{\alpha,n-1}$.

Property 72.4. "To test H_0: $\sigma^2 = \sigma_0^2$ versus H_1: $\sigma^2 \neq \sigma_0^2$ at the α level of significance, reject H_0 if (1) $\chi^2 \leq \chi^2_{\alpha/2,n-1}$ or (2) $\chi^2 \geq \chi^2_{1-\alpha/2,n-1}$." Depending on your test statistic, you will need to use either table of upper critical values or the table of lower critical values, as shown in Properties 72.2 and 72.3.

Source: Broverman, Sam. Actuarial Exam Solutions - CAS Exam 3 - Fall 2005.

Larsen, Richard J. and Morris L. Marx.

An Introduction to Mathematical Statistics and Its Applications. Fourth Edition. Pearson Prentice Hall: 2006. p. 504.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L72-1. Similar to Question 1 from the Casualty Actuarial Society's Fall 2005 Exam 3. This question is a review of Section 53. You are given the following probability distribution for the random variable X: $f(x) = \theta x^{\theta-1}$, for $\theta > 0$ and $0 < x < 1$. You have collected a sample of the following seven values for x:

0.23, 0.46, 0.76, 0.69, 0.13, 0.44, 0.9.

Find P - Q, where P is the maximum likelihood estimate for θ and Q is the method of moments estimate for θ.

Solution S3L72-1. We first determine our likelihood function:

$L(\theta) = (\theta*0.23^{\theta-1})(\theta*0.46^{\theta-1})(\theta*0.76^{\theta-1})(\theta*0.69^{\theta-1})(\theta*0.13^{\theta-1})(\theta*0.44^{\theta-1})(\theta*0.9^{\theta-1})$

$L(\theta) = \theta^7*0.0028561866^{\theta-1}$

L(θ) is maximized at P such that L'(P) = 0.

$L'(\theta) = 7\theta^6*0.0028561866^{\theta-1} + \ln(0.0028561866)\theta^7*0.0028561866^{\theta-1}$

$L'(P) = 0 = 7P^6*0.0028561866^{P-1} + \ln(0.0028561866)P^7*0.0028561866^{P-1}$

$7P^6*0.0028561866^{P-1} = -\ln(0.0028561866)P^7*0.0028561866^{P-1}$

$7 = -\ln(0.0028561866)P$

P = 7/-ln(0.0028561866) = P = 1.194892582.

Now we find Q. Since we have only one parameter, we only need to solve the equation E(X) = E(M). Here, $E(X) = {}_0{}^1\int x*\theta x^{\theta-1}*dx = {}_0{}^1\int \theta x^{\theta}*dx = \theta/(\theta + 1)$.

E(M) = (0.23 + 0.46 + 0.76 + 0.69 + 0.13 + 0.44 + 0.9)/7 = E(M) = 0.5157142857.

Thus, Q/(Q + 1) = 0.5157142857

(1/0.5157142857)Q = (Q + 1)

(1/0.5157142857 - 1)Q = 1

Q = 1/(1/0.5157142857 - 1) = Q = 1.064896755.

Thus, P - Q = 1.194892582 - 1.064896755 **= about 0.1299958268**.

Problem S3L72-2. Similar to Question 2 from the Casualty Actuarial Society's Fall 2005 Exam 3. This question is a review of Section 53. You have a sample of size 17 drawn from an exponential distribution with $f(x) = 0.02e^{-0.02x}$. Find the probability that the 17th order statistic is greater than 75.

Solution S3L72-2. For the largest order statistic Y'_i in any set of order statistics, it is the case that $F_{Y'\_i}(y) = F(y)^i$. Thus, $F_{Y'\_17}(y) = F(y)^{17} = (1 - e^{-0.02x})^{17}$.

We thus find $Pr(Y'_{17} > 75) = 1 - F(75)^{17} = 1 - (1 - e^{-0.02*75})^{17}$ = **about 0.986325217**.

Problem S3L72-3. Similar to Question 3 from the Casualty Actuarial Society's Fall 2005 Exam 3. This question is a review of Section 69. You are working with a Poisson distribution that has an unknown mean, and you perform a hypothesis test regarding that mean by conducting a single observation. At a 15% significance level, you test between the following two hypotheses: H_0: $\lambda = 2$; H_1: $\lambda = 4$. Find the power of this hypothesis test.

Solution S3L72-3. We first need to find the value of the observed x for which the null hypothesis is rejected. A 15% significance level implies that, assuming H_0 is true, we will reject H_0 if we observe the highest 15% of possible values of x.

Since a Poisson random variable is discrete, we need to find the first x at which the probability of getting observations

less than x is greater than 0.85.

Assuming that $\lambda = 2$, $Pr(X \leq 3) = e^{-2} + 2e^{-2} + 2^2e^{-2}/2! + 2^3e^{-2}/3! = 0.8571234605$. So 3 is clearly the first value of x for which $Pr(X \leq 3) > 0.85$. Thus, we reject the null hypothesis for all observed x of 4 or greater.

The power of this test is the probability of rejecting H_0, given that H_1 is true. This is the probability that $\lambda = 4$ and x is 4 or greater.

Thus, the power of the test is $1 - e^{-4} + 4e^{-4} + 4^2e^{-4}/2! + 4^3e^{-4}/3!$ = **about 0.5665298796**.

Problem S3L72-4. Similar to Question 4 from the Casualty Actuarial Society's Fall 2005 Exam 3. This question is a review of Section 69. Geoffrey visits the amusement park once a month, and each time he rides on either 0, 1, 2, or 3 different roller coasters. During his past 10 visits to the amusement park, Geoffrey rode on 0 roller coasters twice, on 1 roller coaster three times, on 2 roller coasters 4 times, and on 3 roller coasters once. Geoffrey tosses a loaded coin to determine whether he should ride on any given roller coaster on any given visit, and the probability that the coin will land heads and Geoffrey rides any given roller coaster is p. Find the maximum likelihood estimate for p.

Solution S3L72-4. The probability p is a binomial probability, since Geoffrey can either ride or not ride any given roller coaster. We have the number of trials in the binomial distribution as 10*3 = 30, since every time Geoffrey visits the amusement park, he subjects 3 roller coasters to the coin toss. By Property 72.1, this is a case where the maximum likelihood estimate is the same as the method of moments estimate. Thus, we find the probability that Geoffrey will ride on any given roller coaster by dividing the number of times he rode roller coasters by the number of times he tossed his coin.

Geoffrey rode on 1*3 + 2*4 + 3*1 = 14 roller coasters during the course of 30 coin tosses, so the method of moments and maximum likelihood estimate for p is 14/30 = **p = 7/15**.

Problem S3L72-5. Similar to Question 5 from the Casualty Actuarial Society's Fall 2005 Exam 3. This question is a review of Section 56. 20 observations are drawn from a normal distribution. You know that for these observations X_i, ${}_{i=1}{}^{n}\sum(X_i) = 400$ and ${}_{=1}{}^{n}\sum(X_i{}^2) = 8150$. Which of these are 95% confidence intervals for the distribution variance, σ^2?

(a) $(14.8265296, \infty)$

(b) $(0, 6.234156)$

Solution S3L72-5. We can find the test statistic:

$\chi^2 = {}_{i=1}{}^{n}\sum(X_i - \mu_{X\_observed})^2/\sigma_0{}^2 = {}_{i=1}{}^{n}\sum(X_i{}^2) - n*\mu_{X\_observed}{}^2$.

Since ${}_{i=1}{}^{n}\sum(X_i) = 400$ and $n = 20$, $\mu_{X\_observed} = 400/20 = 20$.

Thus, $\chi^2 = (8150 - 20*20^2)/\sigma_0{}^2 = 150/\sigma_0{}^2$.

The number of degrees of freedom associated with this Chi-square distribution is 20-1 = 19. The 95% confidence interval for σ^2 will be any interval $(\sigma_B{}^2, \sigma_A{}^2)$ such that

$\Pr(\chi^2{}_{0.95,19} \,^{\circ}\, (150/\sigma_A{}^2, 150/\sigma_B{}^2)) = 0.95$.

We examine the Table of Critical Values for the Chi-Square Distribution.

For 19 degrees of freedom and probability 0.95 of exceeding the critical value,

$\chi^2{}_{0.95,19} = 10.117$.

Thus, $(150/\sigma_A{}^2, 150/\sigma_B{}^2) = (0, 10.117) \rightarrow (\sigma_B{}^2, \sigma_A{}^2) = (150/10.117, 150/0) = (14.8265296, \infty)$ is a 95% confidence interval for σ^2 and (a) is a correct answer.

For 19 degrees of freedom and probability 0.95 of exceeding the critical value,

$\chi^2{}_{0.05,19} = 30.144$, so $(150/\sigma_A{}^2, 150/\sigma_B{}^2) = (30.144, \infty) \rightarrow$

$(\sigma_B{}^2, \sigma_A{}^2) = (150/\infty, 150/30.144) = (0, 4.97611465)$ is a 95% confidence interval for σ^2. But the interval in (b) is (0, 6.234156), which is clearly a larger interval than (0, 4.97611465). Thus, **only (a) is a 95% confidence interval for σ^2**.

Section 73

Exam-Style Questions for Actuarial Exam 3L – Part 4

Some of the problems in this section were designed to be similar to problems from past versions of the Casualty Actuarial Society's Exam 3L and the Society of Actuaries' Exam MLC. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

Source: Broverman, Sam. Actuarial Exam Solutions - CAS Exam 3 - Fall 2005.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L73-1. Similar to Question 6 from the Casualty Actuarial Society's Fall 2005 Exam 3. This question is a review of Section 60. A random variable follows a uniform distribution on the interval [0, W]. You take a sample of 15 observations and obtain an estimate $\hat{W}$ for W by picking $\hat{W}$ to be Y, the largest of the 15 observations.

The probability density function for Y is equal to $f_Y(y) = 15y^{14}/W^{15}$, for $0 \leq y \leq W$. Given that the true value of W is 300, find the mean square error of $\hat{W}$.

Solution S3L73-1. When you use an estimate $\hat{W}$ for a value W, then the mean square errorassociated with the estimate is defined as MSE= $E((\hat{W} - W)^2)$. Here, W = 300, so MSE= $E((\hat{W} - 300)^2) =$
$E(\hat{W}^2 - 600W + 90000) = E(\hat{W}^2) - 600E(\hat{W}) + 90000$. We need to find $E(\hat{W}^2)$ and $E(\hat{W})$.
$E(\hat{W}) = E(Y) = {}_0{}^W\!\int y*(15y^{14}/W^{15})dw = {}_0{}^W\!\int(15y^{15}/W^{15})dw = (15y^{16}/16W^{15})\,|\,_0{}^W = (15W^{16}/16W^{15})$ $= 15W/16 = 15*300/16 = 281.25$.
$E(\hat{W}^2) = E(Y^2) = {}_0{}^W\!\int y^2*(15y^{14}/W^{15})dw = {}_0{}^W\!\int(15y^{16}/W^{15})dw = (15y^{17}/17W^{15})\,|\,_0{}^W =$ $(15W^{17}/17W^{15}) =$
$15W^2/17 = 15*90000/17 = 79411.76471$.
Thus, MSE = 79411.76471 - 600*281.25 + 90000 = **MSE = about 661.764059**.

Problem S3L73-2. Similar to Question 7 from the Casualty Actuarial Society's Fall 2005 Exam 3.

You are analyzing values drawn from a normal distribution. You are given the following values of x corresponding to values of $\Phi(x)$:

Φ(x).......x
0.93.......1.476
0.94.......1.555
0.95.......1.645
0.97.......1.751

You are testing between the following hypotheses: H_0: μ= 16versus H_1: μ= 22.

You know that the variance of the normal distribution, σ^2, is equal to 9.

The probability of a Type I error is 0.03, and the probability of a Type II error is no more than 0.07. Find the minimum sample size required for this experiment.

Solution S3L73-2. Recall that in a sample of size n drawn from a normal distribution with standard deviation σ, the sample standard deviation will be equal to σ/√(n).

The threshold for rejecting H_0 and accepting H_1 can be found in two ways.

Based on the Type I error information, we know that only the highest 0.03 of the possible values, if H_0 is true, would be rejected. Thus, the threshold for rejecting H_0 is Φ(0.97) = 1.751 standard deviations to the right from the mean.

This threshold can be expressed as 16 + 1.751σ/√(n) = 16 + 1.751*3/√(n), since σ = 3.
Based on the Type II error information, the most the threshold for rejecting H_0 can be is Φ(0.93) = 1.476 standard deviations to the left of the mean if H_1 is true.

This threshold can be expressed as 22 - 1.476*3/√(n). Thus, for the minimum sample size n, we set up the equality 16 + 1.751*3/√(n) = 22 - 1.476*3/√(n) → 9.681/√(n) = 6 → n = $(9.681/6)^2$ = n = 2.60338225. Since sample sizes must be whole numbers, the smallest possible sample size is **n = 3.**

Problem S3L73-3. Similar to Question 8 from the Casualty Actuarial Society's Fall 2005 Exam 3. This question is a review of Section 72. The number of failed government programs per country follows a normal distribution with an unknown mean and variance. You have the following sample of countries and failed government programs.

Country...............Number of failed government programs
1............................315
2............................1235
3............................1513
4............................1612
5............................2000

You are testing between the following hypotheses: H_0: σ^2 = 100000 versus H_1: σ^2 < 100000. Within which of the following ranges does the p-value for this hypothesis test fall?

(a) $p < 0.001$
(b) $0.001 \leq p < 0.01$
(c) $0.01 \leq p < 0.025$
(d) $0.025 \leq p < 0.05$
(e) $0.05 \leq p < 0.1$
(f) $0.1 \leq p$

Use the Table of Critical Values for the Chi-Square Distribution.

Solution S3L73-3. We perform a Chi-square test for the variance.

The sample mean is (315 + 1235 + 1513 + 1612 + 2000)/5 = $\mu_{X\_observed}$ = 1335.

We can find the test statistic:

$\chi^2 = {}_{i=1}{}^{n}\sum(X_i - \mu_{X\_observed})^2/\sigma_0{}^2 =$
$(315 - 1335)^2/\sigma_0{}^2 + (1235 - 1335)^2/\sigma_0{}^2 + (1513 - 1335)^2/\sigma_0{}^2 + (1612 - 1335)^2/\sigma_0{}^2 + (2000 - 1335)^2/\sigma_0{}^2 = 1601038/\sigma_0{}^2$. We are given that $\sigma_0{}^2 = 100000$, so $\chi^2 = 16.01038$, with 5 - 1 = 4 degrees of freedom.

We examine the Table of Critical Values for the Chi-Square Distribution in the row pertaining to 4 degrees of freedom to find that to find that the value $\chi^2 = 13.277$ corresponds with p = 0.01 and the value $\chi^2 = 18.467$ corresponds with p = 0.001. Our observed χ^2 is between those two values, so the correct answer is **(b) 0.001 ≤ p < 0.01**.

Problem S3L73-4. Similar to Question 9 from the Casualty Actuarial Society's Fall 2005 Exam 3. This question is a review of Section 57. The number of strange accidents that occur in a given year x is modeled by the function $y = \alpha e^{\beta x}$. You know the following about strange accidents over the course of four years.

Year............Number of strange accidents
1....................3546
2....................3665
3....................3793
4....................4012

Use least-squares linear regression to find α and β and find the number of strange accidents predicted by this function to occur in year 7.

Solution S3L73-4. How can one use linear regression on an exponential function? We can turn y into a linear function by taking its natural logarithm:
$z = \ln(y) = \ln(\alpha e^{\beta x}) \rightarrow z = \ln(y) = \ln(\alpha) + \beta x$.
Now we can find $\beta = (n*{}_{i=1}{}^{n}\sum(x_i z_i) - {}_{i=1}{}^{n}\sum(x_i)\ {}_{i=1}{}^{n}\sum(z_i))/(n*{}_{i=1}{}^{n}\sum(x_i{}^2) - {}_{i=1}{}^{n}\sum(x_i)^2)$.
We know that n = 4.
${}_{i=1}{}^{n}\sum(x_i z_i) = 1*\ln(3546) + 2*\ln(3665) + 3*\ln(3793) + 4*\ln(4012) = {}_{i=1}{}^{n}\sum(x_i z_i) = 82.49766094$.
${}_{i=1}{}^{n}\sum(x_i) = 1 + 2 + 3 + 4 = 10$.

$_{i=1}{}^{n}\sum(z_i)$ = ln(3546) + ln(3665) + ln(3793) + ln(4012) = 32.91811679.
$_{i=1}{}^{n}\sum(x_i{}^2) = 1^2 + 2^2 + 3^2 + 4^2 = 30$.
Thus, β = (4*82.49766094 - 10*32.91811679)/(4*30 - 10^2) = β = 0.0404737915.
Now we can find ln(α) = ($_{i=1}{}^{n}\sum(y_i)$ - β*$_{i=1}{}^{n}\sum(x_i)$)/n = (32.91811679 - 0.0404737915*10)/4 = ln(α) = 8.128344719.
Thus, for x = 7, z = 8.128344719 + 0.0404737915*7 = 8.41166126, and y = e^z = **about 4499.228693.**

Problem S3L73-5. Similar to Question 10 from the Casualty Actuarial Society's Fall 2005 Exam 3. This question is a review of Section 10. The survival function of life (x) is as follows: s(x) = 1 - x/230 for 0 ≤ x ≤ 230, 0 otherwise. Find $\dot{e}_{145}$, the complete expectation of life at age 145.

Solution S3L73-5. We note that the future lifetime of (x) follows a uniform distribution with ω = 230. A life aged 145 has its future lifetime uniformly distributed along the interval [145, 230], implying that the expected lifetime from age 145 is exactly half of that interval, i.e., (230 - 145)/2 = **42.5 years.**

Section 74

Exam-Style Questions for Actuarial Exam 3L – Part 5

Some of the problems in this section were designed to be similar to problems from past versions of the Casualty Actuarial Society's Exam 3L and the Society of Actuaries' Exam MLC. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

Source: Broverman, Sam. Actuarial Exam Solutions - CAS Exam 3 - Fall 2005.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L74-1. Similar to Question 11 from the Casualty Actuarial Society's Fall 2005 Exam 3. Within a particular island tribe, the force of mortality for tribe members is a constant μ. Under these conditions, 30% of any group of newborns dies by age 15. Western doctors visit the tribe and are able to reduce the force of mortality to $\mu/3$. Under these new conditions, what percentage newborns can be expected to die by age 15?

Solution S3L74-1. Currently, we know that $s(15) = e^{-\mu*15} = 0.7$, which means that $\mu = \ln(0.7)/-15 = 0.0237783296$. The new force of mortality is 0.0237783296/3 = 0.007926109. Thus, the new $s(15)$ is $e^{-0.007926109*15} = 0.8879040017$, implying that only 1 - 0.8879040017 = **11.20959983%** of newborns now die by age 15.

Problem S3L74-2. Similar to Question 12 from the Casualty Actuarial Society's Fall 2005 Exam 3. This question is a review of Section 62. Orange crocodiles are subject to a force of mortality $\mu_{x+t} = (x + t - 35)/300$ for $x > 35$ and $t \geq 0$. Find $_{15"}q_{38}$, the probability that an orange crocodile aged 38 will die between ages 53 and 54.

Solution S3L74-2. $_{15"}q_{38} = {}_{15}p_{38} - {}_{16}p_{38}$. To find each of the "p" values, we use the formula

$_np_x = \exp(-_0{}^n\int\mu_{x+t}*dt)$.Thus, $_{15}p_{38} = \exp(-_0{}^{15}\int(38 + t - 35)/300*dt) = \exp(-_0{}^{15}\int(3 + t)/300*dt) =$

$\exp(-(3 + t)^2/600"\ _0{}^{15}) = \exp(-0.525)$.

Likewise, $_{16}p_{38} = \exp(-(3 + t)^2/600"\ _0{}^{16}) = \exp(-0.5866666667)$.

Thus, $_{15"}q_{38} = \exp(-0.525) - \exp(-0.5866666667) = $ $\mathbf{_{15"}q_{38} = 0.035377239}$.

Problem S3L74-3. Similar to Question 13 from the Casualty Actuarial Society's [Fall 2005 Exam 3]. You are given the following survival function for blue zebras: $s(x) = 1 - x/140$ for $0 \leq x \leq 140$ and 0 otherwise. Find $_{0.3}q_{56.25}$, the probability that a blue zebra aged 56.25 years will die within the next 0.3 years.

Solution S3L74-3. The lifetime of blue zebras follows a uniform distribution, which implies a uniform distribution of deaths for fractional ages as well. The future lifetime of a blue zebra aged 56.25 years is uniformly distributed over the interval [56.25, 140], with p. d. f. 1/(140-56.25) = 1/83.75. The probability of death within the first 0.3 years is simply 0.3/83.75 = **about 0.0035820896.**

Problem S3L74-4. Similar to Question 14 from the Casualty Actuarial Society's [Fall 2005 Exam 3]. This question is a review of Section 62. For each two independent lives (x) and (y), you are given that the force of mortality $\mu_x = 1/(230 - x)$ for $x < 230$. Find $_{20}p_{\overline{100:130}}$, the probability that the last-survivor status of lives (100) and (130) will survive for 20 years.

Solution S3L74-4. $_{20}p_{\overline{100:130}}$ =1 - Pr(both lives die within 20 years) = $1 - (_{20}q_{100})(_{20}q_{130})$ =

$1 - (1 - {_{20}p_{100}})(1 - {_{20}p_{130}})$. Now we can use the formula $_np_x = \exp(-{_x^{x+n}}\int\mu(t)*dt)$.

$_{20}p_{100} = \exp(-{_{100}^{120}}\int(1/(230 - x))*dt) = \exp(\ln(230 - x)''\ {_{100}^{120}}) = \exp(\ln(110) - \ln(130)) =$ 0.8461538462.

$_{20}p_{130} = \exp(-{_{130}^{150}}\int(1/(230 - x))*dt) = \exp(\ln(230 - x)''\ {_{130}^{150}}) = \exp(\ln(80) - \ln(100)) = 0.8$.

Thus, $_{20}p_{\overline{100:130}}$ = 1 - (1 - 0.8461538462)(1 - 0.8) = $\mathbf{_{20}p_{\overline{100:130}}}$ **= about 0.9692307692**.

Problem S3L74-5. Similar to Question 15 from the Casualty Actuarial Society's [Fall 2005 Exam 3]. Xland is ruled by two co-presidents, who are elected for the duration of their joint survivor status. When either of the co-presidents dies, a new election is called. The future lifetime of each co-president is independent of the other and has the associated survival function $s(x) = 1 - 0.025x$ for $x < 20$. Find the probability that a new election will be called within 25 years.

Solution S3L74-5. A new election will be called if either co-president dies within the next 25 years. The probability of this happening is the complement of the probability that both co-presidents survive during the next 25 years. Each of the lifetimes is uniformly distributed with ω = 40. The probability of each of the co-presidents surviving past 25 years is (40 - 25)/40 = 15/40. Thus, the probability that a new election will be called within 25 years is $1 - (15/40)^2$ = **0.859375.**

Section 75

Exam-Style Questions for Actuarial Exam 3L – Part 6

The following equations are useful.

Property 75.1. ${}_0{}^n\int {}_tp_x*\mu_x(t)dt = {}_nq_x$.

Property 75.2. If Y = g(X), then $f_Y(y) = f_X(g^{-1}(y))*g^{-1}{}'(y)$.

Property 75.3. If losses are modeled by the random variable X, and an insurance company pays all losses minus a deductible d, then the expected value of payouts made by the insurance company is E(X) - E(X Λ d), where X Λ d is the random variable X truncated at d, i.e., all values of X except those values that exceed d.

Property 75.4. For a random variable X that follows an exponential distribution with mean θ, $E(X \Lambda d) = \theta(1 - e^{-d/\theta})$.

Source: Broverman, Sam. Actuarial Exam Solutions - CAS Exam 3 - Fall 2005.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L75-1. Similar to Question 16 from the Casualty Actuarial Society's Fall 2005 Exam 3. This question is a review of Section 66. Two independent lives, (x) and (y), each have the following function associated with them: ${}_tp_x = {}_tp_y = (1 - t/60)^3$ for t < 60. Find the difference between the expected time of the second death and the expected time of the first death.

Solution S3L75-1. We are asked to find $\dot{e}_{--xy} - \dot{e}_{xy} = {}_0{}^{60}\int {}_tp_{--xy}*dt - {}_0{}^{60}\int {}_tp_{xy}*dt =$
${}_0{}^{60}\int({}_tp_x + {}_tp_y - {}_tp_{xy})*dt - {}_0{}^{60}\int {}_tp_{xy}*dt = {}_0{}^{60}\int({}_tp_x + {}_tp_y - 2{}_tp_{xy})*dt = 2\ {}_0{}^{60}\int((1 - t/60)^3 - (1 - t/60)^6)dt$
=
$2(60(1 - t/60)^7/7 - 60(1 - t/60)^4/4) \mid {}_0{}^{60}$ = **90/7 = about 12.85714286 years**.

Problem S3L75-2. Similar to Question 17 from the Casualty Actuarial Society's Fall 2005 Exam 3. 1000 badly managed financial institutions are in operation at the beginning of year 1. At the end of each year, the institutions are either nationalized by the government (cause of decrement 1) or go bankrupt (cause of decrement 2) or continue in business until at least the next year.

You know the following data:

Year.......# Institutions Nationalized (1)........# Institutions Bankrupt (2)
1................24...123
2................314.......................................231
3................124..4

4................131..20

To further exacerbate moral hazard problems, the government decides to give a $1 billion bailout to each financial institution that goes bankrupt, at the end of the year of its bankruptcy. The annual effective interest rate is 0.09. Calculate the actuarial present value of the bailout amount to an institution that has survived for one year.

Solution S3L75-2. If an institution has survived for one year, it is one of 1000 - 24 - 123 = 853 remaining institutions. At the current time, it has a 231/853 probability of bankruptcy in year 2, a 4/853 probability of bankruptcy in year 3, and a 20/853 probability of bankruptcy in year 4. Thus, the APV of the bailout amount to the institution is

$1000000000(231/853(1/1.09) + 4/853(1/1.09^2) + 20/853(1/1.09^3))$ = **about $270,500,580.40**

Problem S3L75-3. Similar to Question 18 from the Casualty Actuarial Society's Fall 2005 Exam 3. The annual interest rate is zero, and an entity having a 30-year term life insurance policy is subject to only two causes of decrement - (1): shark attacks and (2): death from waiting in line for socialized healthcare. The insurance policy pays a premium of 5 at the time of any death from shark attacks and a premium of 10 at the time of any death from waiting in line for socialized healthcare. The force of mortality from shark attacks is $\mu_{x+t}^{(1)} = 0.2t$ and the force of mortality from waiting in line for socialized healthcare is $\mu_{x+t}^{(2)} = 0.6t$. The insurance is paid for by a net single premium. Find that premium.

(a) $(35/4)(1 - e^{-720})$
(b) $(70/4)(1 - e^{-360})$
(c) $(35/4)(1 - e^{-360})$
(d) $1 - e^{-360}$
(e) $1 - e^{-720}$

Solution S3L75-3. We want to find the APV of the insurance policy, which will be the single premium. Typically, $\bar{A}^1_{x:n\neg} = {}_0{}^n\int v^t * {}_tp_x * \mu_x(t)dt$, but here i = 0, so $v^t = 1$ and

$\bar{A}^1_{x:n\neg} = {}_0{}^n\int {}_tp_x * \mu_x(t)dt = {}_nq_x$ by Property 75.1.

What we want to find is not $\bar{A}^1_{x:n\neg}$, but rather the sum of (APV payment due to shark attacks) + (APV payment due to waiting in line for socialized healthcare) =

$5 * {}_0{}^{30}\int {}_tp_x{}^{(\tau)} * \mu_{x+t}{}^{(1)}dt + 10 * {}_0{}^{30}\int {}_tp_x{}^{(\tau)} * \mu_{x+t}{}^{(1)}dt.$

We find ${}_tp_x{}^{(\tau)} = \exp(-{}_0{}^{30}\int \mu_{x+t}{}^{(\tau)}dt)$, where $\mu_{x+t}{}^{(\tau)} = 0.2t + 0.6t = 0.8t$. Thus, ${}_tp_x{}^{(\tau)} = \exp(-{}_0{}^{30}\int 0.8t dt)$ =
$\exp(-0.4t^2 \mid {}_0{}^{30}) = {}_tp_x{}^{(\tau)} = \exp(-360)$.
We have ${}_nq_x{}^{(1)} = (1/4){}_nq_x{}^{(\tau)}$ and ${}_nq_x{}^{(1)} = (3/4){}_nq_x{}^{(\tau)}$.

Thus, our desired answer is

$5*{}_0{}^{30}\int{}_tp_x{}^{(\tau)}*\mu_{x+t}{}^{(1)}dt + 10*{}_0{}^{30}\int{}_tp_x{}^{(\tau)}*\mu_{x+t}{}^{(1)}dt = 5(1/4)_nq_x{}^{(\tau)} + 10(3/4)_nq_x{}^{(\tau)}$

$= 5(1/4)(1 - {}_tp_x{}^{(\tau)}) + 10(3/4)(1 - {}_tp_x{}^{(\tau)})$ = **$(35/4)(1 - e^{-360})$**. Thus, **(c) is the correct answer.**

Problem S3L75-4. Similar to Question 19 from the Casualty Actuarial Society's Fall 2005 Exam 3. The random variable X follows an exponential distribution with $f_X(x) = e^{-x}$. The random variable $Y = X^{1/\tau}$. Find the p.d.f. of Y, $f_Y(y)$.

Solution S3L75-4. We use the formula $f_Y(y) = f_X(g^{-1}(y))*g^{-1}{}'(y)$. Since g(X) here is $X^{1/\tau}$, it follows that $g^{-1}(Y) = Y^\tau$. Thus, $g^{-1}(y) = y^\tau$ and $g^{-1}{}'(y) = \tau y^{\tau-1}$. Hence, $\mathbf{f_Y(y) = \tau y^{\tau-1}e^{-y^\wedge\tau}}$.

Problem S3L75-5. Similar to Question 20 from the Casualty Actuarial Society's Fall 2005 Exam 3. Losses from wild gerbil attacks follow an exponential distribution with mean θ. The expected payout from an insurance policy covering wild gerbil attacks with a deductible of $300 is $50000. Find the expected payout from an insurance policy covering wild gerbil attacks with a deductible of $7000.

(a) $50000(1 - e^{-7000/\theta})$
(b) $\theta(1 - e^{-7000/\theta})$
(c) θ
(d) $50000e^{-70/3\theta}$
(e) $50000e^{-6700/\theta}$

Solution S3L75-5. By Property 75.3, the expected value of a policy with deductible d is E(X) - E(X Λ d). E(X) = θ and, by Property 75.4, $E(X \Lambda d) = \theta(1 - e^{-d/\theta})$.
Thus, $E(X) - E(X \Lambda d) = \theta - \theta(1 - e^{-d/\theta}) = \theta e^{-d/\theta}$. However, $\theta e^{-7000/\theta}$ is not one of the answers provided.
However, we do know that $50000 = \theta e^{-300/\theta}$, and therefore $\theta e^{-7000/\theta} = \theta e^{-300/\theta} * e^{-6700/\theta} = 50000e^{-6700/\theta}$. Thus, **(e): $50000e^{-6700/\theta}$ is the correct answer.**

Section 76

Exam-Style Questions for Actuarial Exam 3L – Part 7

Some of the problems in this section were designed to be similar to problems from past versions of the Casualty Actuarial Society's Exam 3L and the Society of Actuaries' Exam MLC. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

Source: Broverman, Sam. Actuarial Exam Solutions - CAS Exam 3 - Fall 2005.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L76-1. Similar to Question 21 from the Casualty Actuarial Society's Fall 2005 Exam 3. This question is a review of Section 47. The amount of carelessly lost money this year follows a Pareto distribution with $\theta = 100000$ and $\alpha = 3$. Due to government continually expanding credit and printing money, annual inflation is 50%. Find the ratio of the probability that the amount of lost money next year will exceed \$500000 to the probability that the amount of lost money this year will exceed \$500000.

Solution S3L76-1. We use the formula $s(x) = \theta^\alpha/(x + \theta)^\alpha$. This year, $\theta = 100000$ and $x = 500000$, so $s(500000) = 100000^3/(600000)^3 = 1/216$.
Next year, θ will increase to $100000*1.5 = 150000$. Thus, $s(500000)$ will become $150000^3/(650000)^3 = 27/2197$.
We want to find $(27/2197)/(1/216) =$ **5832/2197 = about 2.654528903**.

Problem S3L76-2. Similar to Question 22 from the Casualty Actuarial Society's Fall 2005 Exam 3. This question is a review of Section 51. A widget insurance company compensates Reynaldo for damages to his widget property. Annual damages to Reynaldo's widgets (L) follow a Pareto distribution with $\theta = 1240$ and $\alpha = 4$. The widget insurance company pays Reynaldo an annual amount equal to $(6400 - L)/21$, provided that this amount is positive. Find the annual payment Reynaldo can expect to receive from this insurance company.

Solution S3L76-2. We use the formula $E(L \wedge K) =(\theta/(\alpha - 1))(1 - (\theta/(K + \theta))^{(\alpha - 1)})$. Here, $K = 6400$, since this is the highest value of L for which $(6400 - L)/21$ is positive.
Thus, $E(L \wedge 6400) = (1240/(3 - 1))(1 - (1240/(6400 + 1240))^{(4 - 1)}) = 617.3492018$.
Reynaldo can expect to receive a payment of $E((6400 - L)/21) = (6400 - E(L \wedge 6400))/21 =$

(6400 - 617.3492018)/21 = **about 275.3543237.**

Problem S3L76-3. Similar to Question 23 from the Casualty Actuarial Society's Fall 2005 Exam 3. An entity at time 0 has a 0.2 probability of being assigned to state X, a 0.5 probability of being assigned to state Y, and a 0.3 probability of being assigned to state Z. The following transition matrices represent transitions at times 1 and 2.

Q_1:
...X......Y.....Z
X(0.1....0.5...0.4)
Y(0.4...0.3...0.3)
Z(0......0.7...0.3)

Q_2:
...X......Y.....Z
X(0.3....0.2...0.5)
Y(0.1...0.7...0.2)
Z(0.2...0.4...0.4)

Find the probability that an entity at time 0 will transition from state X to state Z at time 2.

Solution S3L76-3. There are three ways in which an entity at time 0 can transition from state X to state Z at time 2: X → X→ Z, Y → X→ Z, and Z → X→ Z.
Pr(X → X→ Z) = 0.2*0.1*0.5 = 0.01.
Pr(Y → X→ Z) = 0.5*0.4*0.5 = 0.1.
Pr(Z → X→ Z) = 0.3*0*0.5 = 0.
Thus, the desired answer is 0.01 + 0.1 = **0.11**.

Problem S3L76-4. Similar to Question 24 from the Casualty Actuarial Society's Fall 2005 Exam 3. Monthly claims follow a Poisson distribution with λ = 0.234. You are given the following distribution of losses per claim.

Loss (x)...........F(x)
50..................0.1
240.................0.4
2140...............0.6
21400..............0.8
215400.............1

An insurance policy pays all losses except for a deductible of $240 per loss. Find the probability that the insurance company will make at least one payment this month.

Solution S3L76-4. The company will only make a payment on losses greater than $240. That is, it will only make a payment on 0.6 of all possible losses. So the losses that the company will pay for follow a Poisson distribution with mean 0.6*0.234 = 0.1404. The probability that at least one payment will be made is thus 1 - $e^{-0.1404}$ = **about 0.1309894384.**

Problem S3L76-5. Similar to Question 25 from the Casualty Actuarial Society's Fall 2005 Exam 3. The number of leopard attacks in a year follows a Poisson distribution with mean 35. The number of piranha attacks in a year follows a Poisson distribution with mean 120. Find the probability that 6 piranha attacks will have occurred before one leopard attack occurs.

Solution S3L76-5. Since the number of leopard attacks (L) and the number of piranha attacks (P) follow independent Poisson processes, we can add L and P into a new Poisson process, with the mean of L + P being 35 + 120 = 155. The probability that an event within that process is a piranha attack is 120/155. The probability that the next six events will all be piranha attacks is $(120/155)^6$ = **about 0.2153264038.**

Section 77

Exam-Style Questions for Actuarial Exam 3L – Part 8

Some of the problems in this section were designed to be similar to problems from past versions of the Casualty Actuarial Society's Exam 3L and the Society of Actuaries' Exam MLC. They use original exam questions as their inspiration - and the specific inspiration for each problem is cited so as to give students a chance to see the original. All of the original problems are publicly available, and students are encouraged to refer to them. But all of the values, names, conditions, and calculations in the problems here are the original work of Mr. Stolyarov.

Source: Broverman, Sam. Actuarial Exam Solutions - CAS Exam 3 - Fall 2005.

Original Problems and Solutions from The Actuary's Free Study Guide

Problem S3L77-1. Similar to Question 26 from the Casualty Actuarial Society's Fall 2005 Exam 3. The number of spotted crocodile sightings during a 65-hour period follows a Poisson distribution with intensity function $\lambda(t) = (t/92)^{5/19}$, for $0 \leq t \leq 65$, where t is measured in hours.

Find the probability that there are no spotted crocodile sightings between the beginning of hour 25 and the beginning of hour 28.

Solution S3L77-1. The mean number of no spotted crocodile sightings between the beginning of hour 25 and the beginning of hour 28 is
$_{25}{}^{28}\int\lambda(t)dt = {}_{25}{}^{28}\int(t/92)^{5/19}dt = (19/24)t^{24/19}/(92)^{5/19}\Big|_{25}^{28} =$
$(19/24)(92)^{-5/19}(28^{24/19} - 25^{24/19}) = 2.161863984$. Thus, the probability that there are no spotted crocodile sightings between the beginning of hour 25 and the beginning of hour 28 is $e^{-2.161863984}$ = **about 0.115110357.**

Problem S3L77-2. Similar to Question 27 from the Casualty Actuarial Society's Fall 2005 Exam 3. The number of new businesses forming per day follows a Poisson distribution with mean 15. The number of new employees per business follows a negative binomial distribution with r = 5 and β = 0.4. Find the variance of the number of new employees in new businesses formed within a 20-day period.

Solution S3L77-2. We let N be the random variable representing the number of new businesses formed within a 20-day period. Then E(N) = 15*20 = 300.

We let X be the random variable representing the number of new employees per new business. Then E(X) = rβ = 5*0.4 = 2 and Var(X) = rβ(1 + β) = 5*0.4*1.4 = 28.
Thus, $E(X^2) = Var(X) + E(X)^2 = 28 + 4 = 32$.

The number of new employees in new businesses formed within a 20-day period follows a compound Poisson process S, where Var(S) = E(N)*E(X^2) = 300*32 = **Var(S) = 9600**.

Problem S3L77-3. Similar to Question 28 from the Casualty Actuarial Society's Fall 2005 Exam 3. There are three entrances to a government building. As a result of tightened security regulations, each person entering the building must submit to a thorough inspection. On average, 12 inspections are completed per hour at each entrance, and the number of inspections follows a Poisson process.

Benjamin arrives in front of the building at a time when all three entrances are occupied by inspections. There are 4 people waiting in front of him to be inspected. Find the probability that Benjamin will wait more than 20 minutes before his inspection begins.

Solution S3L77-3. The average number of inspections completed per hour at all three entrances is 12*3 = 36. In 20 minutes, then, the average number of inspections completed is 36/3 = 12. Benjamin will wait more than 20 minutes for his inspection if only 4 or fewer inspections are completed during the course of 20 minutes.

The probability of this happening is Pr(0 or 1 or 2 or 3 or 4 inspections completed in 20 minutes) = $e^{-12} + 12e^{-12} + (12^2/2)e^{-12} + (12^3/6)e^{-12} + (12^4/24)e^{-12}$**= about 0.0076003907.**

Problem S3L77-4. Similar to Question 29 from the Casualty Actuarial Society's Fall 2005 Exam 3. The time between corruption hearings for a certain legislative body is exponentially distributed with a mean of 0.2 months. Every time there is a corruption hearing, there is a 0.4 probability that one member of the legislative body will be convicted and promoted for his/her corruption. Find the probability that within one month, no legislators will be convicted and promoted for corruption.

Solution S3L77-4. The average number of corruption hearings per month is 1/0.2 = 5 (this is the mean of a Poisson process, since times between events are exponentially distributed). The average number of convictions/promotions per month is 5*0.4 = 2. This, too, is the mean of a Poisson process. Thus, the probability that

no convictions/promotions will happen in a month is **e^{-2} = about 0.1353352832**.

Problem S3L77-5. Similar to Question 31 from the Casualty Actuarial Society's Fall 2005 Exam 3. Attempts to repeal disastrous regulations are made according to a Poisson process with annual mean of 20. Each attempt is independent, and the probability of each attempt's success is 0.05. Find the standard deviation of the number of disastrous regulations repealed per year.

Solution S3L77-5. Let S denote the number of disastrous regulations repealed per year. Then S follows a compound Poisson process, where N is the number of repeal attempts and X is the number of repeals per attempt (0 or 1). It is given that E(N) = 20. Moreover, E(X) = 0.05*1 = 0.05 and Var(X) = 0.05*0.95 = 0.0475, because X follows a binomial distribution with n = 1. Thus, E(X^2) = Var(X) + $E(X)^2$ = 0.0475 + 0.05^2 = 0.05 and so Var(S) = E(N)*E(X^2) = 20*0.05 = 1 and so $1^{1/2}$ = **SD(S) = 1**.

About Mr. Stolyarov

Gennady Stolyarov II (G. Stolyarov II) is an actuary, science-fiction novelist, independent philosophical essayist, poet, amateur mathematician, composer, and Editor-in-Chief of The Rational Argumentator, a magazine championing the principles of reason, rights, and progress.

In December 2013, Mr. Stolyarov published *Death is Wrong*, an ambitious children's book on life extension illustrated by his wife Wendy. *Death is Wrong* can be found on Amazon in paperback and Kindle formats.

Mr. Stolyarov has contributed articles to the Institute for Ethics and Emerging Technologies (IEET), The Wave Chronicle, Le Quebecois Libre, Brighter Brains Institute, Immortal Life, Enter Stage Right, Rebirth of Reason, The Liberal Institute, and the Ludwig von Mises Institute. Mr. Stolyarov also published his articles on Associated Content (subsequently the Yahoo! Contributor Network) from 2007 until its closure in 2014, in an effort to assist the spread of rational ideas. He held the highest Clout Level (10) possible on the Yahoo! Contributor Network and was one of its Page View Millionaires, with over 3.1 million views.

Mr. Stolyarov holds the professional insurance designations of Associate of the Society of Actuaries (ASA), Associate of the Casualty Actuarial Society (ACAS), Member of the American Academy of Actuaries (MAAA), Chartered Property Casualty Underwriter (CPCU), Associate in Reinsurance (ARe), Associate in Regulation and Compliance (ARC), Associate in Personal Insurance (API), Associate in Insurance Services (AIS), Accredited Insurance Examiner (AIE), and Associate in Insurance Accounting and Finance (AIAF).

Mr. Stolyarov has written a science fiction novel, *Eden against the Colossus*, a philosophical treatise, *A Rational Cosmology*, a play, *Implied Consent*, and a free self-help treatise, *The Best Self-Help is Free*. You can watch his YouTube Videos. Mr. Stolyarov can be contacted at gennadystolyarovii@gmail.com.